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Research Article

Differential Subordinations of Arithmetic and Geometric Means of Some Functionals Related to a Sector

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Some general theorems on differential subordinations of some functionals connected with arithmetic and geometric means related to a sector are proved. These results unify a number of well known results concerning inclusion relation between the classes of analytic functions built with using arithmetic and geometric means.

1. Introduction

For $r > 0$ let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$. Let $\mathbb{D} = \mathbb{D}_1$.

Let the functions f and F be analytic in the unit disc $\mathbb{D}$. A function f is called subordinate to F, written $f \prec F$, if F is univalent in $\mathbb{D}$, $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Let D be a domain in $\mathbb{C}^2$ and $\psi : \mathbb{C}^2 \supset D \to \mathbb{C}$ be an analytic function, and let p be a function analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$, $z \in \mathbb{D}$ and h be a function analytic and univalent in $\mathbb{D}$. The function p is said to satisfy the first-order differential subordination if

$$\psi\big(p(z), zp'(z)\big) \prec h(z), \qquad \psi\big(p(0), 0\big) = h(0), \quad z \in \mathbb{D}. \tag{1.1}$$

The general theory of the differential subordinations has been studied intensively by many authors. A survey of this theory can by found in the monograph by Miller and Mocanu [1].

For $\beta \in (0,2]$ let

$$h_\beta(z) = \left(\frac{1+z}{1-z}\right)^\beta, \quad z \in \mathbb{D}. \tag{1.2}$$

It is clear that h_β maps univalently $\mathbb{D}$ onto the sector of the angle $\beta\pi$ symmetrical with respect to the real axis with the vertex at the origin.

In this paper we are interested in the following problem referring to (1.1) to find the constant $c_k(n,\gamma,\alpha,\beta)$ so that to the following relation is true:

$$\left\{p(z)\left[1+\alpha\frac{zp'(z)}{p^k(z)}\right]^\gamma \prec h_{c_k(n,\alpha,\gamma,\beta)}(z), z \in \mathbb{D}\right\} \Longrightarrow p \prec h_\beta, \tag{1.3}$$

with suitable assumptions on function p and constants n, α, γ, β. For selected parameters n, γ, α, β the theorems presented here reduce to the well-known theorems proved by various authors. Particularly, results of this type can be applied to examine inclusion relation between subclasses of analytic functions defined with using arithmetic or geometric means of some functionals, for example, the class of α-convex functions or γ-starlike functions.

The lemma below that slightly generalizes a lemma proved by Miller and Mocanu [2] will be required in our investigation.

Lemma 1.1 (see [2]). *Let $q : \overline{\mathbb{D}} \to \mathbb{C}$ be a function analytic and univalent on $\mathbb{D}$, injective on $\partial\mathbb{D}$ and $q(0) = 1$. Let*

$$p(z) = 1 + \sum_{k=n}^{\infty} c_k z^k, \quad z \in \mathbb{D}, \tag{1.4}$$

be analytic in $\mathbb{D}, p \not\equiv 1$. Suppose that there exists a point $z_0 \in \mathbb{D}$ such that $p(z_0) \in \partial q(\mathbb{D})$ and

$$p(\{z \in \mathbb{C} : |z| < |z_0|\}) \subset q(\mathbb{D}). \tag{1.5}$$

If $\xi_0 = q^{-1}(p(z_0))$ and $q'(\xi_0)$ exists, then there exists an $m \geq n$ for which

$$z_0 p'(z_0) = m\xi_0 q'(\xi_0). \tag{1.6}$$

2. Main Results

In the first theorem which follows directly from Theorem 2.2 [3] we prove that $c_k(n,\alpha,\gamma,\beta) \geq \beta$. Let us start with the following definition.

Definition 2.1 (see [3]). Let $\gamma \in [0,1]$ and Φ be a function analytic in domain $D \subset \mathbb{C}$. By $\mathcal{A}(\gamma, \Phi)$ will be denoted the class of functions p analytic in $\mathbb{D}$ with $p(0) = 1$, $p \not\equiv 1$ and $p(\mathbb{D}) \subset D$ such that the function

$$Q(z) = p(z)\left[1 + \frac{zp'(z)}{p(z)}\Phi(p(z))\right]^{\gamma}, \quad Q(0) = 1, \ z \in \mathbb{D}, \tag{2.1}$$

is well defined in $\mathbb{D}$.

Theorem 2.2 (see [3]). *Let $\gamma \in [0,1]$, h a convex function such that $0 \in \overline{h(\mathbb{D})}$, $h(0) = 1$, Φ a function analytic in a domain $D \subset \mathbb{C}$ such that $h(\mathbb{D}) \subset D$, and $\operatorname{Re}\Phi(h(z)) > 0$ for $z \in \mathbb{D}$. If $p \in \mathcal{A}(\gamma, \Phi)$ and*

$$p(z)\left[1 + \frac{zp'(z)}{p(z)}\Phi(p(z))\right]^{\gamma} \prec h(z), \quad z \in \mathbb{D}, \tag{2.2}$$

then

$$p \prec h. \tag{2.3}$$

Definition 2.3. Let $k,\ n \in \mathbb{N}$, $\alpha \geq 0$ and $\gamma \in [0,1]$. By $\mathcal{A}_k(n, \alpha, \gamma)$ will be denoted the class of functions p analytic in $\mathbb{D}$ of the form (1.4) such that the function

$$Q(z) = p(z)\left[1 + \alpha\frac{zp'(z)}{p^k(z)}\right]^{\gamma}, \quad Q(0) = 1, \ z \in \mathbb{D}, \tag{2.4}$$

is well defined in $\mathbb{D}$.

