

Research Article

A Multiplicity Result for Quasilinear Problems with Nonlinear Boundary Conditions in Bounded Domains

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We study the following quasilinear problem with nonlinear boundary condition $-\Delta_p u - \lambda a(x)u|u|^{p-2} = b(x)u|u|^{\gamma-2}$, in Ω and $(1 - \alpha)|\nabla u|^{p-2}(\partial u / \partial n) + \alpha u|u|^{p-2} = 0$, on $\partial\Omega$, where $\Omega \subseteq \mathbb{R}^N$ is a connected bounded domain with smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n . Δ_p is the p -Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the functions a and b are sign changing continuous functions in Ω , $1 < p < \gamma < p^*$, where $p^* = Np / (N - p)$ if $N > p$ and ∞ otherwise. The properties of the first eigenvalue $\lambda_1^+(\alpha)$ and the associated eigenvector of the related eigenvalue problem have been studied in (Khademloo, In press). In this paper, it is shown that if $\lambda \leq \lambda_1^+(\alpha)$, the original problem admits at least one positive solution, while if $\lambda_1^+(\alpha) < \lambda < \lambda^*$, for a positive constant λ^* , it admits at least two distinct positive solutions. Our approach is variational in character and our results extend those of Afrouzi and Khademloo (2007) in two aspects: the main part of our differential equation is the p -Laplacian, and the boundary condition in this paper also is nonlinear.

1. Introduction and Results

In this paper, we consider the problem

$$\begin{aligned} -\Delta_p u - \lambda a(x)u|u|^{p-2} &= b(x)u|u|^{\gamma-2}, & x \in \Omega, \\ (1 - \alpha)|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \alpha u|u|^{p-2} &= 0, & x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a connected bounded domain with a smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n . Δ_p is the p -Laplacian operator defined by $\Delta_p u =$

$\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The functions a and b are assumed to be sign changing in Ω . Here, we say a function $a(x)$ changes sign if the measure of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive. $\lambda \geq 0$ is a real parameter and exponent γ is assumed to satisfy the condition $1 < p < \gamma < p^*$, where $p^* = Np/(N-p)$ if $N > p$ and ∞ otherwise.

A host of literature exists for this type of problem when $p = 2$. For the works concerning with problems similar to (1.1) in the case $p = 2$, we refer to [1–3] and references therein.

The growing attention in the study of the p -Laplace operator is motivated by the fact that it arises in various applications, for example, non-Newtonian fluids, reaction diffusion problems, flow through porous media, glacial sliding, theory of superconductors, biology, and so forth (see [4, 5] and the references therein).

In this paper, we obtain new existence results by using a variational method based on the properties of eigencurves, that is, properties of the map $\lambda \rightarrow \mu(\alpha, \lambda)$, where $\mu(\alpha, \lambda)$ denotes the principal eigenvalue of the problem

$$\begin{aligned} -\Delta_p u - \lambda a(x)u|u|^{p-2} &= \mu u|u|^{p-2}, & x \in \Omega, \\ (1 - \alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \alpha u|u|^{p-2} &= 0, & x \in \partial\Omega. \end{aligned} \quad (1.2)$$

Similar to [6] our method works provided that the eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda a(x)u|u|^{p-2}, & x \in \Omega, \\ (1 - \alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \alpha u|u|^{p-2} &= 0, & x \in \partial\Omega, \end{aligned} \quad (1.3)$$

has principal eigenvalues and it can be shown that this occurs on an interval $[\alpha_0, 1]$ where $\alpha_0 \leq 0$. Thus, we are able to obtain existence results for problem (1.2) even in the case of nonlinear Neumann boundary conditions where α is small and negative. Our method depends on using eigencurves to produce an equivalent norm on $W^{1,p}(\Omega)$; such an equivalent norm is also introduced in [2]. The results that we obtain in this paper are generalization of the previous results obtained by Pohozaev and Veron [7].

