Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 949143, 25 pages
doi:10.1155/2010/949143

Research Article

A Theoretical Development of Distance Measure for Intuitionistic Fuzzy Numbers

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Received 4 August 2009; Revised 14 January 2010; Accepted 18 February 2010

Academic Editor: Andrzej Skowron

The objective of this paper is to introduce a distance measure for intuitionistic fuzzy numbers. Firstly the existing distance measures for intuitionistic fuzzy sets are analyzed and compared with the help of some examples. Then the new distance measure for intuitionistic fuzzy numbers is proposed based on interval difference. Also in particular the type of distance measure for triangle intuitionistic fuzzy numbers is described. The metric properties of the proposed measure are also studied. Some numerical examples are considered for applying the proposed measure and finally the result is compared with the existing ones.

1. Introduction

The theory of fuzzy set introduced by Zadeh [1] in 1965 has achieved successful applications in various fields. This is because this theory is an extraordinary tool for representing human knowledge, perception, and so forth. Nevertheless, Zadeh himself established in 1973 knowledge which is better represented by means of some generalizations of fuzzy sets. The so-called extensions of fuzzy set theory arise in this way.

Two years after the concept of fuzzy set was proposed, it was generalized by Gogeun and L-fuzzy set [2] was developed. There are also some other extensions of fuzzy sets. Out of several higher-order fuzzy sets, the concept of intuitionistic fuzzy sets (IFSs) proposed by Atanassov [3] in 1986 is found to be highly useful to deal with vagueness. The major advantage of IFS over fuzzy set is that IFSs separate the degree of membership (belongingness) and the degree of nonmembership (nonbelongingness) of an element in the set. Then in 1993, Gau and Buehrer [4] introduced the concept of vague sets, which is another generalization of fuzzy sets. Bustince and Burillo [5] pointed out that the notion of vague set is the same as that of IFSs. Another well-known generalization of ordinary fuzzy sets is the

concept of interval-valued fuzzy set [6–10]. There is a strong relationship between interval-valued fuzzy sets and IFSs.

Among various extensions of fuzzy sets, IFSs have captured the attention of many researchers in the last few decades. This is mainly due to the fact that IFSs are consistent with human behavior, by reflecting and modeling the hesitancy present in real life situations. Therefore in practice, it is realized that human expressions like perception, knowledge, and behavior are better represented by IFSs rather than fuzzy sets. IFS theory is applied to many different fields such as decision making, logic programming, medical diagnosis, and pattern recognition.

In the application of fuzzy sets as well as IFSs, similarity measures play a very important role. But the similarity measures bear a relation to distances in many cases. Therefore the study about the distance measures is very much significant. Developing distance measures is one of the fundamental problems of fuzzy set theory. A lot of research has been done to construct the distance measure between fuzzy sets [11]. Recently some researchers have focused their attention to compute the distances between fuzzy numbers [12–18] also. As important contents in fuzzy mathematics, distance measures between IFSs have also attracted many researchers. Several researchers [19–24] focused on computing the distance between IFSs, which we discuss briefly later in Section 2.4.

It has been observed that all the papers discussed above considered distances between IFSs on finite universe of discourses only. But construction of distance measures between IFSs for countable and uncountable universe of discourse is also necessary. With this point of view, the concept of intuitionistic fuzzy number (IFN) [25–29] with the universe of discourse as the real line was introduced and studied.

Grzegorzewski [28] introduced two families of metrics in a space of IFNs. A method of ranking IFNs based on these metrics was also suggested and investigated in that work. But this distance measure is not effective for some cases. The distance measures proposed by Grzegorzewski [28] compute crisp distance measures for IFNs. But the well-known fact that needs to be remembered here is that [17] "if we are not certain about the numbers themselves, how can we be *certain* about the distances among them." This is the reason why fuzzy distance measure for measuring the distance measure between two fuzzy numbers came into the field. For the same reason it is not reasonable to define crisp distance between IFNs. Our intuition says that the uncertainty or hesitation or lack of knowledge presented in defining IFN should inherently be involved in their corresponding distance measures. Now it can be assumed that an IFN is a collection of points with different degrees of membership and corresponding degree of nonmembership. Therefore the distance between two IFNs is nothing but the distances of pairwise membership and nonmembership functions of the respective points. With this point of view in this paper a new approach is introduced to calculate the distance measure between two IFNs. Here the distance measure is proposed based on interval difference. It is worth noting that the proposed distance measures between IFNs are direct generalizations of the results obtained for the classical fuzzy numbers.

The paper is organized as follows. Section 2 briefly describes the basic definition and notations of IFS, IFN and *LR*-type IFN. Also some preliminary result is presented in this section. A short review of the existing distance measures is described in Section 2.4. Section 3 introduces the new distance measure for IFNs. The distance measure for IFNs and TIFNs is derived in Section 3.1 and Section 3.2, respectively. The metric properties are studied in Section 3.3. In Section 4, the proposed distance method is illustrated with the help of some numerical examples. The paper is concluded in Section 5.

2. Preliminaries

2.1. Intuitionistic Fuzzy Sets—Basic Definition and Notation

Let X denote a universe of discourse. Then a fuzzy set $A^{/}$ in X is defined as a set of ordered pairs:

$$A^{/} = \{\langle x, \mu_{A^{/}}(x)\rangle : x \in \mathrm{X}\}, \tag{2.1}$$

where $\mu_{A^{/}} : \mathrm{X} \to [0,1]$ and $\mu_{A^{/}}(x)$ is the grade of belongingness of x into $A^{/}$ [1]. Thus automatically the grade of nonbelongingness of x into $A^{/}$ is equal to $1 - \mu_{A^{/}}(x)$. However, while expressing the degree of membership of any given element in a fuzzy set, the corresponding degree of nonmembership is not always expressed as a compliment to 1. The fact is that in real life, the linguistic negation does not always identify with logical negation [21]. Therefore Atanassov [19, 30–33] suggested a generalization of classical fuzzy set, called IFS.

An IFS A in X is given by a set of ordered triples:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathrm{X}\}, \tag{2.2}$$

where $\mu_A, \nu_A : X \to [0,1]$ are functions such that $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$. For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively.

It is easily seen that $A_{\mathrm{IFS}} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in \mathrm{X}\}$ is equivalent to (2.1); that is, each fuzzy set is a particular case of the IFS. We will denote a family of fuzzy sets in X by $\mathrm{FS}(X)$, while $\mathrm{IFS}(X)$ stands for the family of all IFSs in X.

For each element $x \in X$ we can compute, so called, the intuitionistic fuzzy index of x in A defined as follows:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x). \tag{2.3}$$

The value of $\pi_A(x)$ is called the degree of indeterminacy (or hesitation) of the element $x \in X$ to the IFS A. It is seen immediately that $\pi_A(x) \in [0,1]$. If $A \in \mathrm{FS}(X)$, then $\pi_A(x) = 0$ for all $x \in X$.

2.2. Intuitionistic Fuzzy Numbers

Different research works [25–29] were done over Intuitionistic Fuzzy Numbers (IFNs). Taking care of those research works in this section the notion of IFNs is studied. IFN is the generalization of fuzzy number and so it can be represented in the following manner.

Definition 2.1 (Intuitionistic fuzzy numbers). An intuitionistic fuzzy subset $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathrm{X}\}$ of the real line R is called an IFN if the following holds.

(i) There exist $m \in R$, $\mu_A(m) = 1$, and $\nu_A(m) = 0$, (m is called the mean value of A).

(ii) μ_A is a continuous mapping from R to the closed interval $[0,1]$ and for all $x \in R$ the relation $0 \le \mu_A(x) + \nu_A(x) \le 1$ holds.

(iii) The membership and nonmembership function of A is of the following form:

$$\mu_A(x) = \begin{cases} 0 & \text{for } -\infty < x \leq m-\alpha, \\ f_1(x) & \text{for } x \in [m-\alpha, m], \\ 1 & \text{for } x = m, \\ h_1(x) & \text{for } x \in [m, m+\beta], \\ 0 & \text{for } m+\beta \leq x < \infty, \end{cases} \tag{2.4}$$

where $f_1(x)$ and $h_1(x)$ are strictly increasing and decreasing functions in $[m-\alpha, m]$ and $[m, m+\beta]$, respectively:

$$\nu_A(x) = \begin{cases} 1 & \text{for } -\infty \leq x \leq m-\alpha', \\ f_2(x) & \text{for } x \in [m-\alpha', m]; \quad 0 \leq f_1(x) + f_2(x) \leq 1, \\ 0 & \text{for } x = m \\ h_2(x) & \text{for } x \in [m, m+\beta']; \quad 0 \leq h_1(x) + h_2(x) \leq 1, \\ 1 & \text{for } m+\beta' \leq x \leq \infty. \end{cases} \tag{2.5}$$

Here m is the mean value of A. α and β are called left and right spreads of membership function $\mu_A(x)$, respectively. α' and β' represented left and right spreads of nonmembership function $\nu_A(x)$, respectively. Symbolically the intuitionistic fuzzy number is represented as $A_{\text{IFN}} = (m; \alpha, \beta; \alpha', \beta')$.