Remark 2.4. (1) Setting

$$\Phi(w) = \Phi_{k,\alpha}(w) = \frac{\alpha}{w^{k-1}}, \quad w \in \mathbb{C} \setminus \{0\}, \tag{2.5}$$

we see that

$$\mathcal{A}_k(1, \alpha, \gamma) = \mathcal{A}(\gamma, \Phi_{k,\alpha}). \tag{2.6}$$

(2) For each $k,\ n,\ \alpha,\ \gamma$ as in Definition 2.3 the class $\mathcal{A}_k(n, \alpha, \gamma)$ is nonempty. To see this take

$$p(z) = 1 + c_n z^n, \quad z \in \mathbb{D}, \tag{2.7}$$

for sufficiently small $c_n \in \mathbb{C}$.

(3) Clearly, for $\gamma = 0$ the class $\mathcal{A}_k(n, \alpha, 0)$ contains all analytic functions p of the form (1.4).

(4) Let $k = \gamma = 1$. Then

$$Q(z) = p(z) + \alpha z p'(z), \quad z \in \mathbb{D}. \tag{2.8}$$

Therefore the class $\mathcal{H}_1(n, \alpha, 1)$ contains all analytic functions p of the form (1.4).

(5) Let p be analytic function in $\mathbb{D}$ of the form (1.4). Suppose that $p(z_0) = 0$ for some $z_0 \in \mathbb{D}$. Then

$$p(z) = (z - z_0)^m p_1(z), \quad z \in \mathbb{D}, \tag{2.9}$$

where $m \geq 1$ and p_1 is analytic function in $\mathbb{D}$ with $p_1(z_0) \neq 0$ for $z \in \mathbb{D}$. Then we have

$$\begin{aligned} z\frac{p'(z)}{p^k(z)} &= z\frac{m(z - z_0)^{m-1}p_1(z) + (z - z_0)^m p_1'(z)}{\left[(z - z_0)^m p_1(z)\right]^k} \\ &= \frac{mp_1(z) + (z - z_0)p_1'(z)}{(z - z_0)^{m(k-1)+1} p_1^k(z)}. \end{aligned} \tag{2.10}$$

Hence we see that for $k = 1$ and $\gamma \in (0, 1)$ or for $k \geq 2$ and $\gamma \in (0, 1]$ the function

$$Q(z) = p(z)\left[1 + \alpha\frac{zp'(z)}{p^k(z)}\right]^\gamma, \quad z \in \mathbb{D}, \tag{2.11}$$

has a pole at z_0. Therefore for such k and γ we see that every $p \in \mathcal{H}_k(n, \alpha, \gamma)$ is nonvanishing in D.

Theorem 2.5. *Let $k \in \mathbb{N}$, $\alpha \geq 0$, $\gamma \in [0, 1]$ and $\beta \in [0, 1]$ be such that $(k - 1)\beta \leq 1$. If $p \in \mathcal{H}_k(1, \alpha, \gamma)$ and*

$$p(z)\left[1 + \alpha\frac{zp'(z)}{p^k(z)}\right]^\gamma \prec h_\beta(z), \quad z \in \mathbb{D}, \tag{2.12}$$

then

$$p \prec h_\beta. \tag{2.13}$$

Proof. The case $\alpha = 0$ is evident so we assume that $\alpha > 0$. For $k \in N$ and $\alpha > 0$ let $\Phi = \Phi_{k,\alpha}$ be defined by (2.5). For $\beta \in [0, 1]$ the function h_β is convex with $0 \in \partial h(\mathbb{D})$. Since $(k - 1)\beta \leq 1$, we have

$$\operatorname{Re}\Phi\left(h_\beta(z)\right) = \operatorname{Re}\Phi\left(\left(\frac{1 + z}{1 - z}\right)^\beta\right) = \alpha \operatorname{Re}\left\{\left(\frac{1 - z}{1 + z}\right)^{(k-1)\beta}\right\} > 0, \quad z \in \mathbb{D}. \tag{2.14}$$

Applying Theorem 2.2 with h_β instead of h we get the assertion. □

Now we prove two theorems were we improve the result of Theorem 2.5. The problem (1.3) will be divided into two cases: $k = 1$ and $k > 1$.

First we consider the case $k = 1$. The theorem below was proved in [4]. To be self-contained we include its proof.