It can be shown that $\mu(\alpha, \lambda)$ has the variational characterization:

$$\mu(\alpha, \lambda) = \inf \left\{ \int_{\Omega} |\nabla u|^p dx + \frac{\alpha}{1 - \alpha} \int_{\partial\Omega} |u|^p d\sigma - \lambda \int_{\Omega} a|u|^p dx; u \in W^{1,p}(\Omega), \int_{\Omega} |u|^p dx = 1 \right\} \quad (1.4)$$

from whence it follows that

- (i) λ is a principal eigenvalue of (1.2) if and only if $\mu(\alpha, \lambda) = 0$,
- (ii) $\alpha \rightarrow \mu(\alpha, \lambda)$ is an increasing function,
- (iii) $\lambda \rightarrow \mu(\alpha, \lambda)$ is a concave function with a unique maximum such that $\mu(\alpha, \lambda) \rightarrow -\infty$ as $\lambda \rightarrow \pm\infty$ [6].

If $\alpha \in (0, 1]$, then $\mu(\alpha, \lambda) > 0$, and so $\lambda \rightarrow \mu(\alpha, \lambda)$ has exactly one negative zero $\lambda_1^-(\alpha)$ and one positive zero $\lambda_1^+(\alpha)$. Thus, $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$ are principal eigenvalues for (1.2).

If $\alpha = 0$, then $\mu(0, 0) = 0$. If $a(x) > 0$, then $\mu(0, \lambda)$ is decreasing, and if $a(x) < 0$, then $\mu(0, \lambda)$ is increasing. Assume now that $a(x)$ changes sign in Ω : if $\int_{\Omega} a(x)dx < 0$, there exists a unique $\lambda_1^+(0) > 0$ such that $\mu(0, \lambda_1^+(0)) = 0$ and $\mu(0, \lambda) > 0$ for $\lambda \in (0, \lambda_1^+(0))$. If $\int_{\Omega} a(x)dx = 0$, then $\mu(0, 0) = 0$ and $\mu(0, \lambda) < 0$ for $\lambda \neq 0$. If $\int_{\Omega} a(x)dx > 0$, then there exists a unique $\lambda_1^-(0) < 0$ such that $\mu(0, \lambda_1^-(0)) = 0$ and $\mu(0, \lambda) > 0$ for $\lambda \in (\lambda_1^-(0), 0)$.

Suppose now that $\int_{\Omega} a(x)dx < 0$ and that α is small and negative. Then, since $\alpha \rightarrow \mu(\alpha, \lambda)$ is increasing, it follows that there still exist principal eigenvalues $\lambda_1^-(\alpha) < \lambda_1^+(\alpha)$ of (2.5), but now both of them are positive.

It can be shown that there exists $\alpha_0 < 0$ such that the above is true for all $\alpha \in (\alpha_0, 0)$, but for $\alpha < \alpha_0$, $\mu(\alpha, \lambda) < 0$ for all λ so that principal eigenvalues no longer exist.

Similar considerations show that when $\int_{\Omega} a(x)dx > 0$, there exists $\alpha_0 < 0$ such that there are principal eigenvalues $\lambda_1^-(\alpha) < \lambda_1^+(\alpha) < 0$ for $\alpha_0 < \alpha < 0$ but when $\int_{\Omega} a(x)dx = 0$, there are no principal eigenvalues for $\alpha < 0$ (see [6]).

It is easy to see that if $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$ exist, $\mu(\alpha, \lambda) > 0$ for all $\lambda \in (\lambda_1^-(\alpha), \lambda_1^+(\alpha))$.

