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Research Article

New Exact Solutions for an Oldroyd-B Fluid in a Porous Medium

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New exact solutions for unsteady magnetohydrodynamic (MHD) flows of an Oldroyd-B fluid have been derived. The Oldroyd-B fluid saturates the porous space. Two different flow cases have been considered. The analytical expressions for velocity and shear stress fields have been obtained by using Laplace transform technique. The corresponding solutions for hydrodynamic Oldroyd-B fluid in a nonporous space appeared as the limiting cases of the obtained solutions. Similar solutions for MHD Newtonian fluid passing through a porous space are also recovered. Graphs are sketched for the pertinent parameters. It is found that the MHD and porosity parameters have strong influence on velocity and shear stress fields.

1. Introduction

The behavior of viscoelastic materials in particularly the response of many polymeric liquids are best described by Oldroyd-B fluid model [1]. Further, these, fluids are also quite useful in chemical and process industry due to the fact that they encounter both the memory and elastic effects exhibited by most polymers and biological liquids. In early 1970, Waters and King [2, 3] have obtained first time the exact solutions to these fluids using Laplace transform technique. Later, several authors [4–20] have discussed the different motions of Oldroyd-B fluids in different geometries using various approaches for the solution.

Recently, the concept of studing the fluid flows through porous medium has gained much attention largely due to their several technological and industrial applications such as geothermal energy extrusion, oil recovery, food processing, ground water flow, irrigation problems, and the biophysical sciences where the human lungs, for example, are modeled as

a porous layer. The literature survey revealed that very few attempts are available in which the flows of electrically conducting Oldroyd-B fluid through a porous medium are studied [21–26]. Therefore, the basic objective of the present paper is to study the motion of Oldroyd-B fluid taking into account both magnetic and porosity effects.

To the best of authors knowledge so far no study has been reported in which the MHD flow of an Oldroyd-B fluid passing through a porous medium has been considered for the following two cases: (i) flow due to impulsive motion of the plate and (ii) flow due to uniform motion of the plate. Therefore the main objective of this paper is to make such an attempt. The rest of the paper is arranged as follows. The mathematical formulation of the problem is given in Section 2. Section 3 comprises the solution of the problem. The graphical results are displayed and discussed in the last section.

2. Problem Formulation

For the MHD flow through a porous medium, the continuity and momentum equations are given by (Tan and Masuoka [22])

$$\operatorname{div} \mathbf{V} = 0, \tag{2.1}$$

$$\rho\left(\frac{d\mathbf{V}}{dt}\right) = \operatorname{div} \mathbf{T} + \mathbf{J} \times \mathbf{B} - \frac{\mu\phi}{k}\left(1 + \lambda_r \frac{\partial}{\partial t}\right)\mathbf{V}, \tag{2.2}$$

where $\mathbf{V} = (u, v, w)$ denotes the velocity vector, ρ is the fluid density, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the total magnetic field, $\mathbf{T}$ is the Cauchy stress tensor, d/dt is the material time derivative, ϕ $(0 < \phi < 1)$ is the porosity, and $k > 0$ is the permeability of the porous medium.

The Cauchy stress tensor $\mathbf{T}$ for an incompressible Oldroyd-B fluid is given by [1]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{2.3}$$

in which the extra stress tensor $\mathbf{S}$ satisfies

$$\mathbf{S} + \lambda\left[\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T\right] = \mu\left[\mathbf{A}_1 + \lambda_r\left(\frac{\partial \mathbf{A}_1}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{A}_1 - \mathbf{L}\mathbf{A}_1 - \mathbf{A}_1\mathbf{L}^T\right)\right], \tag{2.4}$$

where $-p\mathbf{I}$ is the spherical part of the stress due to the constraint of incompressibility, μ is the dynamic viscosity, $\mathbf{L}$ is the velocity gradient, $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, and λ and $\lambda_r (< \lambda)$ are, respectively, the relaxation and retardation times.

The second term on the right side of (2.2) is called the Lorentz force and can be written as

$$\mathbf{J} \times \mathbf{B} = \left(-\sigma B_0^2 u, 0, 0\right), \tag{2.5}$$

in which B_0 is the magnitude of a uniform magnetic field $\mathbf{B}_0$ applied in a direction normal to the fluid motion and σ is the electrical conductivity of the fluid.