Theorem 2.6. *Fix* $n \in \mathbb{N}$, $\alpha \geq 0$ *and* $\gamma \in [0,1]$. *Let* $\beta \in (0, \beta_1(n,\alpha,\gamma)]$, *where* $\beta = \beta_1(n,\alpha,\gamma)$ *is the solution of the equation*

$$\beta + c_1(n,\alpha,\gamma,\beta) = 4 - \gamma, \tag{2.15}$$

with

$$c_1(n,\alpha,\gamma,\beta) = \beta + \frac{2\gamma}{\pi}\arctan(n\alpha\beta). \tag{2.16}$$

If $p \in \mathcal{H}_1(n,\alpha,\gamma)$ *and*

$$p(z)\left[1 + \alpha\frac{zp'(z)}{p(z)}\right]^{\gamma} \prec h_{c_1(n,\alpha,\gamma,\beta)}(z), \quad z \in \mathbb{D}, \tag{2.17}$$

then

$$p \prec h_\beta. \tag{2.18}$$

Proof. (1) Assume that $\alpha > 0$ and $\gamma \in (0,1]$ since the cases $\gamma = 0$ or $\alpha = 0$ are evident.

Suppose, on the contrary, that p is not subordinate to h_β. Then, by the minimum principle for harmonic mappings there exists $r_0 \in (0,1)$ such that

$$p(\mathbb{D}_{r_0}) \subset h_\beta(\mathbb{D}), \tag{2.19}$$

and one of the following cases hold:

$$\max\{\operatorname{Arg} p(z) : z \in \mathbb{D}_{r_0}\} = \max\{\operatorname{Arg} p(z) : |z| = r_0\} = \beta\frac{\pi}{2}, \tag{2.20}$$

or

$$\min\{\operatorname{Arg} p(z) : z \in \mathbb{D}_{r_0}\} = \min\{\operatorname{Arg} p(z) : |z| = r_0\} = -\beta\frac{\pi}{2}, \tag{2.21}$$

or

$$p(z_0) = 0, \tag{2.22}$$

for some $z_0 \in \partial\mathbb{D}_{r_0}$.

(2) Assume that (2.20) holds. Then there exists $z_0 \in \partial\mathbb{D}_{r_0}$ such that

$$\operatorname{Arg} p(z_0) = \beta\frac{\pi}{2}. \tag{2.23}$$

Let $\xi_0 = h_\beta^{-1}(p(z_0))$. Thus

$$p(z_0) = h_\beta(\xi_0) = \left(\frac{1+\xi_0}{1-\xi_0}\right)^\beta \neq 0. \tag{2.24}$$

Therefore $\xi_0 \neq \pm 1$ and

$$\frac{1+\xi_0}{1-\xi_0} = xi \tag{2.25}$$

for $x > 0$, that is,

$$\xi_0 = \frac{xi-1}{xi+1}. \tag{2.26}$$

Since $\xi_0 \neq \pm 1$, so $h'_\beta(\xi_0)$ exists. Hence and by Lemma 1.1 there exists an $m \geq n$ for which

$$z_0 p'(z_0) = m\xi_0 h'_\beta(\xi_0). \tag{2.27}$$

(3) Consequently,

$$\begin{aligned} p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right]^\gamma &= h_\beta(\xi_0)\left[1+m\alpha\frac{\xi_0 h'_\beta(\xi_0)}{h_\beta(\xi_0)}\right]^\beta \\ &= (xi)^\beta\left[1+\frac{m\alpha\beta(1+x^2)}{2x}i\right]^\gamma. \end{aligned} \tag{2.28}$$

In view of the fact that $x > 0$ let us take

$$\arg\left\{1+\frac{m\alpha\beta(1+x^2)}{2x}i\right\} \in \left[0,\frac{\pi}{2}\right). \tag{2.29}$$

Hence and from (2.28) we have

$$\begin{aligned}\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right]^{\gamma}\right\} &= \arg\left\{(xi)^{\beta}\left[1+\frac{m\alpha\beta(1+x^2)}{2x}i\right]^{\gamma}\right\} \\ &= \beta\frac{\pi}{2}+\gamma\arg\left\{1+\frac{m\alpha\beta(1+x^2)}{2x}i\right\} \\ &= \beta\frac{\pi}{2}+\gamma\arctan\left(\frac{m\alpha\beta(1+x^2)}{2x}\right).\end{aligned} \tag{2.30}$$

By the above and by the fact that $m \geq n$ we have

$$\begin{aligned}\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right]^{\gamma}\right\} &\geq \beta\frac{\pi}{2}+\gamma\arctan\left(\frac{n\alpha\beta(1+x^2)}{2x}\right) \\ &\geq \beta\frac{\pi}{2}+\gamma\arctan(n\alpha\beta) = c_1(n,\alpha,\gamma,\beta)\frac{\pi}{2}.\end{aligned} \tag{2.31}$$

On the other hand, (2.30) yields

$$\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right]^{\gamma}\right\} = \beta\frac{\pi}{2}+\gamma\arctan\left(\frac{m\alpha\beta(1+x^2)}{2x}i\right) \leq (\beta+\gamma)\frac{\pi}{2}. \tag{2.32}$$

Finally, the above and (2.31) lead to

$$c_1(n,\alpha,\gamma,\beta)\frac{\pi}{2} \leq \arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right]^{\gamma}\right\} \leq (\beta+\gamma)\frac{\pi}{2} \leq 2\pi - c_1(n,\alpha,\gamma,\beta)\frac{\pi}{2}, \tag{2.33}$$

for all $\beta \in (0, \beta_1(n,\alpha,\gamma)]$.