It is to be noted here that the IFN $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in R\}$, that is, $A_{\text{IFN}} = (m; \alpha, \beta; \alpha', \beta')$ is a conjunction of two fuzzy numbers: $A^+ = (m; \alpha, \beta)$ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and $A^- = (m; \alpha', \beta')$ with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$.

Definition 2.2 (LR-type Intuitionistic fuzzy number). An IFN A_{IFN} is LR-type IFN such that for membership and nonmembership functions $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ holds and may be defined as follows.

Membership function is of the form as follows:

$$\mu_A(x) = \begin{cases} 0 & \text{for } -\infty < x \leq m-\alpha, \\ L\left(\dfrac{m-x}{\alpha}\right) & \text{for } m-\alpha \leq x \leq m, \quad \alpha > 0, \\ 1 & \text{for } x = m, \\ R\left(\dfrac{x-m}{\beta}\right) & \text{for } m \leq x \leq m+\beta, \quad \beta > 0, \\ 0 & \text{for } m+\beta \leq x < \infty. \end{cases} \tag{2.6}$$

Nonmembership function is of the form:

$$\nu_A(x) = \begin{cases} 1 & \text{for } -\infty \le x \le m - \alpha', \\ 1 - L\left(\dfrac{m-x}{\alpha^{/}}\right) & \text{for } \ m - \alpha' \le x \le m, \quad \alpha' > 0, \\ 0 & \text{for } \ x = m, \\ 1 - R\left(\dfrac{x-m}{\beta^{/}}\right) & \text{for } \ m \le x \le m + \beta', \quad \beta' > 0, \\ 1 & \text{for } \ m + \beta' \le x \le \infty. \end{cases} \tag{2.7}$$

Provided $L(1) = R(1) = 0$, L is for left, and R is for right reference, m is the mean value of A. $\alpha$and β are called left and right spreads of membership functions, respectively. α' and β' represented left and right spreads of nonmembership functions, respectively. Symbolically, we write $A_{\text{IFN}} = (m; \alpha, \beta; \alpha', \beta')_{\text{LR}}$. Here for $L(x)$ and $R(x)$ different functions may be chosen. For example, $L(x) = R(x) = \max(0, 1 - |x|^p), p \ge 0$, and so forth (Figure 1).

Definition 2.3 (Triangle Intuitionistic fuzzy number). An IFN $A_{\text{IFN}} = (m;\ \alpha,\ \beta;\ \alpha',\ \beta')$ may be defined as a triangle intuitionistic fuzzy number (TIFN) if and only if its membership and nonmembership functions take the following form:

$$\mu_A(x) = \begin{cases} 0 & \text{for } -\infty < x \le m - \alpha, \\ 1 - \dfrac{m-x}{\alpha} & \text{for } \ m - \alpha \le x \le m, \quad \alpha > 0, \\ 1 & \text{for } \ x = m, \\ 1 - \dfrac{x-m}{\beta} & \text{for } \ m \le x \le m + \beta, \quad \beta > 0, \\ 0 & \text{for } \ m + \beta \le x < \infty, \end{cases} \tag{2.8}$$

$$\nu_A(x) = \begin{cases} 1 & \text{for } -\infty \le x \le m - \alpha', \\ \dfrac{m-x}{\alpha'} & \text{for } \ m - \alpha' \le x \le m, \quad \alpha' > 0, \\ 0 & \text{for } \ x = m, \\ \dfrac{x-m}{\beta'} & \text{for } \ m \le x \le m + \beta', \quad \beta' > 0, \\ 1 & \text{for } \ m + \beta' \le x \le \infty. \end{cases} \tag{2.9}$$

Now for a TIFN, we can prove the following result.

Proposition 2.4. *Let one consider a TIFN of the form $A_{TIFN} = (m; \alpha, \beta; \alpha', \beta')$; then $\alpha' > \alpha$ and $\beta' > \beta$.*

Proof. The membership and nonmembership functions of A_{TIFN} is given above in (2.8) and (2.9), respectively.

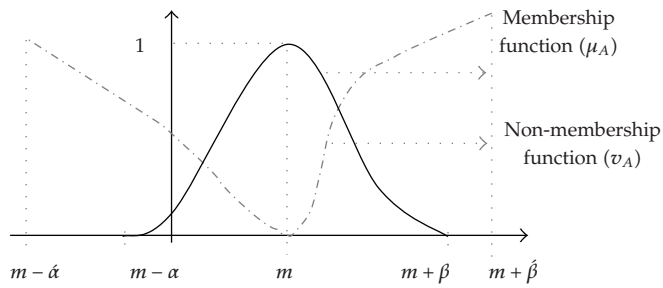

Figure 1: The LR- representation of intuitionistic fuzzy number A

From (2.8) and (2.9), for $m-\alpha \le x \le m$, we can write

$$\begin{aligned} v_A(x)+1-\frac{m-x}{\alpha} &\le 1;\ \left(\text{since}\ \ \mu_A(x)+\nu_A(x)\le 1\right) \\ &\Longleftrightarrow \nu_A(x)\le \frac{m-x}{\alpha}; \\ &\Longleftrightarrow \frac{m-x}{\alpha'}\le \frac{m-x}{\alpha}; \\ &\Longleftrightarrow (m-x)\left(\frac{1}{\alpha'}-\frac{1}{\alpha}\right)\le 0. \end{aligned} \tag{2.10}$$

Since $x \le m$, therefore the following can be written:

$$\left(\frac{1}{\alpha'}-\frac{1}{\alpha}\right)\le 0 \Longleftrightarrow \left(\alpha-\alpha'\right)\le 0 \Longleftrightarrow \alpha\le\alpha'. \tag{2.11}$$

Similarly for $m \le x \le m+\beta$, we can write

$$\begin{aligned} \nu_A(x)+1-\frac{x-m}{\beta} &\le 1;\ \left(\text{since}\ \ \mu_A(x)+\nu_A(x)\le 1\right) \\ &\Longleftrightarrow \nu_A(x)\le \frac{x-m}{\beta}; \\ &\Longleftrightarrow \frac{x-m}{\beta'}\le \frac{x-m}{\beta}; \\ &\Longleftrightarrow (x-m)\left(\frac{1}{\beta'}-\frac{1}{\beta}\right)\le 0. \end{aligned} \tag{2.12}$$

Since $x \ge m$, therefore the following can be written:

$$\left(\frac{1}{\beta'}-\frac{1}{\beta}\right)\le 0 \Longleftrightarrow \left(\beta-\beta'\right)\le 0 \Longleftrightarrow \beta\le\beta'. \tag{2.13}$$

Therefore symbolically a TIFN is represented as $A_{\text{TIFN}} = (m;\ \alpha,\ \beta; \alpha',\ \beta' : \alpha' > \alpha, \beta' > \beta)$. □

Definition 2.5 (Positive Intuitionistic fuzzy number). (i) An IFN denoted as $A_{\text{IFN}} = (m;\alpha,\beta;\alpha^{/},\beta^{/})$ is said to be positive if both $m-\alpha \geq 0$ and $m-\alpha' \geq 0$.

(ii) A TIFN denoted as $A_{\text{TIFN}} = (m;\alpha,\ \beta;\alpha',\beta' : \alpha' > \alpha, \beta' > \beta)$ is called positive if $m-\alpha' \geq 0$ (follows from Proposition 2.4).

2.3. ε-Cut Representation of IFN

Let us consider an IFN $A_{\text{IFN}} = (m;\alpha,\beta;\alpha',\beta')$ defined on the real line R as described before. The ε cut of IFN A_{IFN} is defined by

$$A_{\varepsilon} = \left\{\left\langle x, \mu_A(x), \nu_A(x)\right\rangle \mid \mu_A(x) \geq \varepsilon \ \&\ \nu_A(x) \leq 1-\varepsilon\right\} \quad \forall \varepsilon \in [0,1]. \tag{2.14}$$

The ε cut representation of IFN A_{IFN} generates the following pair of intervals and is denoted by

$$[A_{\text{IFN}}]_{\varepsilon} = \left\{\left[A_{\mu}^{L}(\varepsilon), A_{\mu}^{R}(\varepsilon)\right]; \left[A_{1-\nu}^{L}(\varepsilon), A_{1-\nu}^{R}(\varepsilon)\right]\right\}, \tag{2.15}$$

where the interval $[A_{\mu}^{L}(\varepsilon), A_{\mu}^{R}(\varepsilon)]$ is defined in the following way:

$$\begin{aligned} A_{\mu}^{R}(\varepsilon) &= \begin{cases} \sup\ \{x \mid x \in \mu_{\varepsilon}\} & \text{if } \varepsilon > 0, \\ \sup\ \{x \mid x \in [m-\alpha, m+\beta]\} & \text{if } \varepsilon = 0, \end{cases} \\ A_{\mu}^{L}(\varepsilon) &= \begin{cases} \inf\ \{x \mid x \in \mu_{\varepsilon}\} & \text{if } \varepsilon > 0, \\ \inf\ \{x \mid x \in [m-\alpha, m+\beta]\} & \text{if } \varepsilon = 0. \end{cases} \end{aligned} \tag{2.16}$$