Thus, we assume that the following conditions hold:

- (A₁) $\alpha \in (0, 1)$ or that $\int_{\Omega} a(x)dx \neq 0$ and $\alpha \in (\alpha_0, 1]$,
- (A₂) $a(x) \in L^{\infty}(\Omega)$,
- (B₁) $b(x) \in L^{\infty}(\Omega)$,
- (B₂) $b^+ \neq 0$,
- (B₃) $\int_{\Omega} b(x)(u_1^+)^{\gamma} dx < 0$,

where u_1^+ is the positive principal eigenfunction corresponding to $\lambda_1^+(\alpha)$.

With these constructions we have the following.

Proposition 1.1. *Assume (A₁), then for every $\lambda \in (0, \lambda_1^+(\alpha))$,*

$$\|u\|_{\alpha, \lambda} := \left(\int_{\Omega} (|\nabla u|^p - \lambda a(x)|u|^p) dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} |u|^p d\sigma \right)^{1/p} \tag{1.5}$$

defines a norm in $W^{1,p}(\Omega)$ which is equivalent to the usual norm of $W^{1,p}(\Omega)$, that is,

$$\|u\| = \left(\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p dx \right)^{1/p}. \tag{1.6}$$

Proof. See [2]. □

Now we can state our main results.

Theorem 1.2. *Assume (A₁), (A₂), (B₁), and (B₂). Then, for every $\lambda \in (0, \lambda_1^+(\alpha))$, problem (1.2) admits at least one positive solution $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.*

Theorem 1.3. *Assume (A₁), (A₂), (B₁), (B₂), and (B₃). Then, problem (1.2) has at least one positive solution $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ for $\lambda = \lambda_1^+(\alpha)$.*

Theorem 1.4. *Assume (A₁), (A₂), (B₁), (B₂), and (B₃). Then, there exists $\lambda^* > \lambda_1^+(\alpha)$ such that problem (1.2) admits at least two distinct positive weak solutions in $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, whenever $\lambda_1^+(\alpha) < \lambda < \lambda^*$.*

When $\lambda = 0$ and $\alpha = 0$, we have the following.

Corollary 1.5. *Assume $\int_{\Omega} b(x)dx < 0$. Then, the problem*

$$\begin{aligned} -\Delta_p u &= b(x)u|u|^{\gamma-2}, & x \in \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} &= 0, & x \in \partial\Omega, \end{aligned} \quad (1.7)$$

has a positive solution.

Throughout this paper, c denotes a positive constant. We will use fibering method in a similar way to those in [8]. A brief description of the method and the proof of Theorem 1.2 are presented in Section 2. We then study the cases $\lambda = \lambda_1^+(\alpha)$ and $\lambda > \lambda_1^+(\alpha)$ in Sections 3 and 4.

2. The Case When $\lambda < \lambda_1^+(\alpha)$

In this section, motivated by Pohozaev [8], we will introduce the fibering map as our framework for the study of problem (1.2).

For a (weak) solution of problem (1.2), we mean a function $u \in X = W^{1,p}(\Omega)$ such that for every $v \in W^{1,p}(\Omega)$, there holds

$$\int_{\Omega} \left(|\nabla u|^{p-2} \nabla u \nabla v - \lambda a(x) |u|^{p-2} uv \right) dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} |u|^{p-2} uv \, d\sigma = \int_{\Omega} b(x) |u|^{\gamma-2} uv \, dx. \quad (2.1)$$

Now let us define the variational functional corresponding to problem (1.2). We set $I_{\lambda} : X \rightarrow \mathbb{R}$ as

$$I_{\lambda}(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p - \lambda a(x) |u|^p) dx + \frac{\alpha}{p(1-\alpha)} \int_{\partial\Omega} |u|^p \, d\sigma - \frac{1}{\gamma} \int_{\Omega} b(x) |u|^{\gamma} dx. \quad (2.2)$$

It is easy to see that $I_{\lambda} \in C^1(X, \mathbb{R})$, and for $v \in X$, there holds

$$\begin{aligned} (I'_{\lambda}(u), v) &= \int_{\Omega} \left(|\nabla u|^{p-2} \nabla u \nabla v - \lambda a(x) |u|^{p-2} uv \right) dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} |u|^{p-2} uv \, d\sigma \\ &= \int_{\Omega} b(x) |u|^{\gamma-2} uv \, dx. \end{aligned} \quad (2.3)$$