Thus we arrive at a contradiction with (2.17) so $p \prec h_{\beta}$.

(4) When (2.21) holds, we see that $x < 0$ in (2.25). Next we finish the proof by similar argumentations like in the above.

(5) Assume now that (2.22) holds. In view of Remark 2.4 this is possible only when $k = \gamma = 1$.

(a) For $\beta < 1$ the boundary $\partial h_{\beta}(\mathbb{D})$ has the corner at 0 of the angle $\beta\pi < \pi$. Since $p(\partial\mathbb{D}_{r_0})$ is an analytic curve, in view of (2.19) the case $p(z_0) = 0$ does not hold for $\beta < 1$.

(b) Let now $\beta \geq 1$.

Assume that $p'(z_0) \neq 0$. Since $z_0p'(z_0)$ is an outer normal to the curve $p(\partial\mathbb{D}_{r_0})$ at $p(z_0)$, by (2.19) we see that

$$(3-\beta)\frac{\pi}{2} = \frac{3}{2}\pi - \beta\frac{\pi}{2} \leq \arg\{z_0p'(z_0)\} \leq \beta\frac{\pi}{2}+\frac{\pi}{2} = (\beta+1)\frac{\pi}{2}. \tag{2.34}$$

Hence taking into account that

$$3-\beta \le c_1(n,\alpha,1,\beta), \qquad \beta+1 \le 4-c_1(n,\alpha,1,\beta),$$
$$p(z_0)+\alpha z_0 p'(z_0) = \alpha z_0 p'(z_0), \tag{2.35}$$

we deduce that

$$c_1(n,\alpha,1,\beta)\frac{\pi}{2} \le \arg\{p(z_0)+\alpha z_0 p'(z_0)\} \le (\beta+\gamma)\frac{\pi}{2} \le 2\pi - c_1(n,\alpha,1,\beta)\frac{\pi}{2}, \tag{2.36}$$

for all $\beta \in (0,\beta_1(n,\alpha,\gamma)]$. In this way we arrive at a contradiction with (2.17) so $p \prec h_\beta$.

If $p'(z_0)=0$, then

$$p(z_0)+\alpha z_0 p'(z_0) = 0 \notin h_\beta(\mathbb{D}), \tag{2.37}$$

and once again we contradict (2.17). □

Special Cases

(1) The case $n=1,\ \alpha=1$ was proved in [5].

(2) The case $\gamma=1$ was proved in [6].

Corollary 2.7 (see [6]). *Let $n \in \mathbb{N},\ \alpha \ge 0$. Let $\beta \in (0,\beta_1(n,\alpha,1)]$, where $\beta=\beta_1(n,\alpha,1)$ is the solution of the equation*

$$\beta + c_1(n,\alpha,1,\beta) = 3, \tag{2.38}$$

with

$$c_1(n,\alpha,1,\beta) = \beta + \frac{2}{\pi}\arctan(n\alpha\beta). \tag{2.39}$$

If p is analytic function in $\mathbb{D}$ of the form (1.4) *and*

$$p(z)+\alpha z p'(z) \prec h_{c_1(n,\alpha,1,\beta)}(z), \quad z \in \mathbb{D}, \tag{2.40}$$

then

$$p \prec h_\beta. \tag{2.41}$$

(3) The case $\gamma=1,\ \ \alpha=1$ was remarked [7].

(4) The case $\gamma=1,\ \ \alpha=1,\ \ n=1$ was proved in detail in [8].

Now we consider the problem (1.3) for $k \ge 2$.

Theorem 2.8. *Let $k \ge 2$, $n \in \mathbb{N}$, $\alpha \ge 0$, $\gamma \in [0,1]$ and let $\beta \in (0,1/(k-1)]$. If $p \in \mathcal{A}_k(n,\alpha,\gamma)$ and*

$$p(z)\left[1+\alpha\frac{zp'(z)}{p^k(z)}\right]^\gamma \prec h_{c_k(n,\alpha,\gamma,\beta)}(z), \quad z \in \mathbb{D}, \tag{2.42}$$

then

$$p \prec h_\beta, \tag{2.43}$$

where

$$\begin{aligned} &c_k(n,\alpha,\gamma,\beta) \\ &= \beta + \frac{2\gamma}{\pi} \times \arctan\left[\frac{n\alpha\beta\cos((k-1)\beta\pi/2)}{(1+(k-1)\beta)^{(1+(k-1)\beta)/2}(1-(k-1)\beta)^{(1-(k-1)\beta)/2} + n\alpha\beta\sin((k-1)\beta\pi/2)}\right]. \end{aligned} \tag{2.44}$$

Proof. (1) We repeat argumentation from Parts 1 and 2 of the proof of Theorem 2.6.