By taking the velocity field of the form

$$\mathbf{V} = (u(y,t), 0, 0), \tag{2.6}$$

the continuity equation is automatically satisfied and the momentum equation in the absence of external pressure gradient, together with (2.3)–(2.5) having in mind the initial condition

$$\mathbf{S}(y,0) = \mathbf{0}, \tag{2.7}$$

finally gives

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} &= \left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial T(y,t)}{\partial t} - \left(1+\lambda\frac{\partial}{\partial t}\right)\sigma B_0^2 u - \frac{\mu\phi}{k}\left(1+\lambda_r\frac{\partial}{\partial t}\right)u \\ &\quad + \left(1+\lambda\frac{\partial}{\partial t}\right)T(y,t) \\ &= \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t},\end{aligned} \tag{2.8}$$

where $T(y,t) = S_{xy}(y,t)$ is the tangential stress.

3. Solution of the Problem

3.1. Flow due to Impulsive Motion of the Plate

Consider the unsteady flow of an incompressible Oldroyd-B fluid occupying the upper porous half-space of (x,y) plane. The fluid is bounded by a rigid plate at $y = 0$ such that the positive Y-axis is taken normal to the plate and X-axis is taken parallel to the plate. The Oldroyd-B fluid is assumed to be electrically conducting under the influence of a uniform magnetic field $\mathbf{B}_0$ applied in a direction normal to the flow. Initially, both the fluid and the plate are at rest. At time $t = 0^+$, the plate is suddenly jerked and the motion in the fluid is induced in the direction parallel to X-axis. Under these assumptions the flow is governed by (2.8) along with the following initial and boundary conditions:

$$u(y,0) = 0, \qquad T(y,0) = 0, \quad y > 0, \tag{3.1}$$

$$u(0,t) = U_0, \quad t > 0. \tag{3.2}$$

Furthermore, the natural conditions

$$u(y,t), T(y,t) \longrightarrow 0, \quad \text{as } y \longrightarrow \infty, \tag{3.3}$$

have to be also satisfied.

Using the following dimensionless variables [26]:

$$\tau = \frac{t}{\lambda}, \qquad \xi = \frac{y}{c\lambda}, \qquad U = \frac{u}{U_0}, \qquad S = \frac{T}{\rho c U_0}, \tag{3.4}$$

the governing (2.8) take the following form

$$\begin{gathered} \frac{\partial U}{\partial \tau} = \frac{\partial S}{\partial \xi} - M^2 U - \frac{1}{K}\left(1 + \alpha \frac{\partial}{\partial \tau}\right) U, \\ \left(1 + \frac{\partial}{\partial \tau}\right) S = \left(1 + \alpha \frac{\partial}{\partial \tau}\right) \frac{\partial U}{\partial \xi}. \end{gathered} \tag{3.5}$$

The corresponding initial and boundary conditions are

$$U(\xi, 0) = 0, \qquad S(\xi, 0) = 0, \tag{3.6}$$

$$U(0, \tau) = 1, \quad \tau > 0, \tag{3.7}$$

$$U(\xi, \tau),\ S(\xi, \tau) \longrightarrow 0, \quad \text{as } \xi \longrightarrow \infty, \tag{3.8}$$

where

$$M^2 = \frac{\sigma B_0^2 \lambda}{\rho}, \qquad \frac{1}{K} = \frac{\mu \phi \lambda}{\rho k}, \qquad \alpha = \frac{\lambda_r}{\lambda}, \qquad c = \sqrt{\frac{\mu}{\rho \lambda}}. \tag{3.9}$$

Applying Laplace transform to (3.5), using the initial conditions (3.6), we attain the following system in the transformed q-plane:

$$\begin{gathered} \frac{\partial \overline{S}(\xi, q)}{\partial \xi} = \left(M^2 + \frac{1}{K}\right) \overline{U}(\xi, q) + \left(\frac{\alpha}{K} + 1\right) q \overline{U}(\xi, q), \\ (1 + q) \overline{S}(\xi, q) = (1 + \alpha q) \frac{\partial \overline{U}(\xi, q)}{\partial \xi}, \end{gathered} \tag{3.10}$$

where

$$\overline{U}(0, q) = \frac{1}{q}; \qquad \overline{U}(\xi, q), \overline{S}(\xi, q) \longrightarrow 0, \quad \text{as } \xi \longrightarrow 0. \tag{3.11}$$

