

Research Article

The Generalized Janowski Starlike and Close-to-Starlike Log-Harmonic Mappings

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Motivated by the success of the Janowski starlike function, we consider here closely related functions for log-harmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ defined on the open unit disc U . The functions are in the class of the generalized Janowski starlike log-harmonic mapping, $S_{lh}^*(A, B, \alpha)$, with the functional $zh(z)$ in the class of the generalized Janowski starlike functions, $S^*(A, B, \alpha)$. By means of these functions, we obtained results on the generalized Janowski close-to-starlike log-harmonic mappings, $CST_{lh}(A, B, \alpha)$.

1. Introduction

The class $S^*(A, B)$ was investigated by Janowski [1] in early 1970, and since then various other subclasses in relation with this Janowski class have been introduced and studied. In that direction, the log-harmonic mappings which have been studied extensively for the past 3 decades, (see [2–10]) were also associated with the Janowski class. Perhaps, the Janowski starlike log-harmonic univalent functions were first introduced by Polatoğlu and Deniz [11].

A function f is said to be log-harmonic on the open unit disc $U = \{z : |z| < 1\}$ if it satisfies the nonlinear elliptic partial differential equation:

$$\frac{\overline{f_z}}{f} = a \frac{f_z}{f}, \quad (1.1)$$

where the second dilatation function $a \in \mathcal{H}(U)$ (set of all analytic functions defined on U) such that $|a(z)| < 1$ for all $z \in U$. For analytic functions h and g in U , the function f can be expressed as

$$f(z) = h(z)\overline{g(z)} \quad (1.2)$$

if f is a nonvanishing log-harmonic mapping and

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \quad (1.3)$$

if f vanishes at $z = 0$ but is not identically zero (for $\operatorname{Re} \beta > -1/2$, $g(0) = 1$, and $h(0) \neq 0$).

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent log-harmonic mapping, where $0 \notin f(U)$ or equivalently $0 \notin hg(U)$. Then f is starlike log-harmonic mapping if

$$\operatorname{Re} \left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) > 0. \quad (1.4)$$

Results on starlike log-harmonic mapping of order α was given in [6].

Motivated by [11], the class of the generalized Janowski log-harmonic starlike functions was introduced in [12]. For real numbers A and B , with $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, the family of analytic functions of the form

$$p(z) = 1 + p_1z + p_2z^2 + \dots \quad (1.5)$$

is in $P(A, B, \alpha)$ if and only if

$$p(z) = \frac{1 + [(1 - \alpha)A + \alpha B]\phi(z)}{1 + B\phi(z)}, \quad (1.6)$$

where the function ϕ is analytic in U with $\phi(0) = 0$ and $|\phi(z)| < 1$. The following lemma is also essential for $p(z)$ to be in $P(A, B, \alpha)$.

Lemma 1.1 (see [13]). *The function $p(z) \in P(A, B, \alpha)$ if and only if*

$$\left| p(z) - \frac{1 - [(1 - \alpha)A + \alpha B]Br^2}{1 - B^2r^2} \right| \leq \frac{(1 - \alpha)(A - B)r}{1 - B^2r^2} \quad (1.7)$$

for $|z| \leq r < 1$.

Let $S^*(A, B, \alpha)$ denote the class of the generalized Janowski starlike functions of the analytic functions $s(z) = z + s_2z^2 + \dots$ such that $s(z) \in S^*(A, B, \alpha)$ if and only if

$$\frac{zs'(z)}{s(z)} = p(z) \quad (1.8)$$

and $p(z) \in P(A, B, \alpha)$ for $z \in U$.

