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Research Article

The Semi-Difference Entire Sequence Space $cs \cap d_1$

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Let Γ denote the space of all entire sequences. Let Λ denote the space of all analytic sequences. In this paper, we introduce a new class of sequence space, namely, the semi-difference entire sequence space $cs \cap d_1$. It is shown that the intersection of all semi-difference entire sequence spaces $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma(\Delta) \subset I$.

1. Introduction

A complex sequence, whose kth term is x_k, is denoted by $\{x_k\}$ or simply x. Let w be the set of all sequences and ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the classes of bounded, convergent, and null sequence, respectively. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ.

Given a sequence $x = \{x_k\}$, its nth section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$. Let $\delta^{(n)} = (0, 0, \ldots, 1, 0, 0, \ldots)$, 1 in the nth place and zeros elsewhere, $s^{(k)} = (0, 0, \ldots, 1, -1, 0, \ldots)$, 1 in the nth place, -1 in the $(n+1)$th place and zeros elsewhere. An FK-space (Fréchet coordinate space) is a Fréchet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1, 2, 3, \ldots)$ are continuous.

We recall the following definitions (one may refer to Wilansky [1]).

An FK-space is a locally convex Fréchet space which is made up of sequences and has the property that coordinate projections are continuous. A metric space (X, d) is said to have AK (or sectional convergence) if and only if $d(x^{(n)}, x) \to x$ as $n \to \infty$, (see [1]). The space is said to have AD (or) be an AD-space if ϕ is dense in X, where ϕ denotes the set of all finitely nonzero sequences. We note that AK implies AD (one may refer to Brown [2]).

If X is a sequence space, we define

(i) X' = the continuous dual of X;

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty,$ for each $x \in X\}$;

(iii) $X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k$ is convergent, for each $x \in X\}$;

(iv) $X^{\gamma} = \{a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty,$ for each $x \in X\}$;

(v) let X be an FK-space $\supset \phi$. Then, $X^f = \{f(\delta^{(n)}) : f \in X'\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the α (or Köthe-Töeplitz) dual of X, β—(or generalized Köthe-Töeplitz) dual of X, γ dual of X. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$, then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha$, β, or γ.

Let $p = (p_k)$ be a sequence of positive real numbers with $\sup_k p_k = G$ and $D = \max\{1, 2^{G-1}\}$. Then, it is well known that for all $a_k, b_k \in C$, the field of complex numbers, for all $k \in N$,

$$|a_k + b_k|^{p_k} \leq D\left(|a_k|^{p_k} + |b_k|^{p_k}\right). \tag{1.1}$$

Lemma 1.1 (Wilansky [1, Theorem 7.2.7]). *Let X be an FK-space $\supset \phi$. Then,*

(i) $X^{\gamma} \subset X^f$;

(ii) *if X has AK,* $X^{\beta} = X^f$;

(iii) *if X has AD,* $X^{\beta} = X^{\gamma}$.

2. Definitions and Preliminaries

Let $\Delta : w \to w$ be the difference operator defined by $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$. Let

$$\Gamma = \left\{x \in w : \lim_{k \to \infty}\left(|x_k|^{1/k}\right) = 0\right\},$$
$$\Lambda = \left\{x \in w : \sup_k\left(|x_k|^{1/k}\right) < \infty\right\}. \tag{2.1}$$

Define the sets $\Gamma(\Delta) = \{x \in w : \Delta x \in \Gamma\}$ and $\Lambda(\Delta) = \{x \in w : \Delta x \in \Lambda\}$.

The spaces $\Gamma(\Delta)$ and $\Lambda(\Delta)$ are the metric spaces with the metric

$$d(x, y) = \inf\left\{\rho > 0 : \sup_k\left(\left|\Delta x_k - \Delta y_k\right|^{1/k}\right) \leq 1\right\}. \tag{2.2}$$

Because of the historical roots of summability in convergence, conservative space and matrices play a special role in its theory. However, the results seem mainly to depend on a weaker assumption, that the spaces be semi-conservative. (See Wilansky [1]).