Here μ_{ε} is defined by $\mu_{\varepsilon} = \{x \mid \mu_A(x) \geq \varepsilon\}$.
And in a similar manner the interval $[A_{1-\nu}^{L}(\varepsilon), A_{1-\nu}^{R}(\varepsilon)]$ can be defined as follows:

$$\begin{aligned} A_{1-\nu}^{L}(\varepsilon) &= \begin{cases} \inf\ \{x \mid x \in v_{\varepsilon}\} & \text{if } \varepsilon > 0, \\ \inf\ \{x \mid x \in [m-\alpha', m+\beta']\} & \text{if } \varepsilon = 0, \end{cases} \\ A_{1-\nu}^{R}(\varepsilon) &= \begin{cases} \sup\ \{x \mid x \in \nu_{\varepsilon}\} & \text{if } 1-\varepsilon > 0, \\ \sup\ \{x \mid x \in [m-\alpha', m+\beta']\} & \text{if } 1-\varepsilon = 0, \end{cases} \end{aligned} \tag{2.17}$$

where ν_{ε} is defined by $\nu_{\varepsilon} = \{x \mid \nu_A(x) \leq 1-\varepsilon\} = \{x \mid 1-\nu_A(x) \geq \varepsilon\}$.

2.4. Overview of the Existing Distance Measures

2.4.1. Notes on the Distance Measures for IFSs

Let us consider two $A, B \in \text{IFS}(X)$ with membership functions $\mu_A(x)$, $\mu_B(x)$ and nonmembership functions $\nu_A(x)$, $\nu_B(x)$, respectively.

Atanassov [19] suggested the distance measures as follows.

The normalized Hamming distance $d_A(A,B)$ is

$$d_A(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left[\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|\nu_A(x_i) - \nu_B(x_i)\right|\right]. \tag{2.18}$$

The normalized Euclidean distance $\rho_A(A,B)$ is

$$\rho_A(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2\right]}. \tag{2.19}$$

Then in 2000, it was shown by Szmidt and Kacprzyk [20] that on computing distance for IFSs, all the three parameters, the degree of membership μ, the degree of non membership ν and the hesitation π describing IFSs, should be taken into account. And therefore they modified the concept of distances proposed by Atanassov [19]. The definition of distances presented by Szmidt and Kacprzyk [20] is given as following:

The normalized Hamming distance $d_{\mathrm{SK}}(A,B)$ is

$$d_{\mathrm{SK}}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left[\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|\nu_A(x_i) - \nu_B(x_i)\right| + \left|\pi_A(x_i) - \pi_B(x_i)\right|\right]. \tag{2.20}$$

The normalized Euclidean distance $\rho_{\mathrm{SK}}(A,B)$ is

$$\rho_{\mathrm{SK}}(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2\right] + \left(\pi_A(x_i) - \pi_B(x_i)\right)^2}. \tag{2.21}$$

Developing the above distance measures, Szmidt and Kacprzyk [20] claim that as their distance measures for IFSs are calculated incorporating all the three parameters describing IFSs, it reflects distances in three-dimensional spaces. On the other hand, the distance measures proposed by Atanassov [19] are the orthogonal projections of the real distances. And in this respect in their opinion their distance measures for IFSs are better than that of Atanassov.

But Grzegorzewski [21] was not convinced with the point of view of Szmidt and Kacprzyk [20]. And based on Hausdorff metric, Grzegorzewski [21] proposed another group of distance measures for IFSs as follows.

Normalized Hamming distance $d_{\mathrm{G}}(A,B)$ is

$$d_{\mathrm{G}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\mu_A(x_i) - \mu_B(x_i)\right|, \left|\nu_A(x_i) - \nu_B(x_i)\right|\right\}. \tag{2.22}$$

The normalized Euclidean distance $\rho_G(A,B)$ is

$$\rho_G(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \max\left\{\left(\mu_A(x_i)-\mu_B(x_i)\right)^2, \left(\nu_A(x_i)-\nu_B(x_i)\right)^2\right\}}. \tag{2.23}$$

Obviously, the above distance measures proposed by Grzegorzewski [21] are easy for application. But in reality it may not fit so well. For example, let us consider three IFSs $A,B,C \in \mathrm{IFS}(X)$ where $X = \{x_1\}$ and using the notation in (2.2) IFSs A, B, and C are of the following form: $A = \{\langle x_1, 1, 0\rangle\}$, $B = \{\langle x_1, 0, 1\rangle\}$, and $C = \{\langle x_1, 0, 0\rangle\}$. If we use the ten-person-voting model to interpret, it would be noted that $A = \{\langle x_1, 1, 0\rangle\}$ represents ten personwho all vote for a person; $B = \{\langle x_1, 0, 1\rangle\}$ represents ten persons who all vote against him; whereas $C = \{\langle x_1, 0, 0\rangle\}$ represents ten personswho all hesitate. So it is quite reasonable for us to think that the difference between A and C is lesser than the difference between A and B. But for the distances defined above, the difference between A and C is just equal to the difference between A and B, which is not so reasonable for us. This is the shortcomings of the distance measures proposed by Grzegorzewski [21].

Again in 2005, Wang and Xin [22] first with help of some examples had shown that the distance measure proposed by Szmidt and Kacprzyk [20] is not reasonable for some cases and then developed the following distance measures:

$$\begin{aligned} d_{\mathrm{WX}}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \Bigg[& \frac{\left|\mu_A(x_i)-\mu_B(x_i)\right| + \left|\nu_A(x_i)-\nu_B(x_i)\right|}{4} \\ & + \frac{\max\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|, \left|\nu_A(x_i)-\nu_B(x_i)\right|\right)}{2}\Bigg], \end{aligned} \tag{2.24}$$

$$\rho^p_{\mathrm{WX}}(A,B) = \frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left[\frac{\left|\mu_A(x_i)-\mu_B(x_i)\right|}{2} + \frac{\left|\nu_A(x_i)-\nu_B(x_i)\right|}{2}\right]^p}. \tag{2.25}$$

Though the above distance measures satisfy the properties of a distance measures, but in practice it is realized that the second one is not suitable for some cases. For example, consider three IFSs $A,B,C \in \mathrm{IFS}(X)$ where $X = \{x_1\}$ and A,B, and C are of the following form: $A = \{\langle x_1, 1, 0\rangle\}$, $B = \{\langle x_1, 0, 0\rangle\}$, and $C = \{\langle x_1, 0.5, 0.5\rangle\}$. If we use the ten-person-voting model to interpret, it would benoted that $A = \{\langle x_1, 1, 0\rangle\}$ represents ten personswho all vote for a person; $B = \{\langle x_1, 0, 0\rangle\}$ represents ten person all hesitate; whereas $C = \{\langle x_1, 0.5, 0.5\rangle\}$ represents half of ten person all vote for a person and the rest vote against him; no one is in hesitation. So it is quite reasonable for us to think that the difference between A and C is lesser than the difference between A and B. But for the second distance defined above, the difference between A and C is just equal to the difference between A and B, which is not so reasonable for us. This is the shortcomings of the distance measure (2.25) proposed by Wang and Xin [22].

Now in 2005, Huang et al. [23] suggested several distance measures for IFSs. At first they developed a group of distance measures to unify the distances proposed by Atanassov

[19] and Grzegorzewski [21]. After that they proposed the following group of distance measures for IFSs:

$$\begin{aligned} d_{\mathrm{LW}}(A,B) &= \frac{1}{n}\frac{\sum_{i=1}^{n} 2\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\nu_A(x_i)-\nu_B(x_i)\right|\right)}{2+\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\nu_A(x_i)-\nu_B(x_i)\right|\right)}, \\ \rho_{\mathrm{LW}}(A,B) &= \frac{2\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\nu_A(x_i)-\nu_B(x_i)\right|\right)}{2n+\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\nu_A(x_i)-\nu_B(x_i)\right|\right)}, \\ \rho'_{\mathrm{LW}}(A,B) &= \frac{2\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|^p+\left|\nu_A(x_i)-\nu_B(x_i)\right|^p\right)^{1/p}}{(2n)^{1/p}+\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|^p+\left|\nu_A(x_i)-\nu_B(x_i)\right|^p\right)^{1/p}}. \end{aligned} \tag{2.26}$$

In 2006, based on L_p metric Hung and Yang [24] defined the following distance measure:

$$d^p_{\mathrm{HY}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|^p+\left|\nu_A(x_i)-\nu_B(x_i)\right|^p\right)^{1/p}. \tag{2.27}$$

Now after analyzing the above four distance measures we can say that these measures only reflect the difference between $\mu_A(x)$ and $\nu_A(x)$ and their influence to measure the distances; they do not reflect the influence of degree of indeterminacy or hesitation.

