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Research Article

Padé Approximants in Complex Points Revisited

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The class of complexsymmetric functions contains the Stieltjes functions. The aim of this work is to give some new results concerning the location of zeros and poles of Padé approximants using the Taylor series of functions developed in neighborhoods of complex points and their conjugate points.

1. Introduction

In 1976, Chisholm et al. [1] published a paper concerning the location of poles and zeros of Padé approximants of $\ln(1-z)$ developed at the complex point ζ : $\ln(1-z)$ $=\ln(1-\zeta)-\sum_{n=1}^{\infty} 1/n(z-\zeta/1-\zeta)^n$. They claimed that all poles and zeros of diagonal Padé approximants $[n/n]$ interlace on the cut $z=\zeta+t(1-\zeta), t\in]1,\infty[$. Unfortunately, this result is only partially true, for poles. Klarsfeld remarked in 1981 [2] that the zeros do not follow this rule. The study of this problem was the starting motivation of the present work. We consider the general class of complexsymmetric functions f, that is, functions satisfying the following condition:

$$f(\overline{z})=\overline{f(z)}. \tag{1.1}$$

In particular, if ζ and $\overline{\zeta}$ are two complex conjugate points, then

$$f(z)=\sum_{n=0}^{\infty} c_n(z-\zeta)^n, \qquad f(z)=\sum_{n=0}^{\infty}\overline{c_n}\left(z-\overline{\zeta}\right)^n, \tag{1.2}$$

that is, the coefficients of these two series are also complex conjugates, This property is the basic element of all our proofs.

Let us introduce some definitions and notations.

The 1-point Padé approximant (PA), or simply Padé approximant $[m/n]$ to f at the point ζ is a rational function P_m/Q_n if and only if

$$f(z) - \frac{P_m(z)}{Q_n(z)} = O\left((z-\zeta)^{m+n+1}\right). \tag{1.3}$$

Because the existence of PA defined by (1.3) implies the invertibility of Q_n, the equivalent definition is

$$Q_n(z) f(z) - P_m(z) = O\left((z-\zeta)^{m+n+1}\right), \qquad Q_n(\zeta) = 1. \tag{1.4}$$

This definition leads to the following linear system for the coefficients of polynomials Q_n and P_m:

$$Q_n(z) = 1 + q_1(z-\zeta) + \cdots + q_n(z-\zeta)^n, \tag{1.5}$$

$$P_m(z) = p_0 + p_1(z-\zeta) + \cdots + p_m(z-\zeta)^m, \tag{1.6}$$

$$\begin{pmatrix} c_m & c_{m-1} & c_{m-2} & \cdots & c_{m-n+1} \\ c_{m+1} & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ c_{m+n-1} & c_{m+n-2} & c_{m+n-3} & \cdots & c_m \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix} = -\begin{pmatrix} c_{m+1} \\ c_{m+2} \\ \cdot \\ \cdot \\ \cdot \\ c_{m+n} \end{pmatrix}, \quad c_{-|k|} \equiv 0, \tag{1.7}$$

$$k = 0, 1, \ldots, m : -P_k + \sum_{j=1}^{n} c_{k-j} q_j = -c_k. \tag{1.8}$$

The PA exists iff the system (1.7) has a solution. The following so-called Padé form [3] defines PA if it exists (and other rational functions in the square blocs in the Padé table if (1.7) has no solution); for simplicity, it is written for the case $\zeta = 0$,

$$\frac{P_m(z)}{Q_n(z)} = \frac{\begin{vmatrix} C_{(m)}(z) & zC_{(m-1)}(z) & z^2C_{(m-2)}(z) & \ldots & z^nC_{(m-n)}(z) \\ c_{m+1} & c_m & c_{m-1} & \ldots & c_{m-n+1} \\ c_{m+2} & c_{m+1} & c_m & \ldots & c_{m-n+2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{m+n} & c_{m+n-1} & c_{m+n-2} & \ldots & c_m \end{vmatrix}}{\begin{vmatrix} 1 & z & z^2 & \ldots & z^n \\ c_{m+1} & c_m & c_{m-1} & \ldots & c_{m-n+1} \\ c_{m+2} & c_{m+1} & c_m & \ldots & c_{m-n+2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{m+n} & c_{m+n-1} & c_{m+n-2} & \ldots & c_m \end{vmatrix}}, \tag{1.9}$$

where

$$C_{(k)}(z) = \begin{cases} \sum_{j=0}^{k} c_j z^j & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases} \tag{1.10}$$