For univalent log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ with $g(0) = 1$ and $h(0) \neq 0$, f is in the class of the generalized Janowski starlike log-harmonic mapping denoted by $S_{\text{lh}}^*(A, B, \alpha)$ if

$$\left| p(z) - \frac{1 - [(1 - \alpha)A + \alpha B]Br^2}{1 - B^2r^2} \right| \leq \frac{(1 - \alpha)(A - B)r}{1 - B^2r^2}, \quad (1.9)$$

where

$$p(z) = \frac{h(z)g(z) + zh'(z)g(z) - zg'(z)h(z)}{h(z)g(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}. \quad (1.10)$$

Also observe that if $f \in S_{\text{lh}}^*(A, B, \alpha)$, then

$$\operatorname{Re} \left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) \geq \frac{1 - [(1 - \alpha)A + \alpha B]}{1 - B}. \quad (1.11)$$

In the present work, we consider the log-harmonic mapping $f(z) = zh(z)\overline{g(z)}$ in the generalized Janowski starlike functions with the functional $zh(z) \in S^*(A, B, \alpha)$. We also study the class of generalized Janowski close-to-starlike in the next section.

2. The Generalized Janowski Starlike Log-Harmonic

Theorem 2.1. *If $zh(z) \in S^*(A, B, \alpha)$, then*

$$\begin{aligned} (1 - Br)^{(1-\alpha)(A-B)/B} \leq |h(z)| \leq (1 + Br)^{(1-\alpha)(A-B)/B} \quad \text{for } B \neq 0, \\ e^{-(1-\alpha)Ar} \leq |h(z)| \leq e^{(1-\alpha)Ar} \quad \text{for } B = 0. \end{aligned} \quad (2.1)$$

Proof. Since $zh(z) \in S^*(A, B, \alpha)$, Lemma 1.1 yields that for $B \neq 0$ we have

$$\frac{1 - [(1 - \alpha)A + \alpha B]r}{1 - Br} \leq \operatorname{Re} \left(\frac{z(zh(z))'}{zh(z)} \right) \leq \frac{1 + [(1 - \alpha)A + \alpha B]r}{1 + Br} \quad (2.2)$$

or

$$\frac{-(1 - \alpha)(A - B)r}{1 - Br} \leq \operatorname{Re} \left(\frac{zh'(z)}{h(z)} \right) \leq \frac{(1 - \alpha)(A - B)r}{1 + Br}. \quad (2.3)$$

Simple calculations yield

$$\frac{-(1 - \alpha)(A - B)}{-B} \log(1 - Br) \leq \log|h(z)| \leq \frac{(1 - \alpha)(A - B)}{B} \log(1 + Br), \quad (2.4)$$

and the result follows immediately.

For $B = 0$, Lemma 1.1 yields

$$1 - (1 - \alpha)Ar \leq \operatorname{Re} \left(\frac{z(zh(z))'}{zh(z)} \right) \leq 1 + (1 - \alpha)Ar, \quad (2.5)$$

and the proof is completed similarly. \square

Theorem 2.2. Let $f(z) = zh(z)\overline{g(z)} \in S_{\text{lh}}^*(A, B, \alpha)$ with $zh(z) \in S^*(A, B, \alpha)$. Then one has

$$\begin{aligned} \frac{(1 - Br)^{(1-\alpha)(A-B)/B}}{(1 + Br)^{(1-\alpha)(A-B)/B}} \leq |g(z)| &\leq \frac{(1 + Br)^{(1-\alpha)(A-B)/B}}{(1 - Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \\ e^{-2(1-\alpha)Ar} \leq |g(z)| &\leq e^{2(1-\alpha)Ar} \quad \text{for } B = 0. \end{aligned} \quad (2.6)$$

Proof. It follows from [12] that for $f(z) = zh(z)\overline{g(z)} \in S_{\text{lh}}^*(A, B, \alpha)$, we have

$$\begin{aligned} (1 - Br)^{(1-\alpha)(A-B)/B} \leq \left| \frac{h(z)}{g(z)} \right| &\leq (1 + Br)^{(1-\alpha)(A-B)/B} \quad \text{for } B \neq 0, \\ e^{-(1-\alpha)Ar} \leq \left| \frac{h(z)}{g(z)} \right| &\leq e^{(1-\alpha)Ar} \quad \text{for } B = 0. \end{aligned} \quad (2.7)$$