Since $C_0^{\infty}(\Omega) \subset X$, we know that critical points of I_{λ} are weak solutions of the problem (1.2). Thus, to prove our main theorems, it suffices to show that I_{λ} admits critical point. We will do this in this paper. Our main tool is the theory of the fibering maps. First, we will introduce this map for $I_{\lambda} : X \rightarrow \mathbb{R}$.

Let $\phi_u(t) = I_{\lambda}(tu)$ ($t > 0$). We refer to such maps as fibering maps. It is clear that if u is a local minimizer of I_{λ} , then ϕ_u has a local minimum at $t = 1$.

Lemma 2.1. *Let $u \in X - \{0\}$ and $t > 0$. Then, $(I'_{\lambda}(tu), tu) = 0$ if and only if $\phi'_u(t) = 0$.*

Proof. The result is an immediate consequence of the fact that

$$\phi'_u(t) = (I'_\lambda(tu), u) = \frac{1}{t} (I'_\lambda(tu), tu). \quad (2.4)$$

□

Thus critical points of I_λ correspond to stationary points of the maps ϕ_u that can be given by

$$\phi_u(t) = I_\lambda(tu) = \frac{t^p}{p} A_\lambda(u) - \frac{t^\gamma}{\gamma} B(u), \quad (2.5)$$

where

$$\begin{aligned} A_\lambda(u) &= \int_{\Omega} (|\nabla u|^p - \lambda a(x)|u|^p) dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} |u|^p d\sigma, \\ B(u) &= \int_{\Omega} b(x)|u|^\gamma dx. \end{aligned} \quad (2.6)$$

Hence,

$$\phi'_u(t) = t^{p-1} A_\lambda(u) - t^{\gamma-1} B(u). \quad (2.7)$$

Thus, if $A_\lambda(u)$ and $B(u)$ have the same sign, ϕ_u has exactly one turning point at

$$t(u) = \left(\frac{A_\lambda(u)}{B(u)} \right)^{1/(\gamma-p)}, \quad (2.8)$$

provided that $B(u) \neq 0$, and if $A_\lambda(u)$ and $B(u)$ have opposite signs, ϕ_u has no turning points. Now substituting (2.8) into (2.5), we get

$$\begin{aligned} I_\lambda(t(u)u) &= \frac{1}{p} \left(\frac{A_\lambda(u)}{B(u)} \right)^{p/(\gamma-p)} A_\lambda(u) - \frac{1}{\gamma} \left(\frac{A_\lambda(u)}{B(u)} \right)^{\gamma/(\gamma-p)} B(u) \\ &= \frac{1}{p} \frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} - \frac{1}{\gamma} \frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} = \left(\frac{1}{p} - \frac{1}{\gamma} \right) \frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} := J_\lambda(u). \end{aligned} \quad (2.9)$$

Lemma 2.2. *Suppose that u_0 is a critical point of J_λ , where $A_\lambda(u_0)$ and $B(u_0)$ have the same sign. Then, $I'_\lambda(t(u_0)u_0) = 0$.*

Proof. Let $u_0 \in X$, $B(u_0) \neq 0$, then

$$\begin{aligned} (I'_\lambda(t(u_0)u_0), v) &= \int_{\Omega} \left(|\nabla(t(u_0)u_0)|^{p-2} \nabla(t(u_0)u_0) \nabla v - \lambda a(x) |t(u_0)u_0|^{p-2} t(u_0)u_0 v \right) dx \\ &\quad - \int_{\Omega} b(x) |t(u_0)u_0|^{r-2} t(u_0)u_0 v dx + \frac{\alpha}{1-\alpha} \int_{\partial\Omega} |t(u_0)u_0|^{p-2} t(u_0)u_0 v d\sigma \\ &= t(u_0)^{p-1} (A'_\lambda(u_0), v) - t(u_0)^{r-1} (B'(u_0), v) = \frac{1}{t(u_0)} (J'_\lambda(u_0), v) = 0 \end{aligned} \quad (2.10)$$

for all $v \in X$. This completes the proof. \square

The following lemma shows that the critical points of the functional I_λ can be found by using the conditional variational problem associated with I_λ .