Let f be a function defined at the points $\zeta_1, \ldots, \zeta_N \in \mathbb{C}$ and having at these points the following power expansions:

$$i = 1, 2, \ldots, N : \sum_{k=0}^{p_i - 1} c_k(\zeta_i)(z - \zeta_i)^k + O\big((z - \zeta_i)^{p_i}\big). \tag{1.11}$$

An N-point Padé approximant (NPA) $[m/n]$ at the points $\zeta_1, \ldots, \zeta_N$ noted

$$\left[\frac{m}{n}\right]_{\zeta_1 \zeta_2 \cdots \zeta_N}^{p_1 p_2 \cdots p_N}(z) = \frac{P_m(z)}{Q_n(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{1 + b_1 z + \cdots + b_n z^n}, \tag{1.12}$$

where

$$p := p_1 + p_2 + \cdots + p_N = m + n + 1 \tag{1.13}$$

is defined by

$$i = 1, 2, \ldots, N : f(z) - \left[\frac{m}{n}\right]_{\zeta_1\ \zeta_2\ \cdots\ \zeta_N}^{p_1\ p_2\ \cdots\ p_N}(z) = O\big((z - \zeta_i)^{p_i}\big). \tag{1.14}$$

This leads to the following definition like (1.4):

$$i = 1, 2, \ldots, N : Q_n(z) f(z) - P_m(z) = O\big((z - \zeta_i)^{p_i}\big), \tag{1.15}$$

representing $m + n + 1$ linear equations.

2. Zeros and Poles of Padé Approximants of $\ln(1 - z)$

This function studied in [1] is related to the Stieltjes function

$$f(z) = \int_0^1 \frac{dx}{1 - xz} = -\frac{1}{z} \ln(1 - z) \tag{2.1}$$

defined in the cut-plane $\mathbb{C} \setminus [1, \infty[$. The zeros and poles of PA defined by a power series of f expanded at the real points interlace on the cut $]1, \infty[$. What does happen if PA is defined at the complex point ζ? Klarsfeld remarked [2] that Chisholm result [1] is wrong and showed that poles of the PA of $\ln(1-z)$ follow the cut $z = \zeta + t(1-\zeta), t \geq 1$, as mentioned in the introduction, but not the zeros. We generalize this result to all $m \geq n$. For convenience, let us introduce the following notations:

$$\text{if } \zeta = 0: \quad f(z) = \ln(1-z) = \sum_{n=0}^{\infty} c_n z^n = -\sum_{n=1}^{\infty} \frac{1}{n} z^n \tag{2.2}$$

$$\text{if } \zeta \neq 0: f(z) = \ln(1-\zeta) + \ln\left(1 - \frac{z-\zeta}{1-\zeta}\right) = \sum_{n=0}^{\infty} c_n^* (z-\zeta)^n = \ln(1-\zeta) - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z-\zeta}{1-\zeta}\right)^n, \tag{2.3}$$

then we have

$$c_0^* = \ln(1-\zeta), \quad n \geq 1 : c_n^* = \frac{c_n}{(1-\zeta)^n}. \tag{2.4}$$

Theorem 2.1. *Let $[m/n]$ and $[m/n]^*$, $m \geq n$, be the Padé approximants of the function $f(z) = \ln(1-z)$ developed at the points $z = 0$ and $z = \zeta$, respectively, then if $z_k (k = 1, 2, \ldots, n)$ denotes a pole of $[m/n]$, then*

$$z_k^* = \zeta + z_k (1-\zeta) \tag{2.5}$$

denotes the pole of $[m/n]^*$. In other words, the poles of $[m/n]^*$ locate on the cut

$$z = \zeta + t(1-\zeta), \quad t \geq 1, \tag{2.6}$$

directed by the straight line joining the point of development of $f : z = \zeta$ with the branch point $z = 1$. The zeros of $[m/n]^*$ locate out of this line.