(2) We have

$$\begin{aligned} p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^\gamma &= h_\beta(\xi_0)\left[1+m\alpha\frac{\xi_0h'_\beta(\xi_0)}{h^k_\beta(\xi_0)}\right]^\beta \\ &= (xi)^\beta\left[1+\frac{m\alpha\beta(1+x^2)}{2x}\frac{i}{(xi)^{(k-1)\beta}}\right]^\gamma \\ &= (xi)^\beta\left[1+\frac{m\alpha\beta(1+x^2)}{2x^{1+(k-1)\beta}}i^{1-(k-1)\beta}\right]^\gamma. \end{aligned} \tag{2.45}$$

Since $x>0$ and $1-(k-1)\beta \geq 0$, we can take

$$\arg\left\{1+\frac{m\alpha\beta(1+x^2)}{2x^{1+(k-1)\beta}}i^{1-(k-1)\beta}\right\} \in \left[0, \frac{\pi}{2}\right). \tag{2.46}$$

Hence

$$\begin{aligned} &\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^\gamma\right\} \\ &= \arg\left\{(xi)^\beta\left[1+\frac{m\alpha\beta(1+x^2)}{2x^{1+(k-1)\beta}}i^{1-(k-1)\beta}\right]^\gamma\right\} \\ &= \beta\frac{\pi}{2} + \gamma\arg\left\{1+\frac{m\alpha\beta(1+x^2)}{2x^{1+(k-1)\beta}}\sin\left(\frac{(k-1)\beta\pi}{2}\right) + i\frac{m\alpha\beta(1+x^2)}{2x^{1+(k-1)\beta}}\cos\left(\frac{(k-1)\beta\pi}{2}\right)\right\} \\ &= \beta\frac{\pi}{2} + \gamma\arctan\left[\frac{(m\alpha\beta(1+x^2)/2x^{1+(k-1)\beta})\cos((k-1)\beta\pi/2)}{1+(m\alpha\beta(1+x^2)/2x^{1+(k-1)\beta})\sin((k-1)\beta\pi/2)}\right]. \end{aligned} \tag{2.47}$$

Thus, from (2.46) and by the fact that $m \geq n$ we obtain

$$\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^\gamma\right\} \geq \beta\frac{\pi}{2}+\gamma\arctan\left[\frac{n\alpha\beta a(x)\cos((k-1)\beta\pi/2)}{1+n\alpha\beta a(x)\sin((k-1)\beta\pi/2)}\right], \tag{2.48}$$

where

$$a(x)=\frac{1+x^2}{2x^{1+(k-1)\beta}}, \quad x\neq 0. \tag{2.49}$$

We have

$$a'(x)=\frac{(1-(k-1)\beta)x^2-(1+(k-1)\beta)}{2x^{2+(k-1)\beta}}, \quad x\neq 0. \tag{2.50}$$

(3) Assume now that $(k-1)\beta<1$. Observe that the function a attains its minimum at the point

$$x_0=\sqrt{\frac{1+(k-1)\beta}{1-(k-1)\beta}}. \tag{2.51}$$

Moreover

$$a(x_0)=\frac{1}{(1+(k-1)\beta)^{(1+(k-1)\beta)/2}(1-(k-1)\beta)^{(1-(k-1)\beta)/2}}. \tag{2.52}$$

Hence, and from (2.48), we have

$$\begin{aligned}
&\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^\gamma\right\}\\
&\geq \beta\frac{\pi}{2}+\gamma\arctan\left\{\frac{n\alpha\beta a(x_0)\cos((k-1)\beta\pi/2)}{1+n\alpha\beta a(x_0)\sin((k-1)\beta\pi/2)}\right\}=\beta\frac{\pi}{2}\\
&\quad+\gamma\arctan\left[\frac{n\alpha\beta\cos((k-1)\beta\pi/2)}{(1+(k-1)\beta)^{(1+(k-1)\beta)/2}(1-(k-1)\beta)^{(1-(k-1)\beta)/2}+n\alpha\beta\sin((k-1)\beta\pi/2)}\right]\\
&=c_k(n,\alpha,\gamma,\beta)\frac{\pi}{2}.
\end{aligned} \tag{2.53}$$

On the other hand, using the fact that $0 \le (k-1)\beta\pi/2 < \pi/2$, from (2.47) we obtain

$$
\begin{aligned}
&\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^{\gamma}\right\}\\
&\quad=\beta\frac{\pi}{2}+\gamma\arctan\left[\frac{\left(m\alpha\beta(1+x^2)/2x^{1+(k-1)\beta}\right)\cos((k-1)\beta\pi/2)}{1+\left(m\alpha\beta(1+x^2)/2x^{1+(k-1)\beta}\right)\sin((k-1)\beta\pi/2)}\right]\le(\beta+\gamma)\frac{\pi}{2}.
\end{aligned}
\tag{2.54}
$$

Finally, the above and (2.53) yield

$$
c_k(n,\alpha,\gamma,\beta)\frac{\pi}{2}\le\arg\left\{p(z_0)\left[1+\alpha\frac{z_0p'(z_0)}{p^k(z_0)}\right]^{\gamma}\right\}\le(\beta+\gamma)\frac{\pi}{2}\le 2\pi-c_k(n,\alpha,\gamma,\beta)\frac{\pi}{2},
\tag{2.55}
$$

for all $\beta\in(0,1/(k-1)]$.