Lemma 2.3. *Suppose that H is a well-defined functional on X and u_0 is a minimizer of J_λ on*

$$S := \{u \in X, H(u) = c, (H'(u), u) \neq 0\} \quad (2.11)$$

for some $c \neq 0$. Then, $J'_\lambda(u_0) = 0$.

Proof. If u_0 is a local minimizer of J_λ on S , then u_0 is a solution of the optimization problem:

$$\text{minimize } J_\lambda(u) \quad \text{subject to } \gamma(u) = 0, \quad (2.12)$$

where $\gamma(u) = H(u) - c$. Hence, by the theory of the Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J'_\lambda(u_0) = \mu \gamma'(u_0)$. Thus,

$$(J'_\lambda(u_0), u_0) = \mu (\gamma'(u_0), u_0). \quad (2.13)$$

Note that the functional J_λ is 0-homogeneous and the Gateaux derivative of J_λ at the point $v \in X$, $B(v) \neq 0$, in direction v , is zero, that is, $(J'_\lambda(u_0), u_0) = 0$. It then follows that $\mu = 0$ due to $u_0 \in S$. Hence, the proof is complete. \square

The following scheme for the investigation of the solvability of (1.2) is based on previous lemmas. First, we will prove the existence of nonzero critical points of J_λ under the constraint given by suitable functional H . This will be an actual critical point of J_λ and it will generate critical point of the Euler functional I_λ which will coincide with the weak solution of problem (1.2).

Proof of Theorem 1.2. Suppose that (\mathcal{A}_1) , (\mathcal{A}_2) , (\mathcal{B}_1) , (\mathcal{B}_2) satisfy. It follows from variational characterization of $\mu(\alpha, \lambda)$ that $A_\lambda(u) \geq 0$ for all $u \in X$ and $0 \leq \lambda < \lambda_1^+(\alpha)$. Moreover, we have

$$(A'_\lambda(u), u) = pA_\lambda(u), \quad u \in X. \quad (2.14)$$

Hence, for using Lemma 2.3, it is sufficient to consider the case $A_\lambda(u) = c \neq 0$, for example $c = 1$. In this case, we have

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) \frac{1}{B(u)^{p/(\gamma-p)}}. \tag{2.15}$$

From the necessary condition for the existence of $t(u)$, we have $B(u) > 0$. It follows that we must consider a critical point with $B(u) > 0$.

The functional $J_\lambda(u)$ is nonnegative and so bounded below, hence we can look for positive local minimizer for $J_\lambda(u)$ on X .

Let us consider variational problem:

$$\mathcal{M}_\lambda = \sup\{B(u); A_\lambda(u) = 1, B(u) > 0\}. \tag{2.16}$$

Note that for $0 \leq \lambda < \lambda_1^+(\alpha)$, this set is not empty, and from Lemma 2.3, the solution of this problem is a minimizer of $J_\lambda(u)$ on X .

Suppose $\{u_n\}$ is the maximizing sequence of this problem. It follows from the equivalent property in Proposition 1.1, $\{u_n\}$ is bounded and so we may assume that $u_n \rightharpoonup u_0$ in X . Since X may be compactly embedded in $L^p(\Omega)$, $L^{\gamma}(\Omega)$, and $L^p(\partial\Omega)$, we have $u_n \rightarrow u_0$ in $L^p(\Omega)$, $L^{\gamma}(\Omega)$ and $L^p(\partial\Omega)$. Hence,

$$B(u_n) \rightarrow B(u_0) = \mathcal{M}_\lambda > 0. \tag{2.17}$$

Moreover, we have

$$A_\lambda(u_0) \leq \liminf A_\lambda(u_n) = 1. \tag{2.18}$$

Here, the weak lower semicontinuity of the equivalent norm was used.