Eliminating $\overline{S}(\xi, q)$ between (3.10) and solving the resulting equation for $\overline{U}(\xi, q)$ using conditions (3.11), we get

$$\overline{U}(\xi, q) = \frac{1}{q} \exp\left(-\frac{\xi \sqrt{a3}}{\sqrt{\alpha}} \sqrt{\frac{q^2 + a_4 q + a_5}{q + \alpha_0}}\right), \tag{3.12}$$

or

$$\overline{U}(\xi,q)=\frac{1}{q}\exp\left(-\frac{\xi\sqrt{a3}\sqrt{w(q)}}{\sqrt{\alpha}}\right)\quad \text{where } w(q)=\frac{q^2+a_4q+a_5}{q+\alpha_0}. \tag{3.13}$$

From (3.10), it is easy to write

$$\overline{S}(\xi,q)=-\sqrt{a_3\alpha}\frac{q^2+a_4q+a_5}{q^2+q}\frac{\exp\left(-\left(\xi\sqrt{a3}/\sqrt{\alpha}\right)\sqrt{(q^2+a_4q+a_5)/(q+\alpha_0)}\right)}{\sqrt{(q^2+a_4q+a_5)/(q+\alpha_0)}} \tag{3.14}$$

or equivalently

$$\overline{S}(\xi,q)=-\sqrt{a_3\alpha}\frac{q^2+a_4q+a_5}{q^2+q}\frac{\exp\left(-\left(\xi\sqrt{a3}/\sqrt{\alpha}\right)\sqrt{w(q)}\right)}{\sqrt{w(q)}}. \tag{3.15}$$

Now in order to determine $U(\xi,\tau)=\mathcal{L}^{-1}\{\overline{U}(\xi,q)\}$, we are writing (3.13) as follows:

$$\overline{U}(\xi,q)=\overline{u}_1(q)\overline{u}_2(\xi,q), \tag{3.16}$$

where

$$\overline{u}_1(q)=\frac{1}{q},\qquad \overline{u}_2(\xi,q)=\exp\left(-\frac{\xi\sqrt{a3}}{\sqrt{\alpha}}\sqrt{w(q)}\right). \tag{3.17}$$

If we denote $u_1(\tau)=\mathcal{L}^{-1}\{\overline{u}_1(q)\}$ and $u_2(\xi,\tau)=\mathcal{L}^{-1}\{\overline{u}_2(\xi,q)\}$, then it is easy to write

$$u_1(\tau)=1, \tag{3.18}$$

$$\begin{aligned} u_2(\xi,\tau)=&\frac{\xi\sqrt{a_3}}{2\sqrt{\alpha\pi}}\int_0^\infty\frac{\delta(\tau-u)}{u\sqrt{u}}\exp\left(-\frac{\xi^2a_3}{4\alpha u}-a_6u\right)\mathrm{d}u\\ &-\frac{\xi\sqrt{a_3a_7}}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^\infty\frac{\delta(\tau-s-u)}{u\sqrt{s}}\exp\left(-\frac{\xi^2a_3}{4\alpha u}-\alpha_0s-a_6u\right)\\ &\times J_1\left(2\sqrt{a_7us}\right)\mathrm{d}u\ \mathrm{d}s. \end{aligned} \tag{3.19}$$

Using the convolution formula

$$U(\xi,\tau)=\int_0^\tau u_1(\tau-s)u_2(\xi,s)\mathrm{d}s. \tag{3.20}$$

Equations (3.18) and (3.19) give

$$\begin{aligned}
U(\xi,\tau) = {} & \frac{\xi\sqrt{a_3}}{2\sqrt{\alpha\pi}}\int_0^\tau \frac{1}{s\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s\right)\mathrm{d}s \\
& -\frac{\xi\sqrt{a_3a_7}}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^w \frac{1}{(s-w)\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)\right) \\
& \times J_1\left(2\sqrt{a_7(s-w)w}\right)\mathrm{d}w\,\mathrm{d}s.
\end{aligned} \tag{3.21}$$