With these inequalities and Theorem 2.1, we can conclude the following statement. \square

Theorem 2.3. Let $f(z) = zh(z)\overline{g(z)} \in S_{\text{lh}}^*(A, B, \alpha)$ with $zh(z) \in S^*(A, B, \alpha)$. Then one has

$$\begin{aligned} \frac{r(1 - Br)^{2(1-\alpha)(A-B)/B}}{(1 + Br)^{(1-\alpha)(A-B)/B}} \leq |f(z)| &\leq \frac{r(1 + Br)^{2(1-\alpha)(A-B)/B}}{(1 - Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \\ re^{-3(1-\alpha)Ar} \leq |f(z)| &\leq re^{3(1-\alpha)Ar} \quad \text{for } B = 0. \end{aligned} \quad (2.8)$$

Proof. For $f(z) = zh(z)\overline{g(z)}$ and $|z| = r$, it is easy to see that

$$|f(z)| = |zh(z)\overline{g(z)}| = |z||h(z)||\overline{g(z)}| = r|h(z)||g(z)|. \quad (2.9)$$

Thus, we can obtain the results from Theorems 2.1 and 2.2. \square

3. The Generalized Janowski Close-to-Starlike Log-Harmonic

Let P_{lh} be mapping the set of all log-harmonic mappings, and let R be defined on U which are of the form $R(z) = K(z)\overline{J(z)}$, where K and J are in $\mathcal{H}(U)$, $K(0) = J(0) = 1$ and such that $\operatorname{Re} R(z) > 0$ for all $z \in U$. These log-harmonic mappings with positive real part were studied in [5]. Other interesting studies in the same paper were on the close-to starlike log-harmonic mappings. The author then extended the results to close-to starlike of order α log-harmonic mappings [2].

In that direction, we say that $F(z) = zH(z)\overline{G(z)}$ is the generalized Janowski close-to-starlike log-harmonic mapping if there exist a log-harmonic mapping $f(z) = zh(z)\overline{g(z)} \in ST_{lh}^*(A, B, \alpha)$ ($-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$), with respect to the second dilatation function $a \in \mathcal{H}(U)$ and a log-harmonic mapping with positive real part $R \in P_{lh}$ where its second dilatation function is the same such that

$$F(z) = f(z)R(z) \tag{3.1}$$

or equivalently

$$\operatorname{Re} \frac{F(z)}{f(z)} > 0. \tag{3.2}$$

We could also easily derive from (3.1) that

$$\operatorname{Re} \left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F} \right) = \operatorname{Re} \left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) + \operatorname{Re} \left(\frac{zR_z - \bar{z}R_{\bar{z}}}{R} \right). \tag{3.3}$$

The geometrical interpretation is that under a generalized Janowski close-to-starlike log-harmonic mapping, the radius vector of the image of $|z| = r < 1$ never turns back by the amount more than $((1 - \alpha)(A - B)/(1 - B))\pi$. As special cases, we see that

- (i) for $\alpha = 0$ or under the Janowski close-to-starlike log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than $((A - B)/(1 - B))\pi$,
- (ii) for when $A = 1, B = -1$ or under the close-to-starlike of order α log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than $(1 - \alpha)\pi$,
- (iii) for $\alpha = 0, A = 1, B = -1$ or under the close-to-starlike log-harmonic mappings, the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than π .

The following theorem gives us the radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{lh}(A, B, \alpha)$.