Snyder and Wilansky [3] introduced the concept of semi-conservative spaces. Snyder [4] studied the properties of semi-conservative spaces. Later on, in the year 1996 the semi replete spaces were introduced by Rao and Srinivasalu [5].

In a similar way, in this paper, we define semi-difference entire sequence space $cs \cap d_1$, and show that semi-difference entire sequence space $cs \cap d_1$ is $I \subset cs \cap d_1$ and $\Gamma(\Delta) \subset I$.

3. Main Results

Proposition 3.1. $\Gamma \subset \Gamma(\Delta)$ *and the inclusion is strict.*

Proof. Let $x \in \Gamma$. Then, we have

$$|x_k|^{1/k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{3.1}$$

$$\begin{aligned} \frac{|\Delta x_k|^{1/k}}{2} &\leq \frac{1}{2}\left(|x_k|^{1/k}\right) + \frac{1}{2}\left(|x_{k+1}|^{1/k}\right), \quad \text{by (1.1)} \\ &\longrightarrow 0, \quad \text{as } k \longrightarrow \infty \quad \text{by (3.1)}. \end{aligned} \tag{3.2}$$

Let $x \in \Gamma$. Then, we have

$$\left(|x_k|^{1/k}\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.3}$$

Then, $(x_k) \in \Gamma(\Delta)$ follows from the inequality (1.1) and (3.3).

Consider the sequence $e = (1, 1, \ldots)$. Then, $e \in \Gamma(\Delta)$ but $e \notin \Gamma$. Hence, the inclusion $\Gamma \subset \Gamma(\Delta)$ is strict. □

Lemma 3.2. $A \in (\Gamma, c)$ *if and only if*

$$\lim_{n \to \infty} a_{nk} \quad \text{exists for each } k \in N, \tag{3.4}$$

$$\sup_{n,k} \left| \sum_{i=0}^{k} a_{ni} \right| < \infty. \tag{3.5}$$

Proposition 3.3. *Define the set* $d_1 = \{a = (a_k) \in w : \sup_{n,k \in N} |\sum_{j=0}^{k} (\sum_{i=j}^{n} a_i)| < \infty\}$. *Then,* $[\Gamma(\Delta)]^{\beta} = cs \cap d_1$.

Proof. Consider the equation

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} y_j \right) = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} a_j \right) y_k = (Cy)_n, \tag{3.6}$$

where $C = (C_{nk})$ is defined by

$$C_{nk} = \begin{cases} \displaystyle\sum_{j=k}^{n} a_j, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n; \ n, k \in N. \end{cases} \tag{3.7}$$

Thus, we deduce from Lemma 3.2 with (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \Gamma(\Delta)$ if and only if $Cy \in c$ whenever $y = (y_k) \in \Gamma$, that is $C \in (\Gamma, c)$. Thus, $(a_k) \in cs$ and $(a_k) \in d_1$ by Lemma 3.2 and (3.5) and (3.6), respectively. This completes the proof. □

Proposition 3.4. $\Gamma(\Delta)$ *has AK.*

Proof. Let $x = \{x_k\} \in \Gamma(\Delta)$. Then, $(|\Delta x_k|^{1/k}) \in \Gamma$. Hence,

$$\sup_{k \geq n+1} \left(|\Delta x_k|^{1/k} \right) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{3.8}$$

$$\begin{aligned} d\left(x, x^{[n]}\right) &= \inf\left\{ \rho > 0 : \sup_{k \geq n+1} \left(|\Delta x_k|^{1/k} \right) \leq 1 \right\} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \text{ by using (3.8)} \\ &\Longrightarrow x^{[n]} \longrightarrow x \quad \text{as } n \longrightarrow \infty, \\ &\Longrightarrow \Gamma(\Delta) \text{ has AK.} \end{aligned} \tag{3.9}$$

This completes the proof. □

Proposition 3.5. $\Gamma(\Delta)$ *is not solid.*

To prove Proposition 3.5, consider $(x_k) = (1) \in \Gamma(\Delta)$ and $\alpha_k = \{(-1)^k\}$. Then $(\alpha_k x_k) \notin \Gamma(\Delta)$. Hence, $\Gamma(\Delta)$ is not solid.