Similarly the Laplace inverse of (3.15) is derived as

$$\begin{aligned}
S(\xi,\tau) = {} & -\frac{\sqrt{a_3\alpha}}{\sqrt{\pi\tau}}\exp\left(-\frac{a_3\xi^2}{4\alpha\tau}-a_6\tau\right) \\
& +\frac{\sqrt{a_3a_7\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha(\tau-s)}-\alpha_0 s-a_6(\tau-s)\right) \\
& \times J_1\left(2\sqrt{a_7 s(\tau-s)}\right)\mathrm{d}s-\frac{a_5\sqrt{a_3\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s\right)\mathrm{d}s \\
& +\frac{a_5\sqrt{a_7a_3\alpha}}{\sqrt{\pi}}\int_0^\tau\int_0^w \frac{1}{\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)\right) \\
& \times J_1\left(2\sqrt{a_7 w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s-\frac{a_9\sqrt{a_3\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s-(\tau-s)\right)\mathrm{d}s \\
& +\frac{a_9\sqrt{a_7a_3\alpha}}{\sqrt{\pi}}\int_0^\tau\int_0^w \frac{1}{\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)-(\tau-s)\right) \\
& \times J_1\left(2\sqrt{a_7 w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s,
\end{aligned} \tag{3.22}$$

where

$$\begin{gathered}
a_1 = M^2+\frac{1}{K}, \qquad a_2 = M^2+1+\frac{1}{K}+\frac{\alpha}{K}, \qquad a_3 = \frac{\alpha}{K}+1, \\
a_4 = \frac{a_2}{a_3}, \qquad a_5 = \frac{a_1}{a_3}, \qquad a_6 = a_4-\alpha_0, \qquad a_7 = a_5-\alpha_0(a_4-\alpha_0), \\
a_8 = a_4-1, \qquad a_9 = a_8-a_5.
\end{gathered} \tag{3.23}$$

3.2. Flow due to Uniform Motion of the Plate

Here we consider the flow situation in which motion in the fluid is induced by the uniform acceleration of the plate in x-direction. The governing equations, initial conditions, and a part of boundary conditions are same. The only boundary condition (3.2) is replaced by

$$u(0,t) = At. \tag{3.24}$$

Using a similar method of solution as in previous section, the derived expressions for velocity, and shear stress fields are given as follows:

$$\begin{aligned}
U(\xi,\tau) &= \frac{\xi\sqrt{a_3}}{2\sqrt{\alpha\pi}}\int_0^\tau \frac{(\tau-s)}{s\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s\right)\mathrm{d}s \\
&\quad -\frac{\xi\sqrt{a_3a_7}}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^w \frac{(\tau-s)}{(s-w)\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)\right) \\
&\quad \times J_1\left(2\sqrt{a_7(s-w)w}\right)\mathrm{d}w\,\mathrm{d}s, \\
S(\xi,\tau) &= -\frac{a_{11}\sqrt{a_3\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s\right)\mathrm{d}s \\
&\quad +\frac{a_{11}\sqrt{a_3a_7\alpha}}{\sqrt{\pi}}\int_0^\tau\int_0^w \frac{1}{\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)\right) \\
&\quad \times J_1\left(2\sqrt{a_7w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s-\frac{a_5\sqrt{a_3\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{(\tau-s)}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s\right)\mathrm{d}s \\
&\quad +\frac{a_5\sqrt{a_7a_3\alpha}}{\sqrt{\pi}}\int_0^\tau\int_0^w \frac{(\tau-s)}{\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)\right) \\
&\quad \times J_1\left(2\sqrt{a_7w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s-\frac{a_{10}\sqrt{a_3\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{a_3\xi^2}{4\alpha s}-a_6 s-(\tau-s)\right)\mathrm{d}s \\
&\quad +\frac{a_{10}\sqrt{a_7a_3\alpha}}{\sqrt{\pi}}\int_0^\tau\int_0^w \frac{1}{\sqrt{w}}\exp\left(-\frac{a_3\xi^2}{4\alpha(s-w)}-\alpha_0 w-a_6(s-w)-(\tau-s)\right) \\
&\quad \times J_1\left(2\sqrt{a_7w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s,
\end{aligned} \tag{3.25}$$

where $U = u/\lambda A$, $S = T/\rho c\lambda A$, $a_{10} = a_5 + 1 - a_4$ and $a_{11} = a_4 - a_5$.

4. Limiting Cases

In this section we want to reduce the solutions obtained in the previous section to their limiting cases.