Thus we arrive at a contradiction with (2.17) so $p \prec h_\beta$.

(4) For $(k-1)\beta=1$ we have $c_k(n,\alpha,\gamma,\beta)=\beta$.

This ends the proof of the theorem for the case $x>0$.

(5) When (2.21) holds, we see that $x<0$ in (2.25). Next we finish the proof by similar argumentations like in the above.

(6) Since $\beta\le 1/(k-1)<1$, arguing as in Part 5(a) of the proof of Theorem 2.6 we see that the case (2.22) does not hold. □

Special Cases

(1) $\beta = 1/(k-1)$.

Then $c_k(n,\alpha,\gamma,\beta)=\beta$.

Corollary 2.9. *Let* $k\ge 2, n\in\mathbb{N}, \alpha\ge 0$ *and* $\gamma\in[0,1]$. *If* $p\in\mathcal{H}_k(n,\alpha,\gamma)$ *and*

$$
p(z)\left[1+\alpha\frac{zp'(z)}{p^k(z)}\right]^{\gamma}\prec h_{1/(k-1)}(z),\quad z\in\mathbb{D},
\tag{2.56}
$$

then

$$
p\prec h_{1/(k-1)}.
\tag{2.57}
$$

(2) $k=2,\ \beta=1$.

Corollary 2.10. *Let* $n\in\mathbb{N}$, $\alpha\ge 0$ *and* $\gamma\in[0,1]$. *If* $p\in\mathcal{H}_2(n,\alpha,\gamma)$ *and*

$$
\operatorname{Re}\left\{p(z)\left[1+\alpha\frac{zp'(z)}{p^2(z)}\right]^{\gamma}\right\}>0,\quad z\in\mathbb{D},
\tag{2.58}
$$

then

$$\operatorname{Re}\{p(z)\} > 0, \quad z \in \mathbb{D}. \tag{2.59}$$

(3) The case $n = 1,\ \alpha = 1$ was proved in [9].
(4) $\gamma = 1$.

Corollary 2.11. *Let* $k \geq 2,\ n \in \mathbb{N},\ \alpha \geq 0$ *and* $\beta \in (0, 1/(k-1)]$. *If* p *is a function analytic in* $\mathbb{D}$ *of the form* (1.4) *nonvanishing in* $\mathbb{D}$ *and*

$$p(z) + \alpha \frac{zp'(z)}{p^{k-1}(z)} \prec h_{c_k(n,\alpha,1,\beta)}(z), \quad z \in \mathbb{D}, \tag{2.60}$$

then

$$p \prec h_\beta, \tag{2.61}$$

where $c_k(n, \alpha, 1, \beta)$ *is given by* (2.44).

(5) $\gamma = 1,\ k = 2$.

Corollary 2.12. *Let* $n \in \mathbb{N},\ \alpha \geq 0$, *and* $\beta \in (0, 1]$. *If* p *is a function analytic in* $\mathbb{D}$ *of the form* (1.4) *nonvanishing in* $\mathbb{D}$ *and*

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec h_{c_2(n,\alpha,1,\beta)}(z), \quad z \in \mathbb{D}, \tag{2.62}$$

where

$$c_2(n, \alpha, 1, \beta) = \beta + \frac{2}{\pi} \arctan\left[\frac{n\alpha\beta\cos(\beta\pi/2)}{(1+\beta)^{(1+\beta)/2}(1-\beta)^{(1-\beta)/2} + n\alpha\beta\sin(\beta\pi/2)}\right], \tag{2.63}$$

then

$$p \prec h_\beta. \tag{2.64}$$

(6) The case $\gamma = 1,\ k = 2,\ n = 1,\ \alpha = 1$ was proved in [10]. The same result was reproved in [11] and once again in [12].
(7) $\gamma = 1,\ k = 2,\ \beta = 1$.

Corollary 2.13. *Let* $n \in \mathbb{N},\ \alpha \geq 0$, *and* $\beta \in (0, 1]$. *If* p *is an analytic function in* $\mathbb{D}$ *of the form* (1.4) *nonvanishing in* $\mathbb{D}$ *and*

$$\operatorname{Re}\left\{p(z) + \alpha \frac{zp'(z)}{p(z)}\right\} > 0, \quad z \in \mathbb{D}, \tag{2.65}$$

then

$$\operatorname{Re}\{p(z)\} > 0, \quad z \in \mathbb{D}. \tag{2.66}$$

3. Applications

All this type results can be applied in the theory of analytic functions. Some results concerning the inclusion relations between subclasses of analytic functions can be formulated.

Let $\mathcal{A}(n),\ n \in N$, denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{3.1}$$

which is analytic in $\mathbb{D}$. For short, let $\mathcal{A} = \mathcal{A}(1)$. Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

To use theorems and corollaries listed in the previous section we put instead of the function p some functionals over the class $\mathcal{A}(n)$, such as $p(z) = f(z)/z, p(z) = zf'(z)/f(z)$ or the others. In this way the inclusion relations between selected subclasses of analytic functions can be obtained.

3.1. *Arithmetic Means*

(I) $\gamma = 1,\ k = 1$.

(i) $p(z) = f(z)/z,\ z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in N$.