Theorem 3.1. *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{lh}(A, B, \alpha)$ is the largest positive root, $r \in (0, 1]$, such that*

$$(1 - [(1 - \alpha)A + \alpha B]r)(1 - r)(1 + r) - 2r(1 - Br) > 0. \tag{3.4}$$

Proof. For $F(z) = zH(z)\overline{G(z)} \in CST_{lh}(A, B, \alpha)$, we have

$$\operatorname{Re} \left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F} \right) = \operatorname{Re} \left(\frac{zf_z - \bar{z}f_{\bar{z}}}{f} \right) + \operatorname{Re} \left(\frac{zR_z - \bar{z}R_{\bar{z}}}{R} \right), \tag{3.5}$$

and since $f \in S_{\text{lh}}^*(A, B, \alpha)$ and $R \in P_{\text{lh}}$, (3.5) becomes

$$\operatorname{Re}\left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F}\right) \geq \frac{1 - [(1 - \alpha)A + \alpha B]r}{1 - Br} + \frac{-2r}{1 - r^2}. \quad (3.6)$$

Hence,

$$\operatorname{Re}\left(\frac{zF_z - \bar{z}F_{\bar{z}}}{F}\right) > 0 \quad (3.7)$$

if

$$\frac{1 - [(1 - \alpha)A + \alpha B]r}{1 - Br} - \frac{2r}{1 - r^2} > 0. \quad (3.8)$$

□

Corollary 3.2 (see [2]). *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\text{lh}}$ is*

$$r < 2 - \sqrt{3}. \quad (3.9)$$

Corollary 3.3 (see [2]). *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\text{lh}}(\alpha)$ is*

$$r < \frac{2 - \alpha - \sqrt{\alpha^2 - 2\alpha + 3}}{1 - 2\alpha}. \quad (3.10)$$

Corollary 3.4. *The radius of starlikeness for $F(z) = zH(z)\overline{G(z)} \in CST_{\text{lh}}(A, B)$ is the largest positive root, $r \in (0, 1]$, such that*

$$(1 - Ar)(1 - r)(1 + r) - 2r(1 - Br) > 0. \quad (3.11)$$

Proof. The proof is completed by taking $\alpha = 0$ in (3.4). □

We need the following theorem from [5] to prove our next result.

Theorem

Let $R(z) \in P_{\text{lh}}$, and suppose that $a(0) = 0$. Then, for $z \in U$, we have

$$e^{-2|z|/(1-|z|)} \leq |R(z)| \leq e^{2|z|/(1-|z|)}. \quad (3.12)$$

Theorem 3.5. For $F(z) = zH(z)\overline{G(z)} \in CST_{lh}(A, B, \alpha)$ and $f(z) = zh(z)\overline{g(z)}$ with $zh(z) \in S^*(A, B, \alpha)$, one has

$$\frac{r(1-Br)^{2(1-\alpha)(A-B)/B} e^{-2r/(1-r)}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |F(z)| \leq \frac{r(1-Br)^{2(1-\alpha)(A-B)/B} e^{2r/(1-r)}}{(1+Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \quad (3.13)$$

$$re^{-3(1-\alpha)Ar-(2r/(1-r))} \leq |F(z)| \leq re^{3(1-\alpha)Ar-(2r/(1-r))} \quad \text{for } B = 0.$$

Proof. From (3.12) and Theorem 2.3, we have

$$e^{-2r/(1-r)} \leq |R(z)|e^{2r/(1-r)}, \quad |z| = r < 1,$$

$$\frac{r(1-Br)^{2(1-\alpha)(A-B)/B}}{(1+Br)^{(1-\alpha)(A-B)/B}} \leq |f(z)| \leq \frac{r(1+Br)^{2(1-\alpha)(A-B)/B}}{(1-Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0, \quad (3.14)$$

$$re^{-3(1-\alpha)Ar} \leq |f(z)| \leq re^{3(1-\alpha)Ar} \quad \text{for } B = 0,$$

respectively. Also, we know that for $F \in CST_{lh}(A, B, \alpha)$, we have $F(z) = f(z)R(z)$ which then leads to the desired result. \square

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