So after a short review of the existing measures between IFSs, in our opinion, all the measures have some advantages as well as some disadvantages. Therefore we cannot say that one particular distance measure is the best and should replace others. In our opinion all existing distance measures are valuable. From application point of view it can be said that depending on the characteristics of the data and the specific requirements of the problem, we need to decide what measure should be used.

However, after reviewing the existing measures it is seen that the distance measures mentioned above calculate distance measures for IFSs of finite universe of discourse. Therefore the problem of developing distance measures for IFNs was an open problem. Then Grzegorzewski [28] investigated two families of metrics in space of IFNs as given in the next section.

2.4.2. The Distance Measures between IFNs

Consider that $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in R\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in R\}$ are two intuitionistic fuzzy numbers. Now ε cut representation of the IFNs A_{IFN} and B_{IFN} is denoted by

$$\begin{aligned} [A_{\mathrm{IFN}}]_\varepsilon &= \left\{\left[A^L_\mu(\varepsilon), A^R_\mu(\varepsilon)\right]; \left[A^L_{1-\nu}(\varepsilon), A^R_{1-\nu}(\varepsilon)\right]\right\}, \\ [B_{\mathrm{IFN}}]_\varepsilon &= \left\{\left[B^L_\mu(\varepsilon), B^R_\mu(\varepsilon)\right]; \left[B^L_{1-\nu}(\varepsilon), B^R_{1-\nu}(\varepsilon)\right]\right\} \quad \forall \varepsilon \in [0,1]. \end{aligned} \tag{2.28}$$

Then Grzegorzewski [28] proposed the distance measures as follows.

For $1 \leq p \leq \infty$

$$d_G^p(A,B) = \left(\frac{1}{4}\int_0^1 \left|A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon)\right|^p d\varepsilon + \frac{1}{4}\int_0^1 \left|A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right|^p d\varepsilon \right.$$
$$\left. + \frac{1}{4}\int_0^1 \left|A_{1-\nu}^L(\varepsilon) - B_{1-\nu}^L(\varepsilon)\right|^p d\varepsilon + \frac{1}{4}\int_0^1 \left|A_{1-\nu}^R(\varepsilon) - B_{1-\nu}^R(\varepsilon)\right|^p d\varepsilon \right)^{1/p}. \tag{2.29}$$

And for $p = \infty$

$$d_G^p(A,B) = \frac{1}{4}\sup_{0<\varepsilon\leq 1}\left|A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon)\right| + \frac{1}{4}\sup_{0<\varepsilon\leq 1}\left|A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right|$$
$$+ \frac{1}{4}\sup_{0<\varepsilon\leq 1}\left|A_{1-\nu}^L(\varepsilon) - B_{1-\nu}^L(\varepsilon)\right| + \frac{1}{4}\sup_{0<\varepsilon\leq 1}\left|A_{1-\nu}^R(\varepsilon) - B_{1-\nu}^R(\varepsilon)\right|. \tag{2.30}$$

For $1 \leq p \leq \infty$

$$\rho_G^p(A,B) = \max\left(\sqrt[p]{\int_0^1 \left|A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon)\right|^p d\varepsilon}, \sqrt[p]{\int_0^1 \left|A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right|^p d\varepsilon}, \right.$$
$$\left. \sqrt[p]{\int_0^1 \left|A_{1-\nu}^L(\varepsilon) - B_{1-\nu}^L(\varepsilon)\right|^p d\varepsilon}, \sqrt[p]{\int_0^1 \left|A_{1-\nu}^R(\varepsilon) - B_{1-\nu}^R(\varepsilon)\right|^p d\varepsilon}\right). \tag{2.31}$$

And for $p = \infty$

$$\rho_G^p(A,B) = \max\left\{\sup_{0<\varepsilon\leq 1}\left|A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon)\right|, \sup_{0<\varepsilon\leq 1}\left|A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right|, \right.$$
$$\left. \sup_{0<\varepsilon\leq 1}\left|A_{1-\nu}^L(\varepsilon) - B_{1-\nu}^L(\varepsilon)\right|, \sup_{0<\varepsilon\leq 1}\left|A_{1-\nu}^R(\varepsilon) - B_{1-\nu}^R(\varepsilon)\right|\right\}. \tag{2.32}$$

After reviewing the existing measures we realize the need to explore new points of view and the need to develop new distance measures that contain more information if we want them to be more logical. We believe that the distance between two uncertain numbers never generates a crisp value. The uncertainty inherent in the number should be intrinsically connected with their distance value. With this point of view in the next section we define new distance measure for IFNs based on the interval difference.

3. New Distance Measure for IFNs

Human intuition says that the distances between two uncertain numbers should also be an uncertain number. In view of this the distance measure for IFNs is defined here. The proposed distance measure is an extension of the fuzzy distance measure [34] in which the degree

of rejection (that is degree of nonmembership) is considered with degrees of satisfaction (degree of membership). It is also seen that when there is no degree of hesitation; that is, when intuitionistic fuzzy number become fuzzy number, this new distance measure converts to the fuzzy distance measure for normalized fuzzy number [34].

3.1. Construction of the Distance Measure for IFNs

Let us consider two IFNs A_{IFN} and B_{IFN} as follows:

$$A_{\text{IFN}} = (m_1; \alpha_1, \beta_1; \alpha_1{}', \beta_1{}'), \qquad B_{\text{IFN}} = (m_2; \alpha_2, \beta_2; \alpha_2{}', \beta_2{}'). \tag{3.1}$$

Therefore, ε cut representation of the IFNs A_{IFN} and B_{IFN} is denoted by

$$\begin{aligned} [A_{\text{IFN}}]_\varepsilon &= \left\{ \left[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon) \right]; \left[A_{1-\nu}^L(\varepsilon), A_{1-\nu}^R(\varepsilon) \right] \right\}, \\ [B_{\text{IFN}}]_\varepsilon &= \left\{ \left[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon) \right]; \left[B_{1-\nu}^L(\varepsilon), B_{1-\nu}^R(\varepsilon) \right] \right\} \end{aligned} \quad \forall \varepsilon \in [0,1]. \tag{3.2}$$

From mathematical point of view we can say that since A_{IFN} and B_{IFN} are IFNs, therefore their distance measure should also have membership and nonmembership part.

Let us denote the distance measure between A_{IFN} and B_{IFN} as $d_{\text{IFN}} = (d; \theta_1, \sigma_1; \theta_2, \sigma_2)$, where d is the mean value of the distance measure d_{IFN}, θ_1, σ_1 and θ_2, σ_2 are the left spread and right spread of the membership function and nonmembership function of the distance measure d_{IFN}, respectively.

And denote the ε cut of d_{IFN} in the following way:

$$[d_{\text{IFN}}]_\varepsilon = \left\{ \left[d_\mu^L(\varepsilon), d_\mu^R(\varepsilon) \right]; \ \left[d_{1-\nu}^L(\varepsilon), d_{1-\nu}^R(\varepsilon) \right] \right\} \quad \forall \varepsilon \in [0,1]. \tag{3.3}$$

To calculate the value of d, θ_1, σ_1, we have to formulate the membership function of the distance between A_{IFN} and B_{IFN}.

Clearly ε (for $0 \le \varepsilon \le 1$) cut representation of the membership function of A_{IFN} and B_{IFN} is $[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon)]$ and $[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon)]$, respectively.