Assume that $A_\lambda(u_0) < 1$. By using the map:

$$L(t) = A_\lambda(tu_0), \tag{2.19}$$

we have $L(1) < 1$ and $\lim_{t \rightarrow \infty} L(t) = \infty$ and so $L(t_0) = 1$ for some $t_0 > 1$, that is, $A_\lambda(t_0u_0) = 1$. So we derive

$$B(t_0u_0) = t_0^\gamma B(u_0) > \mathcal{M}_\lambda, \tag{2.20}$$

which is a contradiction. Hence, $A_\lambda(u_0) = 1$ and so u_0 is a maximizer. As Lemma 2.2, the result would follow by considering $u_1 = t(u_0)u_0 \in X$. Then, u_1 is a weak solution of (1.2) and $u_1 \geq 0$ in Ω . Now, following the bootstrap argument (used, e.g., in [9]), we prove $u_1 \in L^\infty(\Omega)$. Then, we can apply the Harnack inequality due to Trudinger [5] in order to get $u_1 > 0$ in Ω (cf. [9]). □

3. The Case When $\lambda = \lambda_1^+(\alpha)$

Proof of Theorem 1.3. If $\lambda = \lambda_1^+(\alpha)$, It is easy to see that the set $\{u \in X; A_\lambda(u) = 1\}$ is unbounded and we are forced to require an additional assumption (\mathcal{B}_3) . In this case, again we are looking for a maximizing of the problem:

$$\mathcal{M}_{\lambda_1^+(\alpha)} = \sup \left\{ B(u); A_{\lambda_1^+(\alpha)}(u) = 1, B(u) > 0 \right\}. \quad (3.1)$$

Suppose $\{u_n\}$ is a maximizing sequence of this problem. First, we investigate the case when $\{u_n\}$ is unbounded. Then, we may assume without loss of generality that $\|u_n\| \rightarrow \infty$. So, we obtain

$$1 = A_{\lambda_1^+(\alpha)}(u_n) = \|u_n\|^p A_{\lambda_1^+(\alpha)}(v_n), \quad (3.2)$$

where $v_n = u_n / \|u_n\|$. Thus, we have

$$A_{\lambda_1^+(\alpha)}(v_n) = \frac{A_{\lambda_1^+(\alpha)}(u_n)}{\|u_n\|^p} = \frac{1}{\|u_n\|^p} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

Since $\|v_n\| = 1$, we may assume that $v_n \rightharpoonup v_0$ in X . Again, since X may be compactly embedded in $L^p(\Omega)$, $L^Y(\Omega)$ and $L^p(\partial\Omega)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} a(x)|v_n|^p dx &= \int_{\Omega} a(x)|v_0|^p dx, \\ \lim_{n \rightarrow \infty} \int_{\Omega} b(x)|v_n|^Y dx &= \int_{\Omega} a(x)|v_0|^Y dx, \\ \lim_{n \rightarrow \infty} \int_{\partial\Omega} |v_n|^p d\sigma &= \int_{\partial\Omega} |v_0|^p d\sigma. \end{aligned} \quad (3.4)$$

So,

$$\int_{\Omega} a(x)|v_0|^p dx \geq \frac{1}{\lambda_1^+(\alpha)}, \quad (3.5)$$

And, therefore, $v_0 \neq 0$. Returning to $A_{\lambda_1^+(\alpha)}(v_n)$, we have also

$$0 \leq A_{\lambda_1^+(\alpha)}(v_0) \leq \liminf_{n \rightarrow \infty} A_{\lambda_1^+(\alpha)}(v_n) = 0. \quad (3.6)$$