Proposition 3.6. $(\Gamma(\Delta))^\mu = cs \cap d_1$ *for* $\mu = \alpha, \beta, \gamma, f$.

Proof.

Step 1. $\Gamma(\Delta)$ has AK by Proposition 3.4. Hence, by Lemma 1.1(ii), we get $(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^f$. But $(\Gamma(\Delta))^\beta = cs \cap d_1$. Hence,

$$(\Gamma(\Delta))^f = cs \cap d_1. \tag{3.10}$$

Step 2. Since AK ⇒ AD. Hence, by Lemma 1.1(iii), we get $(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^\gamma$. Therefore,

$$(\Gamma(\Delta))^\gamma = cs \cap d_1. \tag{3.11}$$

Step 3. $\Gamma(\Delta)$ is not normal by Proposition 3.5. Hence by Proposition 2.7 of Kamthan and Gupta [6], we get

$$(\Gamma(\Delta))^\alpha \neq (\Gamma(\Delta))^\gamma \neq cs \cap d_1. \tag{3.12}$$

From (3.10) and (3.11), we have

$$(\Gamma(\Delta))^\beta = (\Gamma(\Delta))^\gamma = (\Gamma(\Delta))^f = cs \cap d_1. \tag{3.13}$$

□

Lemma 3.7 (Wilansky [1, Theorem 8.6.1]). *$Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD-space and Y an FK-space.*

Proposition 3.8. *Let Y be any FK-space $\supset \phi$. Then, $Y \supset \Gamma(\Delta)$ if and only if the sequence $\delta^{(k)}$ is weakly converges in $cs \cap d_1$.*

Proof. The following implications establish the result.

$Y \supset \Gamma(\Delta) \Leftrightarrow Y^f \subset (\Gamma(\Delta))^f$, since $\Gamma(\Delta)$ has AD by Lemma 3.7.

$\Leftrightarrow Y^f \subset cs \cap d_1$, since $(\Gamma(\Delta))^f = cs \cap d_1$.

$\Leftrightarrow$ for each $f \in Y'$, the topological dual of Y.

$\Leftrightarrow f(\delta^{(k)}) \in cs \cap d_1$.

$\Leftrightarrow \delta^{(k)}$ is weakly converges in $cs \cap d_1$.

This completes the proof. □

4. Properties of Semi-Difference Entire Sequence Space $cs \cap d_1$

Definition 4.1. An FK-space ΔX is called "semi-difference entire sequence space $cs \cap d_1$" if its dual $(\Delta X)^f \subset cs \cap d_1$.

In other words ΔX is semi-difference entire sequence space $cs \cap d_1$ if $f(\delta^{(k)}) \in cs \cap d_1$ for all $f \in (\Delta X)'$ for each fixed k.

Example 4.2. $\Gamma(\Delta)$ is semi-difference entire sequence space $cs \cap d_1$. Indeed, if $\Gamma(\Delta)$ is the space of all difference of entire sequences, then by Lemma 4.3, $(\Gamma(\Delta))^f = cs \cap d_1$.