Now, the distance between $[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon)]$ and $[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon)]$ for all $\varepsilon \in [0,1]$ is one of the following:

$$\begin{aligned} &(a) \left[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon) \right] - \left[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon) \right] \quad \text{if } m_1 \ge m_2, \\ \text{or} \quad &(b) \left[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon) \right] - \left[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon) \right] \quad \text{if } m_1 < m_2. \end{aligned} \tag{3.4}$$

In order to consider both the notations together, an indicator variable η is introduced such that

$$\begin{aligned}&\eta\left(\left[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon)\right] - \left[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon)\right]\right) + (1-\eta)\left(\left[B_\mu^L(\varepsilon), B_\mu^R(\varepsilon)\right] - \left[A_\mu^L(\varepsilon), A_\mu^R(\varepsilon)\right]\right)\\ &\quad = \left[L_\mu(\varepsilon), R_\mu(\varepsilon)\right], \quad \text{for } \eta = \begin{cases} 1 & \text{if } m_1 \geq m_2,\\ 0 & \text{if } m_1 \geq m_2,\end{cases}\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}L_\mu(\varepsilon) &= \eta\left(A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon) + A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right) + \left(B_\mu^L(\varepsilon) - A_\mu^R(\varepsilon)\right),\\ R_\mu(\varepsilon) &= \eta\left(A_\mu^L(\varepsilon) - B_\mu^L(\varepsilon) + A_\mu^R(\varepsilon) - B_\mu^R(\varepsilon)\right) + \left(B_\mu^R(\varepsilon) - A_\mu^L(\varepsilon)\right).\end{aligned} \tag{3.6}$$

Thus

$$\left[d_\mu^L(\varepsilon), d_\mu^R(\varepsilon)\right] = \begin{cases} \left[L(\varepsilon), R_\mu(\varepsilon)\right]; & \text{for } L_\mu(\varepsilon) \geq 0,\\ \left[0, \left|L_\mu(\varepsilon)\right| \vee R_\mu(\varepsilon)\right]; & \text{for } L_\mu(\varepsilon) \leq 0 \leq R_\mu(\varepsilon)\end{cases} \quad \forall \varepsilon \in [0,1]. \tag{3.7}$$

Now using (3.7) θ_1, σ_1 are defined as follows:

$$\theta_1 = d_\mu^L(1) - \max\left\{\int_0^1 d_\mu^L(\varepsilon)d\varepsilon, 0\right\}, \qquad \sigma_1 = \int_0^1 d_\mu^R(\varepsilon)\ d\varepsilon - d_\mu^R(1). \tag{3.8}$$

Similarly $\theta_2,\ \sigma_2$ are of the following form:

$$\theta_2 = d_{1-\nu}^L(1) - \max\left\{\int_0^1 d_{1-\nu}^L(\varepsilon)d\varepsilon, 0\right\}, \qquad \sigma_2 = \int_0^1 d_{1-\nu}^R(\varepsilon)d\varepsilon - d_{1-\nu}^R(1), \tag{3.9}$$

where

$$\left[d_{1-\nu}^L(\varepsilon), d_{1-\nu}^R(\varepsilon)\right] = \begin{cases} [L_{1-\nu}(\varepsilon), R_{1-\nu}(\varepsilon)]; & \text{for } \ L_{1-\nu}(\varepsilon) \geq 0,\\ [0, |L_{1-\nu}(\varepsilon)| \vee R_{1-\nu}(\varepsilon)]; & \text{for } \ L_{1-\nu}(\varepsilon) \leq 0 \leq R_{1-\nu}(\varepsilon).\end{cases} \tag{3.10}$$

As $L_{1-\nu}(\varepsilon)$ and $R_{1-\nu}(\varepsilon)$ take the following form:

$$\begin{aligned} L_{1-\nu}(\varepsilon) &= \eta\left(A^L_{1-\nu}(\varepsilon) - B^L_{1-\nu}(\varepsilon) + A^R_{1-\nu}(\varepsilon) - B^R_{1-\nu}(\varepsilon)\right) \\ &\quad + \left(B^L_{1-\nu}(\varepsilon) - A^R_{1-\nu}(\varepsilon)\right), \\ R_{1-\nu}(\varepsilon) &= \eta\left(A^L_{1-\nu}(\varepsilon) - B^L_{1-\nu}(\varepsilon) + A^R_{1-\nu}(\varepsilon) - B^R_{1-\nu}(\varepsilon)\right) \\ &\quad + \left(B^R_{1-\nu}(\varepsilon) - A^L_{1-\nu}(\varepsilon)\right). \end{aligned} \tag{3.11}$$

From (3.7) and (3.10), we can find $d = d^L_\mu(1) = d^R_\mu(1) = d^L_{1-\nu}(1) = d^R_{1-\nu}(1)$. Therefore finally the distance measure between A_{IFN} and B_{IFN} is obtained as

$$d_{\text{IFN}} = (d;\ \theta_1, \sigma_1; \theta_2, \sigma_2). \tag{3.12}$$

3.2. Distance Measure for TIFNs

If A_{IFN} and B_{IFN} are two TIFNs, then their distance measure with the help of the above approach (Section 3.1) should be a TIFN. It can be proved by the following proposition:

Proposition 3.1. *Let one consider two TIFNs as follows:*

$$\begin{aligned} A_{TIFN} &= \left(m_1; \alpha_1, \beta_1; \alpha_1{}^{/}, \beta_1{}^{/} : \alpha_1{}^{/} > \alpha_1, \beta_1{}^{/} > \beta_1\right), \\ B_{TIFN} &= \left(m_2; \alpha_2, \beta_2; \alpha_2{}^{/}, \beta_2{}^{/} : \alpha_2{}^{/} > \alpha_2, \beta_2{}^{/} > \beta_2\right). \end{aligned} \tag{3.13}$$

Then their distance measure $d_{IFN} = (d;\ \theta_1,\ \sigma_1;\ \theta_2,\ \sigma_2)$ is a TIFN.

Proof. Here we have considered two TIFNs A and B. Therefore the ε cut representation of A_{TIFN} and B_{TIFN} is as follows:

$$\begin{aligned} [A_{\text{TIFN}}]_\varepsilon &= \left\{\left[A^L_\mu(\varepsilon), A^R_\mu(\varepsilon)\right]; \left[A^L_{1-\nu}(\varepsilon), A^R_{1-\nu}(\varepsilon)\right]\right\}, \\ [B_{\text{TIFN}}]_\varepsilon &= \left\{\left[B^L_\mu(\varepsilon), B^R_\mu(\varepsilon)\right]; \left[B^L_{1-\nu}(\varepsilon), B^R_{1-\nu}(\varepsilon)\right]\right\} \end{aligned} \quad \forall \varepsilon \in [0,1]. \tag{3.14}$$

Now with the help of (2.8) and (2.9), respectively, we can write

$$\left[A^L_\mu(\varepsilon),\ A^R_\mu(\varepsilon)\right] = \left[m_1 - \alpha_1(1-\varepsilon),\ m_1 + \beta_1(1-\varepsilon)\right], \tag{3.15}$$

$$\left[A^L_{1-\nu}(\varepsilon),\ A^R_{1-\nu}(\varepsilon)\right] = \left[m_1 - \alpha'_1(1-\varepsilon), m_1 + \beta'_1(1-\varepsilon)\right] \quad \forall \varepsilon \in [0,1]\ . \tag{3.16}$$

And in a similar manner

$$\left[B_{\mu}^{L}(\varepsilon), B_{\mu}^{R}(\varepsilon)\right] = \left[m_2 - \alpha_2(1-\varepsilon), m_2 + \beta_2(1-\varepsilon)\right], \tag{3.17}$$

$$\left[B_{1-\nu}^{L}(\varepsilon), B_{1-\nu}^{R}(\varepsilon)\right] = \left[m_2 - \alpha_2'(1-\varepsilon), m_2 + \beta_2'(1-\varepsilon)\right] \quad \forall \varepsilon \in [0,1]. \tag{3.18}$$

Here two possibilities can arise depending on the position of the mean values of A_{TIFN} and B_{TIFN}, which are given as follows.

Case 1. For $\eta = 1$, that is, when $m_1 \geq m_2$, we proceed in the following way.

In (3.6), putting the value from (3.15) and (3.17), we can express $L_{\mu}(\varepsilon)$ and $R_{\mu}(\varepsilon)$ as follows:

$$\begin{aligned} L_{\mu}(\varepsilon) &= \left[(m_1 - m_2) - \alpha_1(1-\varepsilon) - \beta_2(1-\varepsilon)\right], \\ R_{\mu}(\varepsilon) &= \left[(m_1 - m_2) + \alpha_2(1-\varepsilon) + \beta_1(1-\varepsilon)\right]. \end{aligned} \tag{3.19}$$

Similarly from (3.11), with the help of (3.16) and (3.18), $L_{1-\nu}(\varepsilon)$, $R_{1-\nu}(\varepsilon)$ can be written in the following form:

$$\begin{aligned} L_{1-\nu}(\varepsilon) &= \left[(m_1 - m_2) - \alpha_1'(1-\varepsilon) - \beta_2'(1-\varepsilon)\right], \\ R_{1-\nu}(\varepsilon) &= \left[(m_1 - m_2) + \alpha_2'(1-\varepsilon) + \beta_1'(1-\varepsilon)\right]. \end{aligned} \tag{3.20}$$

Now from (3.7) and (3.10) we have the following distances for A_{TIFN} and B_{TIFN} :

$$\begin{aligned} d_{\mu}^{L}(\varepsilon) &= \left[(m_1 - m_2) - \alpha_1(1-\varepsilon) - \beta_2(1-\varepsilon)\right] \quad \text{or} \quad 0, \\ d_{\mu}^{R}(\varepsilon) &= \left[(m_1 - m_2) + \alpha_2(1-\varepsilon) + \beta_1(1-\varepsilon)\right], \end{aligned} \tag{3.21}$$

$$\begin{aligned} d_{1-\nu}^{L}(\varepsilon) &= \left[(m_1 - m_2) - \alpha_1'(1-\varepsilon) - \beta_2'(1-\varepsilon)\right] \quad \text{or} \quad 0, \\ d_{1-\nu}^{R}(\varepsilon) &= \left[(m_1 - m_2) + \alpha_2'(1-\varepsilon) + \beta_1'(1-\varepsilon)\right]. \end{aligned} \tag{3.22}$$