(1) The solutions (3.21) and (3.22) (impulsive motion of the plate), and (3.25) (uniform motion of the plate) for hydrodynamic Oldroyd-B fluid ($M = 0$) in a nonporous space ($1/K = 0$) reduce to the following equations:

$$\begin{aligned} U(\xi,\tau) = {} & \frac{\xi}{2\sqrt{\alpha\pi}}\int_0^\tau \frac{1}{s\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha s}-(1-\alpha_0)s\right)\mathrm{d}s \\ & -\frac{\xi\sqrt{\alpha_0(\alpha_0-1)}}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^w \frac{1}{(s-w)\sqrt{w}}\exp\left(-\frac{\xi^2}{4\alpha(s-w)}-\alpha_0 w-(1-\alpha_0)(s-w)\right) \\ & \times J_1\left(2\sqrt{\alpha_0(\alpha_0-1)(s-w)w}\right)\mathrm{d}w\,\mathrm{d}s, \end{aligned} \tag{4.1}$$

$$\begin{aligned} S(\xi,\tau) = {} & -\frac{\sqrt{\alpha}}{\sqrt{\pi\tau}}\exp\left(-\frac{\xi^2}{4\alpha\tau}-(1-\alpha_0)\tau\right) \\ & +\frac{\sqrt{\alpha_0(\alpha_0-1)\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha(\tau-s)}-\alpha_0 s-(1-\alpha_0)(\tau-s)\right) \\ & \times J_1\left(2\sqrt{\alpha_0(\alpha-1)s(\tau-s)}\right)\mathrm{d}s, \end{aligned} \tag{4.2}$$

$$\begin{aligned} U(\xi,\tau) = {} & \frac{\xi}{2\sqrt{\alpha\pi}}\int_0^\tau \frac{(\tau-s)}{s\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha s}-(1-\alpha_0)s\right)\mathrm{d}s \\ & -\frac{\xi\sqrt{\alpha_0(\alpha_0-1)}}{2\sqrt{\alpha\pi}}\int_0^\tau\int_0^w \frac{(\tau-s)}{(s-w)\sqrt{w}}\exp\left(-\frac{\xi^2}{4\alpha(s-w)}-\alpha_0 w-(1-\alpha_0)(s-w)\right) \\ & \times J_1\left(2\sqrt{\alpha_0(\alpha_0-1)(s-w)w}\right)\mathrm{d}w\,\mathrm{d}s, \end{aligned} \tag{4.3}$$

$$\begin{aligned} S(\xi,\tau) = {} & -\frac{\sqrt{\alpha}}{\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{\xi^2}{4\alpha s}-(1-\alpha_0)s\right)\mathrm{d}s+\frac{\sqrt{\alpha_0(\alpha_0-1)\alpha}}{\sqrt{\pi}} \\ & \times\int_0^\tau\int_0^w \frac{1}{\sqrt{w}}\exp\left(-\frac{\xi^2}{4\alpha(s-w)}-\alpha_0 w-(1-\alpha_0)(s-w)\right) \\ & \times J_1\left(2\sqrt{\alpha_0(\alpha_0-1)w(s-w)}\right)\mathrm{d}w\,\mathrm{d}s. \end{aligned} \tag{4.4}$$

Here we can see that (4.1) and (4.2) are identical to (3.20) and (3.19) from [14]. The criteria for getting (4.1) and (4.2) is different than one used in [14]. This is due to the fact that the inclusion of last term due to the porosity effect in the equation of motion makes it difficult to use the procedure followed in [14]. However, in the present situation we make it able to find the solutions (4.1) and (4.2) in a very easy and interesting way.

(2) Making the limit as $\alpha \to 1$ ($\lambda_r \to \lambda$) into (3.21), (3.22), and (3.25), we get the similar solutions for a Newtonian fluid performing the same motion. Thus the solutions (3.21) and (3.22) corresponding to the flow due to impulsive motion of the plate are given by

$$\begin{aligned} U(\xi,\tau) &= \frac{\xi\sqrt{K+1}}{2\sqrt{K\pi}}\int_0^\tau \frac{1}{s\sqrt{s}}\exp\left(-\frac{(K+1)\xi^2}{4Ks}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s, \\ S(\xi,\tau) &= -\frac{\sqrt{K+1}}{\sqrt{K\pi\tau}}\exp\left(-\frac{(K+1)\xi^2}{4K\tau}-\left(M^2+\frac{1}{K}\right)\tau\right) \\ &\quad -\frac{(KM^2+1)}{\sqrt{(K+1)K\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{(K+1)\xi^2}{4Ks}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s \end{aligned} \tag{4.5}$$