Proof. We can readily verify this theorem looking at the formula (1.9) and considering the numerator P_m^* and the denominator Q_n^* of $[m/n]^*$ and the relation (2.4). The denominator Q_n^* expressed in the variable $(z-\zeta)/(1-\zeta)$ has the same coefficients as Q_n of $[m/n]$ if c_0^* do not occur in its definition, that is, if $m - n + 1 > 0$ and if $m \geq n$. More exactly, we have

$$Q_n^*(z) = \frac{1}{(1-\zeta)^{mn}} Q_n\left(\frac{z-\zeta}{1-\zeta}\right). \tag{2.7}$$

That is, in this case, the poles of $[m/n]^*$ follow the way of poles of $[m/n]$ rotated by some angle around the branch point which gives (2.5) and (2.6). On the contrary, the definition of P_m^* always contains c_0^*, and then the zeros of $[m/n]^*$ locate out of (2.6). □

The location of the zeros of $[m/n]^*$ remains an open problem. However, the following remarks can unlock, may be, this question. The particular case of Gilewicz theorem [3, page 217] says that if $[n/n]_f$ is a PA of some function f and α a constant, then

$$\left[\frac{n}{n}\right]_f + \alpha = \left[\frac{n}{n}\right]_{f+\alpha}. \tag{2.8}$$

This readily leads to the following theorem, where all notations are the same as in Theorem 2.1 except c_0^* which is replaced by an arbitrary constant α.

Theorem 2.2. *Let* $f(z) = \ln(1-z) - \ln(1-\zeta) = \ln(1-z/1-\zeta)$ *and*

$$\left[\frac{n}{n}\right]_f (z) = \frac{P_n(z-\zeta)/(1-\zeta)}{Q_n(z-\zeta)/(1-\zeta)} = \frac{p_0 + p_1(z-\zeta)/(1-\zeta) + \cdots + p_n(z-\zeta/1-\zeta)^n}{1 + q_1(z-\zeta/1-\zeta) + \cdots + q_n(z-\zeta/1-\zeta)^n}, \tag{2.9}$$

then

$$\left[\frac{n}{n}\right]^*(z) = \left[\frac{n}{n}\right]_{f+\alpha}(z) = \frac{P_n^*(z-\zeta)/(1-\zeta)}{Q_n^*(z-\zeta)/(1-\zeta)} = \frac{\alpha Q_n(z-\zeta)/(1-\zeta) + P_n(z-\zeta)/(1-\zeta)}{Q_n(z-\zeta)/(1-\zeta)}. \tag{2.10}$$

If $\alpha = 0$, *then the poles and also the zeros simulate the cut* $\zeta + t(1-\zeta)$, $t \geq 0$. *The problem consists to analyze the behavior of the zeros of* P_n^* *as a function of* α *with* $\alpha \in [0,\ c_0^* = \ln(1-\zeta)]$.

3. Zeros and Poles of Padé Approximants at Complex Conjugate Points

In this section, $[m/n]_f(z-\zeta)$ and $[m/n]_f^*(z-\overline{\zeta})$ denote the Padé approximants of a complexsymmetric function f at the point ζ and its complex conjugate $\overline{\zeta}$, respectively.

Theorem 3.1. *Let* $[m/n]$ *and* $[m/n]^*$ *be Padé approximants of a complexsymmetric function* f *at* ζ *and* $\overline{\zeta}$, *then the zeros and the poles of* $[m/n]$ *are complex conjugates of the corresponding zeros and poles of* $[m/n]^*$.