As given by Section 3.1, using (3.21) we can obtain

$$\theta_1 = d_{\mu}^{L}(1) - \max\left\{\int_0^1 d_{\mu}^{L}(\varepsilon)d\varepsilon, 0\right\} = (m_1 - m_2) - \max\left\{\left[(m_1 - m_2) - (\alpha_1 + \beta_2)/2\right], 0\right\}. \tag{3.23}$$

In a similar way, θ_2 can be evaluated as

$$\theta_2 = (m_1 - m_2) - \max\left\{\left[(m_1 - m_2) - \left(\alpha_1^{/} + \beta_2^{/}\right)/2\right], 0\right\}. \tag{3.24}$$

Now as $\alpha_1' > \alpha_1$ and $\beta_2' > \beta_2$

$$\begin{aligned}
&\Rightarrow (\alpha_1' + \beta_2')/2 > (\alpha_1 + \beta_2)/2 \\
&\Rightarrow \max\left\{\left[(m_1 - m_2) - \left(\alpha_1' + \beta_2'\right)/2\right], 0\right\} \leq \max\{[(m_1 - m_2) - (\alpha_1 + \beta_2)/2], 0\} \\
&\Rightarrow \theta_2 > \theta_1
\end{aligned} \tag{3.25}$$

Similarly for $\sigma_1 = (\alpha_2 + \beta_1)/2$ and $\sigma_2 = (\alpha_2' + \beta_1')/2$, it is clear that $\sigma_2 > \sigma_1$ (as $\alpha_1' > \alpha_1$ and $\beta_2' > \beta_2$). Now from Proposition 2.4, we can conclude that as $\theta_2 > \theta_1$ and $\sigma_2 > \sigma_1$, d_{IFN} is a TIFN.

Case 2. For $\eta = 0$, the proof can be done in a similar manner as for Case 1. Hence the proof is completed. □

Therefore it is now proved that the distance measure between A_{TIFN} and B_{TIFN} is a TIFN denoted by $d_{\text{TIFN}} = (d; \theta_1, \sigma_1; \theta_2, \sigma_2 : \theta_2 > \theta_1, \sigma_2 > \sigma_1)$.

3.3. Metric Properties

The new distance measure satisfies the following properties of a distance metric.

(i) The distance measure proposed in the Section 3.1 is a nonnegative intuitionistic fuzzy number.

(ii) For any two intuitionistic fuzzy numbers A^1_{IFN} and A^2_{IFN} the following holds:

$$d\left(A^1_{\text{IFN}}, A^2_{\text{IFN}}\right) = d\left(A^2_{\text{IFN}}, A^1_{\text{IFN}}\right). \tag{3.26}$$

(iii) For three IFNs $A^1_{\text{IFN}}, A^2_{\text{IFN}}$, and A^3_{IFN}, the distance measure satisfies the triangle inequality:

$$d\left(A^1_{\text{IFN}}, A^2_{\text{IFN}}\right) + d\left(A^2_{\text{IFN}}, A^3_{\text{IFN}}\right) \geq d\left(A^1_{\text{IFN}}, A^3_{\text{IFN}}\right). \tag{3.27}$$

Proof. Proof of property (i) follows from (3.12) and property (ii) can be proved with the help of (3.5).

Proof of property (iii) is given here.

Let A^1_{IFN}, A^2_{IFN}, and A^3_{IFN} be three IFNs. The ε cut representation of IFNs A^1_{IFN}, A^2_{IFN} and A^3_{IFN} is expressed as

$$\left[A^i_{\text{IFN}}\right]_\varepsilon = \left\{\left[A^{iL}_\mu(\varepsilon), A^{iR}_\mu(\varepsilon)\right];\ \left[A^{iL}_{1-\nu}(\varepsilon), A^{iR}_{1-\nu}(\varepsilon)\right]\right\} \quad \text{for } i = 1, 2, 3 \text{ for } \varepsilon \in [0, 1]. \tag{3.28}$$

In order to prove the triangle inequality of the distance measure for the above three IFNs A^1_{IFN}, A^2_{IFN} and A^3_{IFN}, we show below that the triangle inequality for the distance measure

between the membership functions of A^1_{IFN}, A^2_{IFN} and A^3_{IFN} and nonmembership function of A^1_{IFN}, A^2_{IFN}, and A^3_{IFN} should hold separately.

Hence, for membership function of the distance measure, the triangle inequality is established in the following way.

From (3.28) consider the interval number $[A^{iL}_\mu(\varepsilon), A^{iR}_\mu(\varepsilon)]$ that is, the ε cut of membership function of A^1_{IFN}, A^2_{IFN}, and A^3_{IFN}.

Depending on the relative positions of the means of A^1_{IFN}, A^2_{IFN} and A^3_{IFN}, three situations arise.

Situation (I)

When mean of A^1_{IFN} is less than mean of A^2_{IFN} which is less than mean of A^3_{IFN}

$$\frac{A^{1L}_\mu(1) + A^{1R}_\mu(1)}{2} \leq \frac{A^{2L}_\mu(1) + A^{2R}_\mu(1)}{2} \leq \frac{A^{3L}_\mu(1) + A^{3R}_\mu(1)}{2}. \tag{3.29}$$

From (3.7), we have the following distances for

$$\begin{aligned}
&\text{(I.a)} \quad A^1_{\text{IFN}}, A^2_{\text{IFN}} : \begin{cases} d^L_\mu(\varepsilon) = A^{2L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon), \\ d^R_\mu(\varepsilon) = A^{2R}_\mu(\varepsilon) - A^{1L}_\mu(\varepsilon), \end{cases} \\
&\text{(I.b)} \quad A^2_{\text{IFN}}, A^3_{\text{IFN}} : \begin{cases} \overline{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{2R}_\mu(\varepsilon) \quad \text{or} \quad 0, \\ \overline{d}^R_\mu(\varepsilon) = A^{3R}_\mu(\varepsilon) - A^{2L}_\mu(\varepsilon), \end{cases} \\
&\text{(I.c)} \quad A^1_{\text{IFN}}, A^3_{\text{IFN}} : \begin{cases} \widehat{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon) \quad \text{or} \quad 0, \\ \widehat{d}^R_\mu(\varepsilon) = A^{3R}_\mu(\varepsilon) - A^{1L}_\mu(\varepsilon). \end{cases}
\end{aligned} \tag{3.30}$$

Therefore we have to prove that

$$d^R + d^L + \overline{d}^R + \overline{d}^L \geq \widehat{d}^R + \widehat{d}^L. \tag{3.31}$$

From the above three options (I.a), (I.b) and (I.c), the following eight combinations are possible:

(i) $d^L_\mu(\varepsilon) = A^{2L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon)$, $\overline{d}^L_\mu(\varepsilon) = 0$, $\widehat{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon)$,

(ii) $d^L_\mu(\varepsilon) = 0$, $\overline{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{2R}_\mu(\varepsilon)$, $\widehat{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon)$,

(iii) $d^L_\mu(\varepsilon) = 0, \overline{d}^L_\mu(\varepsilon) = 0$, $\widehat{d}^L_\mu(\varepsilon) = A^{3L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon)$,

(iv) $d^L_\mu(\varepsilon) = A^{2L}_\mu(\varepsilon) - A^{1R}_\mu(\varepsilon)$, $\overline{d}^L_\mu(\varepsilon) = 0$, $\widehat{d}^L_\mu(\varepsilon) = 0$,

(v) $d_\mu^L(\varepsilon) = A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon),\ \overline{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon),\ \widehat{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon),$

(vi) $d_\mu^L(\varepsilon) = 0,\ \overline{d}_\mu^L(\varepsilon) = 0,\ \widehat{d}_\mu^L(\varepsilon) = 0,$

(vii) $d_\mu^L(\varepsilon) = A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon),\ \overline{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon),\ \widehat{d}_\mu^L(\varepsilon) = 0,$

(viii) $d_\mu^L(\varepsilon) = 0,\ \overline{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon),\ \widehat{d}_\mu^L(\varepsilon) = 0.$

Now, from the above eight combinations, (vii) and (viii) are not possible.
As from (vii), the following can be seen that.

(i) The membership functions of A_{IFN}^1 and A_{IFN}^2 are disjoint.

(ii) The membership functions of A_{IFN}^2 and A_{IFN}^3 are disjoint.

(iii) The membership functions of A_{IFN}^1 and A_{IFN}^3 intersect.

It is very clear that the above situation cannot be happened.
From (viii), we observe the following.

(i) The membership functions of A_{IFN}^1 and A_{IFN}^2 intersect.

(ii) The memberships functions of A_{IFN}^2 and A_{IFN}^3 are disjoint.

(iii) The memberships functions of A_{IFN}^1 and A_{IFN}^3 intersect.