and the solutions (3.25) due to uniform motion of the plate reduce to

$$\begin{aligned} U(\xi,\tau) &= \frac{\xi\sqrt{K+1}}{2\sqrt{\pi K}}\int_0^\tau \frac{(\tau-s)}{s\sqrt{s}}\exp\left(-\frac{\xi^2(K+1)}{4Ks}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s, \\ S(\xi,\tau) &= -\frac{\sqrt{K+1}}{\sqrt{K\pi}}\int_0^\tau \frac{1}{\sqrt{s}}\exp\left(-\frac{(K+1)\xi^2}{4Ks}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s \\ &\quad -\frac{M^2K+1}{\sqrt{(K+1)K\pi}}\int_0^\tau \frac{(\tau-s)}{\sqrt{s}}\exp\left(-\frac{(K+1)\xi^2}{4Ks}-\left(M^2+\frac{1}{K}\right)s\right)\mathrm{d}s. \end{aligned} \tag{4.6}$$

It is important to note that if we take $M = 1/K = 0$ into (4.5) and make some suitable change of variables, we get the similar solutions as given in [14] ([see (4.2) and (4.3)]).

5. Results and Discussion

This section includes the graphical illustrations of various results from the flow analyzed in this paper. The results have been interpreted for various values of permeability parameter K and MHD parameter M. Special attention has been focused on the permeability parameter K. It is found that the flow analysis strongly depends on these parameters. The graphical results are displayed for velocity and shear stress fields for two different flow cases, namely (i) flow due to impulsive motion of the plate and (ii) flow due to uniform motion of the plate. In all these figures the relaxation parameter λ, the retardation parameter λ_r and dimensionless time τ are, respectively, chosen as 1.0, 2.0 and 0.4. Further, in these Figures 1–4 panels (a) and (b) are displayed for velocity and shear stress fields, respectively.

Figures 1 and 2 are prepared for velocity and shear stress fields for flow induced by the impulsive motion of the plate for various values of permeability parameter K magnetic parameter M. It is noted from Figure 1 that velocity and boundary layer thickness increases

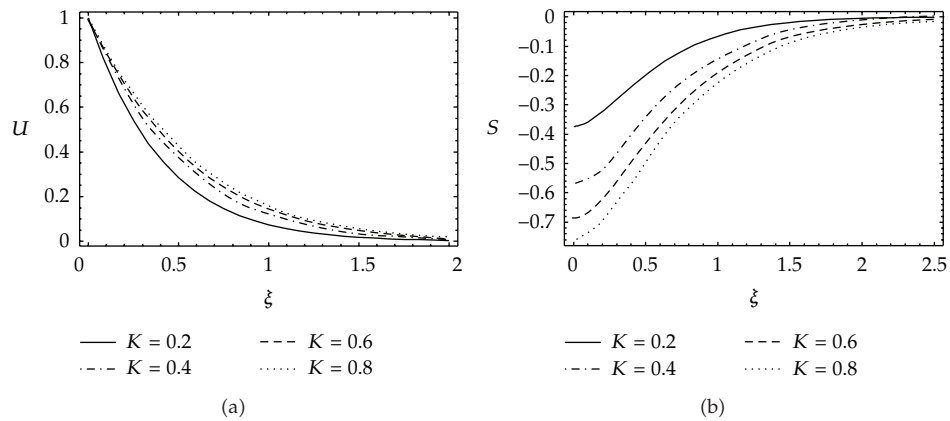

Figure 1: Profiles of (a) velocity and (b) shear stress for different values of K (impulsive motion of the plate).

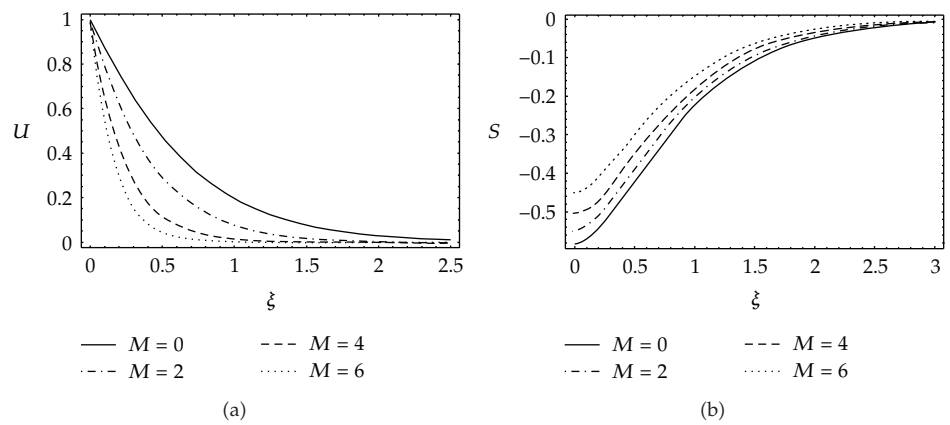

Figure 2: Profiles of (a) velocity and (b) shear stress for different values of M (impulsive motion of the plate).