Due to simplicity of $\lambda_1^+(\alpha)$, there exists $c \neq 0$ such that $v_0(x) = cu_1^+(x)$. Therefore, we have

$$0 < \mathcal{M}_{\lambda_1^+(\alpha)} = \lim_{n \rightarrow \infty} B(u_n) = \lim_{n \rightarrow \infty} \|u_n\|^Y B(v_n), \quad (3.7)$$

that implies $|c|^Y B(u_1^+) \geq 0$, which contradicts (\mathcal{B}_3) . Therefore, $\{u_n\}$ is bounded and so $u_n \rightharpoonup u_0$ in X and $u_n \rightarrow u_0$ in $L^p(\Omega)$, $L^Y(\Omega)$ and $L^p(\partial\Omega)$. Hence, $B(u_n) \rightarrow B(u_0) = \mathcal{M}_{\lambda_1^+(\alpha)} > 0$, and so $u_0 \neq 0$.

Moreover, we obtain $0 \leq A_{\lambda_1^+(\alpha)}(u_0) \leq 1$. Here, the variational characteristic of $\lambda_1^+(\alpha)$ and the weak lower semicontinuity of the norm were used. Let us now consider the case $A_{\lambda_1^+(\alpha)}(u_0) = 0$. Again, we obtain $u_0 = cu_1^+$ for some $c \neq 0$, and so $B(u_0) = |c|^Y B(u_1^+) = \mathcal{M}_{\lambda_1^+(\alpha)} > 0$ which contradicts (\mathcal{B}_3) .

Now we prove that $A_{\lambda_1^+(\alpha)}(u_0) = 1$. Suppose otherwise, then by using $L(t)$ in (2.20), we obtain some $t_0 > 1$ such that $A_{\lambda_1^+(\alpha)}(t_0 u_0) = 1$. By direct calculation, we get a contradiction like (4.1). This yields that u_0 is a nonnegative solution of problem (1.2). Using the same ideas as the proof of Theorem 1.2, we have proved Theorem 1.3. \square

4. The Case When $\lambda > \lambda_1^+(\alpha)$

As it is proved in [10], we will show that for $\lambda > \lambda_1^+(\alpha)$ but close enough to $\lambda_1^+(\alpha)$, we have two distinct positive solutions for problem (1.2). The existence of one of them is obtained by using the following lemma.

Lemma 4.1. *Under the assumption of Theorem 1.4, there exists a maximizer u_1 of the problem:*

$$\sup\{B(u); A_\lambda(u) = 1, B(u) > 0\}, \tag{4.1}$$

whenever $\lambda \in (\lambda_1^+, \lambda_1^+(\alpha) + \delta)$, for some $\delta > 0$. Moreover, $u_1 \in X \cap L^\infty(\Omega)$ is a positive weak solution of problem (1.2).

Proof. First note that using $L(t)$ in (2.20), it is easy to see that u_1 is a maximizer of problem (4.2) if and only if u_1 is a maximizer of the problem:

$$\mathcal{M}_\lambda = \sup\{B(u); A_\lambda(u) \leq 1, B(u) > 0\}. \tag{4.2}$$

Now suppose that the result is false. Then, there exists a sequence $\{\delta_k\}$ such that $\delta_k \rightarrow 0$ and problem (4.1) has no solution for $\lambda_1^+(\alpha) + \delta_k$. For simplicity, we use $\lambda_k = \lambda_1^+(\alpha) + \delta_k$. Let $\{u_n^k\}$ be a maximizing sequence of this problem, that is,

$$A_{\lambda_k}(u_n^k) \leq 1, \quad B(u_n^k) \rightarrow \mathcal{M}_{\lambda_k} > 0. \tag{4.3}$$

We prove that if $\{u_n^k\}$ be bounded or unbounded, we arrive at a contradiction and so the lemma is proved.