For $n \in \mathbb{N},\ \alpha \geq 0,\ \beta \in (0,1]$ let $\mathcal{R}_n(\alpha,\beta)$ denote class of functions $f \in \mathcal{A}(n)$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec h_\beta(z), \quad z \in \mathbb{D}, \tag{3.2}$$

or, equivalently,

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.3}$$

Using Corollary 2.7 we have the following.

Corollary 3.1. *Let $n \in \mathbb{N},\ \alpha \geq 0$, and $\beta \in (0, \beta_1(n,\alpha,1)]$. If $f \in \mathcal{A}(n)$ and*

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\}\right| < c_1(n,\alpha,1,\beta)\frac{\pi}{2}, \tag{3.4}$$

then

$$\left|\arg\left\{\frac{f(z)}{z}\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.5}$$

The above result we can write in the following form.

Corollary 3.2.

$$\mathcal{R}_n(\alpha, c_1(n, \alpha, 1, \beta)) \subset \mathcal{R}_n(0, \beta). \tag{3.6}$$

(ii) $p(z) = f'(z),\ z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in \mathbb{N}$

For $n \in \mathbb{N},\ \alpha \geq 0,\ \beta \in (0, 1]$ let $\mathcal{T}_n(\alpha, \beta)$ denote class of functions $f \in \mathcal{A}(n)$ such that

$$f'(z) + \alpha z f''(z) \prec h_\beta(z), \qquad z \in \mathbb{D}, \tag{3.7}$$

or, equivalently,

$$\left|\arg\{f'(z) + \alpha z f''(z)\}\right| < \beta \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.8}$$

Remark 3.3. (1) The class $\mathcal{T}_1(1, 1)$ was introduced in [13].

(2) The class $\mathcal{T}_1(\alpha, 1)$ coincides with the class $H(\alpha, 1, -1)$ studied in [14].

Observe that $f \in \mathcal{T}_n(\alpha, \beta)$ if and only if $zf' \in \mathcal{R}_n(\alpha, \beta)$.

Using Corollary 2.7 we have the following.

Corollary 3.4. *Let* $n \in \mathbb{N},\ \alpha \geq 0$, *and* $\beta \in (0, \beta_1(n, \alpha, 1)]$. *If* $f \in \mathcal{A}(n)$ *and*

$$\left|\arg\{f'(z) + \alpha z f''(z)\}\right| < c_1(n, \alpha, 1, \beta) \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{3.9}$$

then

$$\left|\arg\{f'(z)\}\right| < \beta \frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.10}$$

Hence we have the following.

Corollary 3.5.

$$\mathcal{T}_n\ (\alpha, c_1(n, \alpha, 1, \beta)) \subset \mathcal{T}_n(0\ , \beta). \tag{3.11}$$

(II) $\gamma = 1,\ k = 2.$

(ii) $p(z) = zf'(z)/f(z),\ z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in \mathbb{N}.$

For $n \in \mathbb{N},\ \alpha \geq 0,\ \beta \in (0, 1]$ let $\mathcal{M}_n(\alpha, \beta)$ denote class of functions $f \in \mathcal{A}(n)$ such that $f(z)f'(z) \neq 0$ for $z \in \mathbb{D}$ and

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec h_\beta(z), \quad z \in \mathbb{D}, \tag{3.12}$$

or, equivalently,

$$\left|\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right|<\beta\frac{\pi}{2},\quad z\in\mathbb{D}. \tag{3.13}$$

Remark 3.6. (1) The class $\mathcal{M}_1(\alpha,1)$, that is, the class of so-called *α-convex* functions was introduced by Mocanu [15].

(2) The class $\mathcal{M}_1(0,1)$ is identical with the class $\mathcal{S}^*$ of *starlike* functions. The class $\mathcal{M}_1(1,1)$ is identical with the class $\mathcal{C}$ of *convex* functions.

(3) The class $\mathcal{M}_1(0,\beta)$ denoted by $\mathcal{S}^*(\beta)$ were defined by Brannan and Kirwan [16] and, independently, by Stankiewicz [17, 18]. Functions in this class are called *strongly starlike of order* β.

The class $\mathcal{M}_1(1,\beta)$ denoted by $\mathcal{C}^*(\beta)$ contains functions called *strongly convex of order* β.

Using Corollary 2.12 we have the following result proved by Marjono and Thomas [19].

Corollary 3.7. *Let $n\in\mathbb{N}$, $\alpha\geq 0$, and $\beta\in(0,1]$. If $f\in\mathcal{A}(n)$ and*

$$\left|\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right|<c_2(n,\alpha,1,\beta)\frac{\pi}{2},\quad z\in\mathbb{D}, \tag{3.14}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\beta\frac{\pi}{2},\quad z\in\mathbb{D}. \tag{3.15}$$

For $\alpha=1$ one has the result due to Nunokawa and Thomas [12]:

Corollary 3.8. *Let $n\in\mathbb{N}$ and $\beta\in(0,1]$. If $f\in\mathcal{A}(n)$ and*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right|<c_2(n,1,1,\beta)\frac{\pi}{2},\quad z\in\mathbb{D}, \tag{3.16}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\beta,\quad z\in\mathbb{D}. \tag{3.17}$$