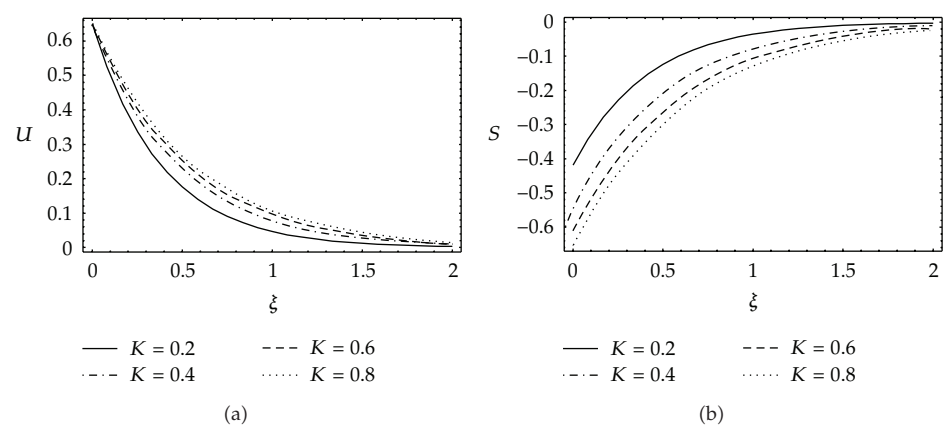

Figure 3: Profiles of (a) velocity and (b) shear stress for different values of K (uniform motion of the plate).

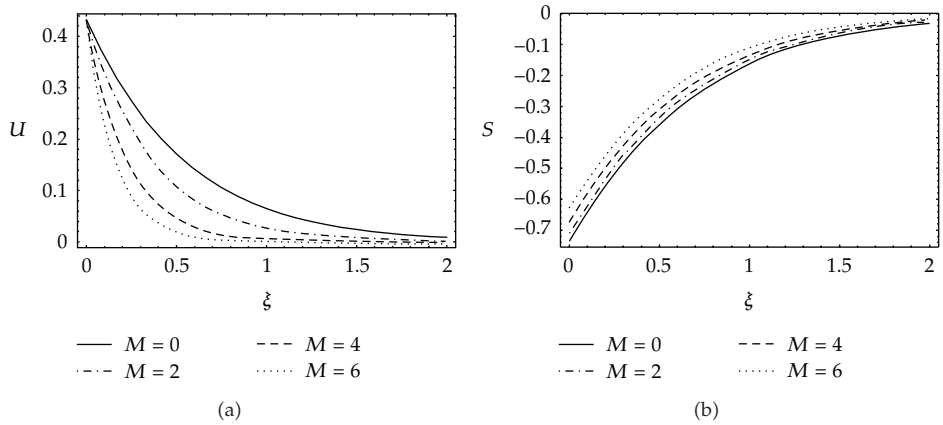

Figure 4: Profiles of (a) velocity and (b) shear stress for different values of M (uniform motion of the plate).

with increasing values of K. It may also be expected due to the fact that increasing values of K reduces the friction forces which assists the fluid considerably to move fast. Further as it was expected that the strongest shear stress occurs near the boundary and decreases rapidly with increasing distance from the plate. Figure 2 is sketched in order to explore the variations of magnetic parameter M. It is observed that the velocity and boundary layer thickness decreases upon increasing the values of M. It is because of the fact that the application of transverse magnetic field will result in a resistive type force (called Lorentz force) similar to drag force and upon increasing the values of M increases the drag force which leads to the deceleration of the flow. However, the magnitude of the shear stress increases with increasing values of M. The magnitude of the shear stress in the close regime of the boundary is larger as compared to the region away from the boundary.

Figures 3 and 4 have been sketched for the situation when the flow is driven by the uniform motion of the plate. It is observed that the variations of velocity and shear stress fields in these figures are qualitatively similar to the figures in case of impulsive motion of the plate. However, when analyzed carefully, it is found that these observations are not the same quantitatively.

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