Lemma 4.3 (Wilansky [1, Theorem 4.3.7]). *Let z be a sequence. Then (z^β, P) is an AK space with $P = (P_k : k = 0, 1, 2, \ldots)$, where $P_0(x) = \sup_m |\sum_{k=1}^m z_k x_k|$, and $P_n(x) = |x_n|$. For any k such that $z_k \neq 0$, P_k may be omitted. If $z \in \phi$, P_0 may be omitted.*

Proposition 4.4. *Let z be a sequence. z^β is a semi-difference entire sequence space $cs \cap d_1$ if and only if z is in $cs \cap d_1$.*

Proof. Suppose that z^β is a semi-difference entire sequence space $cs \cap d_1$. z^β has AK by Lemma 4.3. Therefore $z^{\beta\beta} = (z^\beta)^f$ by Lemma 1 [1]. So z^β is semi-difference entire sequence space $cs \cap d_1$ if and only if $z^{\beta\beta} \subset cs \cap d_1$. But then $z \in z^{\beta\beta} \subset cs \cap d_1$. Hence, z is in $cs \cap d_1$.

Conversely, suppose that z is in $cs \cap d_1$. Then $z^\beta \supset \{cs \cap d_1\}^\beta$ and $z^{\beta\beta} \subset \{cs \cap d_1\}^{\beta\beta} = cs \cap d_1$. But$(z^\beta)^f = z^{\beta\beta}$. Hence, $(z^\beta)^f \subset cs \cap d_1$. Therefore z^β is semi-difference entire sequence space $cs \cap d_1$. This completes the proof. □

Proposition 4.5. *Every semi-difference entire sequence space $cs \cap d_1$ contains Γ.*

Proof. Let ΔX be any semi-difference entire sequence space $cs \cap d_1$. Hence, $(\Delta X)^f \subset cs \cap d_1$. Therefore $f(\delta^{(k)}) \in cs \cap d_1$ for all $f \in (\Delta X)'$. So, $\{\delta^{(k)}\}$ is weakly converges in $cs \cap d_1$ with respect to ΔX. Hence, $\Delta X \supset \Gamma(\Delta)$ by Proposition 3.8. But $\Gamma(\Delta) \supset \Gamma$. Hence, $\Delta X \supset \Gamma$. This completes the proof. □

Proposition 4.6. *ΔX is semi-difference entire sequence space $cs \cap d_1$.*

Proof. Let $\Delta X = \bigcap_{n=1}^{\infty} \Delta X_n$. Then ΔX is an FK-space which contains ϕ. Also every $f \in (\Delta X)'$ can be written as $f = g_1 + g_2 + \ldots + g_m$, where $g_k \in (\Delta X_n)'$ for some n and for $1 \leq k \leq m$. But then $f(\delta^k) = g_1(\delta^k) + g_2(\delta^k) + \cdots + g_m(\delta^k)$. Since ΔX_n $(n = 1, 2, \ldots)$ are semi-difference entire sequence space $cs \cap d_1$, it follows that $g_i(\delta^k) \in cs \cap d_1$ for all $i = 1, 2, \ldots m$. Therefore $f(\delta^k) \in cs \cap d_1$ for all k and for all f. Hence, ΔX is semi-difference entire sequence space $cs \cap d_1$. This completes the proof. □

Proposition 4.7. *The intersection of all semi-difference entire sequence space $cs \cap d_1$ is $I \subset (cs \cap d_1)^{\beta}$ and $\Gamma(\Delta) \subset I$.*

Proof. Let I be the intersection of all semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.4, we see that the intersection

$$I \subset \cap \left\{ z^{\beta} : z \in cs \cap d_1 \right\} = \{cs \cap d_1\}^{\beta}. \tag{4.1}$$

By Proposition 4.6 it follows that I is semi-difference entire sequence space $cs \cap d_1$. By Proposition 4.5, consequently

$$\Gamma_M = \Gamma(\Delta) \subset I. \tag{4.2}$$

From (4.1) and (4.2), we get $I \subset \{cs \cap d_1\}^{\beta}$ and $\Gamma(\Delta) \subset I$. This completes the proof. □

Corollary 4.8. *The smallest semi-difference entire sequence space $cs \cap d_1$ is $I \subset (cs \cap d_1)^{\beta}$ and $\Gamma(\Delta) \subset I$.*

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