Corollary 3.9.

$$\begin{gathered}\mathcal{M}_n(\alpha,c_2(n,\alpha,1,\beta))\subset\mathcal{M}_n(0,\beta),\\ \mathcal{M}_1(\alpha,1)\subset\mathcal{S}^*,\\ \mathcal{C}\subset\mathcal{S}^*.\end{gathered} \tag{3.18}$$

3.2. Geometric Mean

(I) $\alpha = 1,\ k = 1.$

(i) $p(z) = f(z)/z,\ z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in \mathbb{N}.$

For $n \in \mathbb{N}$, $\beta \in (0,1]$, and $\gamma \in [0,1]$ let $\mathcal{L}_n(\gamma,\beta)$ denote class of functions $f \in \mathcal{A}(n)$ such that

$$\left(\frac{f(z)}{z}\right)^{1-\gamma}(f'(z))^{\gamma} \prec h_\beta(z), \quad z \in \mathbb{D}, \tag{3.19}$$

or equivalently

$$\left|\arg\left\{\left(\frac{f(z)}{z}\right)^{1-\gamma}(f'(z))^{\gamma}\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.20}$$

Remark 3.10. The class $\mathcal{L}_1(\gamma,1)$ was introduced in [20].

Applying Theorem 2.6 with $\alpha = 1$ we have the following.

Corollary 3.11. *Let $n \in \mathbb{N}$, $\gamma \in [0,1]$, and $\beta \in (0,\beta_1(n,1,\gamma)]$. If $f \in \mathcal{A}(1)$ and*

$$\left|\arg\left\{\left(\frac{f(z)}{z}\right)^{1-\gamma}(f'(z))^{\gamma}\right\}\right| < c_1(n,1,\gamma,\beta)\frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{3.21}$$

then

$$\left|\arg\left\{\frac{f(z)}{z}\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.22}$$

Corollary 3.12.

$$\mathcal{L}_n(\gamma, c_1(n,1,\gamma,\beta)) \subset \mathcal{L}_n(0,\beta). \tag{3.23}$$

(II) $\alpha = 1,\ k = 2.$

(ii) $p(z) = zf'(z)/f(z),\ z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in \mathbb{N}.$

For $n \in \mathbb{N}$, $\beta \in (0,1]$, and $\gamma \in [0,1]$ let $\mathcal{S}_n^*(\gamma,\beta)$ denote class of functions $f \in \mathcal{A}(n)$ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma} \prec h_\beta(z), \quad z \in \mathbb{D}, \tag{3.24}$$

or, equivalently,

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.25}$$

Remark 3.13. (1) The class $\mathcal{S}_n^*(\gamma,1)$, that is, the class of so-called *γ-starlike* functions was introduced by Lewandowski et al. [21].

(2) Clearly,

$$\begin{aligned}
&\mathcal{S}_1^*(0,1) = \mathcal{S}^*, \qquad \mathcal{S}_1^*(1,1) = \mathcal{C}, \\
&\mathcal{S}_1^*(0,\beta) = \mathcal{M}_1(0,\beta) = \mathcal{S}^*(\beta), \\
&\mathcal{S}_1^*(1,\beta) = \mathcal{M}_1(1,\beta) = \mathcal{C}^*(\beta).
\end{aligned} \tag{3.26}$$

Using Theorem 2.8 we obtain results due to Darus and Thomas [22].

Theorem 3.14. *Let $n \in \mathbb{N}$, $\gamma \in [0,1]$, and $\beta \in (0,1]$. If $f \in \mathcal{A}(n)$ and*

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < c_2(n,1,\gamma,\beta)\frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{3.27}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{3.28}$$

Corollary 3.15.

$$\begin{gathered}
\mathcal{S}_n^*(\gamma; c_2(n,1,\gamma,\beta)) \subset \mathcal{S}_n^*(\beta), \\
\mathcal{S}_1^*(\gamma,1) \subset \mathcal{S}^*.
\end{gathered} \tag{3.29}$$

As further applications of Theorems 2.6 and 2.8 we can use arbitrary well-defined functionals over the class $\mathcal{A}(n)$. We recall two examples:

(1)

$$p(z) = \frac{K_a^{\delta+1} f(z)}{z}, \quad z \in \mathbb{D},\ f \in \mathcal{A}(n),\ n \in \mathbb{N},\ a > 0,\ \delta \ge 0, \tag{3.30}$$

where the integral operator K_a^δ over the class $\mathcal{A}(n)$ was defined by Komatu [23] as follows:

$$K_a^\delta f(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\delta-1} f(zt)dt, \quad z \in \mathbb{D}, \tag{3.31}$$

where Γ is the Gamma function;

(2)

$$p(z) = \frac{L_\lambda f(z)}{z}, \quad z \in \mathbb{D}, f \in \mathcal{A},\ \lambda \geq -1, \tag{3.32}$$

where the operator L_λ over the class $\mathcal{A}$ called *Ruscheweyh derivative* [24] was defined as follows:

$$L_\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad z \in \mathbb{D}. \tag{3.33}$$

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