The situation (viii) is also not possible.

Therefore, for the rest six different cases, the proof of inequality (3.31) is given as follows:

(i) $d_\mu^L(\varepsilon) = A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon), \overline{d}_\mu^L(\varepsilon) = 0, \widehat{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon),$

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R \text{[Proved]}.
\end{aligned} \tag{3.32}$$

(ii) $d_\mu^L(\varepsilon) = 0$, $\overline{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon)$, $\widehat{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)$

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{2R}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R \text{[Proved]}.
\end{aligned} \tag{3.33}$$

(iii) $d_\mu^L(\varepsilon) = 0$, $\overline{d}_\mu^L(\varepsilon) = 0$, $\widehat{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)$,

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R \text{[Proved]}.
\end{aligned} \tag{3.34}$$

(iv) $d_\mu^L(\varepsilon) = A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)$, $\overline{d}_\mu^L(\varepsilon) = 0$, $\widehat{d}_\mu^L(\varepsilon) = 0$,

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R \text{[Proved]}.
\end{aligned} \tag{3.35}$$

(v) $d_\mu^L(\varepsilon) = A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)$, $\overline{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon)$, $\widehat{d}_\mu^L(\varepsilon) = A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)$,

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon)\right] \\
&\qquad + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{2L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{2R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3L}(\varepsilon) - A_\mu^{1R}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R [\text{Proved}].
\end{aligned} \tag{3.36}$$

(vi) $d_\mu^L(\varepsilon) = 0$, $\overline{d}_\mu^L(\varepsilon) = 0$, $\widehat{d}_\mu^L(\varepsilon) = 0$,

$$\begin{aligned}
&d^R + d^L + \overline{d}^R + \overline{d}^L \\
&\quad = \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad = \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon + \int_0^1 \left[A_\mu^{2R}(\varepsilon) - A_\mu^{2L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \int_0^1 \left[A_\mu^{3R}(\varepsilon) - A_\mu^{1L}(\varepsilon)\right] d\varepsilon \\
&\quad \geq \widehat{d}^L + \widehat{d}^R [\text{Proved}].
\end{aligned} \tag{3.37}$$

The proof is similar for the following two cases:

(I) $([A_\mu^{1L}(1) + A_\mu^{1R}(1)]/2) \leq ([A_\mu^{3L}(1) + A_\mu^{3R}(1)]/2) \leq ([A_\mu^{2L}(1) + A_\mu^{2R}(1)]/2)$,

(II) $([A_\mu^{2L}(1) + A_\mu^{2R}(1)]/2) \leq ([A_\mu^{1L}(1) + A_\mu^{1R}(1)]/2) \leq ([A_\mu^{3L}(1) + A_\mu^{3R}(1)]/2)$.

With the help of (3.28), considering the interval number $[A_{1-\nu}^{iL}(\varepsilon), A_{1-\nu}^{iR}(\varepsilon)]$ for all $\varepsilon \in [0,1]$, we can prove the triangle inequality for nonmembership function of the distance measure, in same way as for membership function. □

4. Numerical Illustration

Here we have considered the following numerical examples to illustrate the proposed measure.

Example 4.1. Let us consider two IFSs A and B defined over the universe of discourse X where $X = \{x_1\}$ is as follows: $A = \{\langle x_1, 0.6, 0.3\rangle\}$ and $B = \{\langle x_1, 0.7, 0.2\rangle\}$.

Therefore A can be expressed as a conjunction of two fuzzy numbers $A^+ = (x_1; 0, 0)$ with membership degree 0.6 and $A^- = (x_1; 0, 0)$ with membership degree 0.7. And similarly B can be expressed as a conjunction of $B^+ = (x_1; 0, 0)$ with membership degree 0.7 and $B^- = (x_1; 0, 0)$ with membership degree 0.8.

Applying the proposed distance method we will find the required distance measure as $d = \{\langle 0, 0.6, 0.3\rangle\}$. In this way the proposed distance measure covers the case for $X = \{x_1\}$.

Example 4.2. Let us consider two IFNs say $A_{\text{IFN}} = (5; 1, 2; 2, 3)$ and $B_{\text{IFN}} = (9; 1, 1; 2, 1)$ characterized by their membership functions and nonmembership functions as follows:

$$\mu_A(x) = \begin{cases} 0 & \text{for } -\infty < x \le 4, \\ 1-(5-x)^3 & \text{for } 4 \le x \le 5, \\ 1 & \text{for } x = 5, \\ 1-((x-5)/2\ )^3 & \text{for } 5 \le x \le 7, \\ 0 & \text{for } 7 \le x < \infty, \end{cases}$$

$$\nu_A(x) = \begin{cases} 1 & \text{for } -\infty \le x \le 3, \\ ((5-x)/2)^3 & \text{for } 3 \le x \le 5, \\ 0 & \text{for } x = 5, \\ ((x-5)/2)^3 & \text{for } 5 \le x \le 7, \\ 1 & \text{for } 7 \le x \le \infty, \end{cases} \tag{4.1}$$

$$\mu_B(x) = \begin{cases} 0 & \text{for } -\infty < x \le 8, \\ 1-\sqrt{9-x} & \text{for } 8 \le x \le 9, \\ 1 & \text{for } x = 9, \\ 1-\sqrt{x-9} & \text{for } 9 \le x \le 10, \\ 0 & \text{for } 10 \le x < \infty, \end{cases}$$

$$\nu_B(x) = \begin{cases} 1 & \text{for } -\infty \le x \le 7, \\ \sqrt{(9-x)/2} & \text{for } 7 \le x \le 9, \\ 0 & \text{for } x = 9, \\ \sqrt{(x-9)} & \text{for } 9 \le x \le 10, \\ 1 & \text{for } 10 \le x \le \infty. \end{cases}$$

The ε cut representation of A_{IFN} and B_{IFN} is obtained as follows:

$$\begin{aligned}
[A_{\text{IFN}}]_\varepsilon &= \left\{\left[5-(1-\varepsilon)^{1/3}, 5+2\ (1-\varepsilon)^{1/3}\right]; \left[5-2\ (1-\varepsilon)^{1/3}, 5+2(1-\varepsilon)^{1/3}\right]\right\}, \\
[B_{\text{IFN}}]_\varepsilon &= \left\{\left[9-(1-\varepsilon)^{2}, 9+(1-\varepsilon)^{2}\right]; \left[9-2\ (1-\varepsilon)^{2}, 9+2(1-\varepsilon)^{2}\right]\right\}
\end{aligned} \quad \forall\varepsilon \in [0,1]. \tag{4.2}$$

The ε cut of the distance measure d_{IFN} using (3.7) and (3.10) is obtained as

$$\begin{aligned}
[d_{\text{IFN}}]_\varepsilon = \Big\{ &\left[4-(1-\varepsilon)^{2}-2\ (1-\varepsilon)^{1/3}, 4+(1-\varepsilon)^{2}+\ (1-\varepsilon)^{1/3}\right]; \\
&\left[4-2\ (1-\varepsilon)^{2}-2\ (1-\varepsilon)^{1/3}, 4+(1-\varepsilon)^{2}+2(1-\varepsilon)^{1/3}\right]\Big\} \quad \forall\varepsilon \in [0,1].
\end{aligned} \tag{4.3}$$

Finally the distance measure of A_{IFN} and B_{IFN}, computed by the proposed method is given by $d_{\text{IFN}} = (4; 1.84, 1.08; 2.17, 1.83)$.

Example 4.3. In the following, 2 sets of TIFNs are given to compare the proposed distance method with the existing distance measures presented by Grzegorzewski [28]. A comparison between the results of the proposed distance measure and the results of the existing method is shown in Table 1.

Set 1. Consider $A^1_{\text{TIFN}} = (3; 2, 2; 3, 4)$ and $B_{\text{TIFN}} = (4; 2, 2; 3, 3)$. The membership and nonmembership functions of A^1_{TIFN} and B_{TIFN} are, respectively, of the following form:

$$\begin{aligned}
\mu_{A_1}(x) &= \begin{cases} \dfrac{1}{2\ (x-1)}; & 1 \le x \le 3, \\[2ex] \dfrac{1}{2\ (5-x)}; & 3 \le x \le 5, \end{cases} \\
\nu_{A_1}(x) &= \begin{cases} \dfrac{1}{3\ (3-x)}; & 0 \le x \le 3, \\[2ex] \dfrac{1}{4\ (x-3)}; & 3 \le x \le 7, \end{cases} \\
\mu_{B}(x) &= \begin{cases} \dfrac{1}{2\ (x-2)}; & 2 \le x \le 4, \\[2ex] \dfrac{1}{2\ (6-x)}; & 4 \le x \le 6, \end{cases} \\
\nu_{B}(x) &= \begin{cases} \dfrac{1}{3\ (4-x)}; & 2 \le x \le 4, \\[2ex] \dfrac{1}{3\ (x-4)}; & 4 \le x \le 7. \end{cases}
\end{aligned} \tag{4.4}$$

Table 1: Here a comparison of the proposed distance measure with the existing distance measure through the above set of 2 examples is given.