Proof. Equation (1.1) gives

$$f(z) = \sum_{i=0}^{\infty} c_i (z-\zeta)^i = \sum_{i=0}^{\infty} \overline{c}_i \left(z-\overline{\zeta}\right)^i. \tag{3.1}$$

Now, $[m/n](z-\zeta) = P_m/Q_n$, $Q_n = 1 + q_1(z-\zeta) + \cdots + q_n(z-\zeta)^n$, where q_i are defined by the linear system (1.7). We identify the solutions $\overline{q}_i$ as coefficients of Q_n^* due to (3.1): $Q_n^* = 1+\overline{q_1}(z-\overline{\zeta})+\cdots+\overline{q_n}(z-\overline{\zeta})^n$. Then, the zeros of Q_n^* (poles of $[m/n]^*$) are the complex conjugates of those of Q_n. The same arguments are used for the system (1.8) and for P_m and P_m^* which completes the proof. ∎

4. N-Point Padé Approximants of the Complexsymmetric Functions

The following theorem is the consequence of Theorem 3.1.

Theorem 4.1. *Let f be a complexsymmetric function, then the zeros and poles of 2-point Padé approximant $[m/n]_{\zeta\ \overline{\zeta}}^{l\ \ l'}$ of f are complex conjugates of the zeros and poles of $[m/n]_{\zeta\ \overline{\zeta}}^{l'\ \ l}$, where l and l' are the arbitrary integers satisfying the condition $l + l' = m + n + 1$.*

Proof. In this case, the NPA is defined by two linear systems like (1.7) and (1.8) leading from

$$fQ_n - P_m = O\left((z-\zeta)^l\right), \qquad fQ_n - P_m = O\left(\left(z-\overline{\zeta}\right)^{l'}\right), \tag{4.1}$$

and for the second NPA leading from

$$fQ_n - P_m = O\left((z-\zeta)^{l'}\right), \qquad fQ_n - P_m = O\left(\left(z-\overline{\zeta}\right)^{l}\right). \tag{4.2}$$

The same arguments as those used in the proof of Theorem 3.1 can be used to transform the system (4.1) to (4.2), and then to obtain the result of Theorem 4.1 on the basis of Theorem 3.1. □

Corollary 4.2. *Let f be a complexsymmetric function, then the zeros and poles of NPA $[m/n]_{\zeta_1\ \zeta_2 \cdots \zeta_k\ \overline{\zeta}_1\ \overline{\zeta}_2\ \cdots\ \overline{\zeta}_k}^{l_1\ l_2\ \cdots\ l_k\ l'_1\ l'_2\ \cdots\ l'_k}$ of f are complex conjugates of the zeros and poles of $[m/n]_{\zeta_1\ \zeta_2 \cdots \zeta_k\ \overline{\zeta}_1\ \overline{\zeta}_2 \cdots\ \overline{\zeta}_k}^{l'_1\ l'_2\ \cdots\ l'_k\ l_1\ l_2\ \cdots\ l_k}$, where $l_1 + l_2 + \cdots + l_k + l'_1 + l'_2 + \cdots + l'_k = m + n + 1$.*

We also prove the following

Theorem 4.3. *Let f be a complexsymmetric function, then all coefficients of N-point Padé approximant $[m/n]_{\zeta_1\ \cdots\ \zeta_k\ \overline{\zeta}_1 \cdots\ \overline{\zeta}_k}^{l_1\ \cdots\ l_k\ l_1\ \cdots\ l_k}$ of f are real.*

Proof. In the last corollary, all $l_i = l'_i$, then two NPA are equal, and then, because the coefficients of the first NPA are complex conjugates of the coefficients of the second NPA, they are real. Unfortunately, it is not true that all the zeros and poles of these NPA are real. □

Unfortunately, it is not true that all the zeros and poles of these NPA are real.

5. Conclusion

In many numerical experiments with Padé approximants or N-point Padé approximants at the complex points, we are not able to detect any clear regularity related to the location of poles and zeros. However, it seems that their positions follow some well-defined corridors. In the present time, many general problems in this field remain open.

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