Distance measures for IFNs	Grzegorzewski's method [28] Equation (2.29); $p = 1$	Equation (2.31); $1 \le p < \infty$	The Proposed Method Equation (3.12)
Set 1: $d(A^1_{\text{TIFN}}, B_{\text{TIFN}})$	7/8	1	$d_{\text{TIFN}} = (1;\ 1,2;1,3)$
Set 2: $d(A^2_{\text{TIFN}}, B_{\text{TIFN}})$	5/8	1	$d'_{\text{TIFN}} = (1;1,1.5;1,3)$

Set 2. Consider $A^2_{\text{TIFN}} = (3;1,3;3,4)$ and $B_{\text{TIFN}} = (4;2,2;3,3)$, where the membership and nonmembership functions of $A_{2\text{TIFN}}$ are given as follows:

$$\mu_{A_2}(x) = \begin{cases} (x-2); & 1 \le x \le 3, \\ \dfrac{1}{3(6-x)}; & 3 \le x \le 6, \end{cases}$$
$$\nu_{A_2}(x) = \begin{cases} \dfrac{1}{3(3-x)}; & 0 \le x \le 3, \\ \dfrac{1}{4(x-3)}; & 3 \le x \le 7. \end{cases} \tag{4.5}$$

And B_{TIFN} is defined as in Set 1.

Now we have calculated the distance measure between A^1_{TIFN}, B_{TIFN} and A^2_{TIFN}, B_{TIFN} by applying (2.29), (2.31), and (3.12) and the results are compared in Table 1.

From Table 1 we can see that Set 1 and Set 2 are different sets of TIFNs, but Equation (2.31) gives the same distance value. Therefore from Table 1, it can be said that Equation (2.31) is not so well fit in real life applications.

Applying (2.29) and choosing $p = 1$, we can say from Table 1 that $d(A^2_{\text{TIFN}}, B_{\text{TIFN}}) < d(A^1_{\text{TIFN}},\ B_{\text{TIFN}})$.

Also we utilize (3.12) to find the distance value of A^1_{TIFN}, B_{TIFN} and A^2_{TIFN}, B_{TIFN} respectively. Then we analyze the results by applying the defuzzification procedure proposed by Chang et al. [35] to the both distance measures, respectively, and we obtain $d(A^2_{TIFN}, \frac{a}{b}B_{\text{TIFN}}) < d(A^1_{\text{TIFN}},\ \text{B}_{\text{TIFN}})$. This result is matching with the result obtained from (2.29).

5. Conclusion

IFSs theory provides a flexible framework to cope with imperfect and/or imprecise information often present in real world application. The concept of IFS can be viewed as an alternative approach to define a fuzzy set in the case when available information is not sufficient to define a conventional fuzzy set. In this paper a new distance measure for computing distance for IFNs is introduced. We believe that the distances between two uncertain numbers should be an uncertain number. If the uncertainty is inherent within the numbers, this uncertainty should be intrinsically connected with their distance value. With this view point here a new method to measure the distance for IFNs is presented. What is important, the proposed distance method gives new viewpoints for the study of similarity of IFNs. This will be a topic of our future research work.

Acknowledgments

The authors would like to thank the anonymous reviewers for their constructive suggestions. The first author gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research, India (Award no. 9/81(712)/08-EMR-I).

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
[2] J. A. Goguen, "L-fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 18, pp. 145–174, 1967.
[3] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[4] W.-L. Gau and D. J. Buehrer, "Vague sets," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 23, no. 2, pp. 610–614, 1993.
[5] H. Bustince and P. Burillo, "Vague sets are intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 79, no. 3, pp. 403–405, 1996.
[6] I. Grattan-Guinness, "Fuzzy membership mapped onto intervals and many-valued quantities," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 22, no. 2, pp. 149–160, 1976.
[7] K.-U. Jahn, "Intervall-wertige Mengen," *Mathematische Nachrichten*, vol. 68, pp. 115–132, 1975.
[8] R. Sambuc, *Fonctions phi-floues, Application a l'Aide au diagnostic en pathologie thyroidienne*, Ph.D. thesis, University of Marseille, Marseille, France, 1975.
[9] B. Gorzalczany, "Approximate inference with interval-valued fuzzy sets—an outline," in *Proceedings of the Polish Symposium on Interval and Fuzzy Mathematics*, pp. 89–95, Poznan, Poland, 1983.
[10] I. B. Türksen, "Interval valued fuzzy sets based on normal forms," *Fuzzy Sets and Systems*, vol. 20, no. 2, pp. 191–210, 1986.
[11] J. Kacprzyk, *Multistage Fuzzy Control*, Wiley, Chichester, UK, 1997.
[12] C.-H. Cheng, "A new approach for ranking fuzzy numbers by distance method," *Fuzzy Sets and Systems*, vol. 95, no. 3, pp. 307–317, 1998.
[13] P. Diamond, "Fuzzy least squares," *Information Sciences*, vol. 46, no. 3, pp. 141–157, 1988.
[14] L. Tran and L. Duckstein, "Comparison of fuzzy numbers using a fuzzy distance measure," *Fuzzy Sets and Systems*, vol. 130, no. 3, pp. 331–341, 2002.
[15] R. Xu and C. Li, "Multidimensional least-squares fitting with a fuzzy model," *Fuzzy Sets and Systems*, vol. 119, no. 2, pp. 215–223, 2001.
[16] M.-S. Yang and C.-H. Ko, "On cluster-wise fuzzy regression analysis," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 27, no. 1, pp. 1–13, 1997.
[17] W. Voxman, "Some remarks on distances between fuzzy numbers," *Fuzzy Sets and Systems*, vol. 100, no. 1–3, pp. 353–365, 1998.
[18] C. Chakraborty and D. Chakraborty, "A theoretical development on a fuzzy distance measure for fuzzy numbers," *Mathematical and Computer Modelling*, vol. 43, no. 3-4, pp. 254–261, 2006.
[19] K. T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Application*, vol. 35 of *Studies in Fuzziness and Soft Computing*, Physica, Heidelberg, Germany, 1999.
[20] E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 114, no. 3, pp. 505–518, 2000.
[21] P. Grzegorzewski, "Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric," *Fuzzy Sets and Systems*, vol. 148, no. 2, pp. 319–328, 2004.
[22] W. Wang and X. Xin, "Distance measure between intuitionistic fuzzy sets," *Pattern Recognition Letters*, vol. 26, no. 13, pp. 2063–2069, 2005.
[23] G.-S. Huang, Y.-S. Liu, and X.-D. Wang, "Some new distances between intuitionistic fuzzy sets," in *Proceedings of the International Conference on Machine Learning and Cybernetics (ICMLC '05)*, pp. 2478–2482, Guangzhou, China, August 2005.
[24] W.-L. Hung and M.-S. Yang, "Similarity measures of intuitionistic fuzzy sets based on L_p metric," *International Journal of Approximate Reasoning*, vol. 46, no. 1, pp. 120–136, 2007.
[25] P. Burillo, H. Bustince, and V. Mohedano, "Some definitions of intuitionistic fuzzy number. First properties," in *Proceedings of the 1st Workshop on Fuzzy Based Expert Systems*, D. Lakov, Ed., pp. 53–55, Sofia, Bulgaria, September 1994.
[26] F. Lee, *Fuzzy Information Processing System*, Peking University Press, Beijing, China, 1998.

[27] H. Liu and K. Shi, "Intuitionistic fuzzy numbers and intuitionistic distribution numbers," *Journal of Fuzzy Mathematics*, vol. 8, no. 4, pp. 909–918, 2000.
[28] P. Grzegorzewski, "Distances and orderings in a family of intuitionistic fuzzy numbers," in *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT '03)*, pp. 223–227, Zittau, Germany, September 2003.
[29] V. L. G. Nayagam, G. Venkateshwari, and G. Sivaraman, "Ranking of intuitionistic fuzzy numbers," in *Proceedings of IEEE International Conference on Fuzzy Systems (FUZZ '08)*, pp. 1971–1974, Hong Kong, June 2008.
[30] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
[31] K. T. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 37–45, 1989.
[32] K.T. Atanassov, "New operations defined over the intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 61, no. 2, pp. 137–142, 1994.
[33] K. T. Atanassov, "Operators over interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 64, no. 2, pp. 159–174, 1994.
[34] D. Guha and D. Chakraborty, "A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers," *Applied Soft Computing Journal*, vol. 10, no. 1, pp. 90–99, 2010.
[35] K.-H. Chang, C.-H. Cheng, and Y.-C. Chang, "Reliability assessment of an aircraft propulsion system using IFS and OWA tree," *Engineering Optimization*, vol. 40, no. 10, pp. 907–921, 2008.