The first case: $\{u_n^k\}$ is bounded. Thus, $u_n^k \rightharpoonup u_0^k$ in X for some $u_0^k \in X$ and $u_n^k \rightarrow u_0^k$ in $L^p(\Omega)$, $L^Y(\Omega)$, and $L^p(\partial\Omega)$. Hence, $B(u_0^k) = \mathcal{M}_{\lambda_k} > 0$ and the weak lower semicontinuity of the norm gives $A_{\lambda_k}(u_0^k) \leq 1$. Therefore, u_0^k is a solution of problem (4.5), which is a contradiction.

The case $\{u_n^k\}$ is unbounded. Then we may assume that $\|u_n^k\| \rightarrow \infty$. Let $v_n^k = u_n^k / \|u_n^k\|$. Then, $v_n^k \rightharpoonup v_0^k$ in X for some $v_0^k \in X$ and $v_n^k \rightarrow v_0^k$ in $L^p(\Omega)$, $L^Y(\Omega)$ and $L^p(\partial\Omega)$. This implies that

$$B(u_n^k) = \|u_n^k\|^Y B(v_n^k) \rightarrow \mathcal{M}_{\lambda_k} > 0. \tag{4.4}$$

Therefore, $B(v_0^k) \geq 0$. Furthermore, $\|u_n^k\|^p A_{\lambda_k}(v_n^k) \leq 1$ and so $A_{\lambda_k}(v_n^k) \leq 1/\|v_n^k\|^p$. Thus, we arrive at

$$A_{\lambda_k}(v_0^k) \leq \liminf_{n \rightarrow \infty} A_{\lambda_k}(v_n^k) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.5)$$

which implies

$$\int_{\Omega} |\nabla v_0^k|^p dx \leq 1, \quad (4.6)$$

$$\lambda_k \int_{\Omega} a(x) |v_0^k|^p dx \geq 1. \quad (4.7)$$

It is a direct consequent of the compactly embedding of X in $L^p(\Omega)$, $L^q(\Omega)$ and $L^p(\partial\Omega)$. Now, we pass to the limit for $k \rightarrow \infty$. Then, $\lambda_k \rightarrow \lambda_1^+(\alpha)$ and since $\{v_0^k\}$ is a bounded sequence, we may assume that $v_0^k \rightharpoonup v_0$ in X for some $v_0 \in X$ and $v_0^k \rightarrow v_0$ in $L^p(\Omega)$, $L^q(\Omega)$, and $L^p(\partial\Omega)$.

It follows from (4.10) that

$$\lambda_1^+(\alpha) \int_{\Omega} a(x) |v_0|^p dx \geq 1, \quad (4.8)$$

and from the variational characteristic of $\lambda_1^+(\alpha)$ and (4.6) that

$$0 \leq A_{\lambda_1^+(\alpha)}(v_0) \leq 0, \quad (4.9)$$

which contradicts (\mathcal{B}_3) .

Hence, for some $\delta > 0$, problem (4.5) has at least one nonnegative solution u_1 for any $\lambda \in (\lambda_1^+(\alpha), \lambda_1^+(\alpha) + \delta)$. \square

In order to find the second positive solution, we consider the minimizing problem:

$$\mathcal{N}_{\lambda} = \inf\{A_{\lambda}(u); B(u) = -1\}. \quad (4.10)$$

Note that this set is empty for $\lambda < \lambda_1^+(\alpha)$ (because of the variational characterization of $\lambda_1^+(\alpha)$). Hence, this second problem does not have a solution for $\lambda < \lambda_1^+(\alpha)$.

Lemma 4.2. *Under the assumption of Theorem 1.4, there exists $\varepsilon > 0$ such that for $\lambda \in (\lambda_1^+(\alpha), \lambda_1^+(\alpha) + \varepsilon)$, problem (1.1) has a nonnegative solution u_2 satisfying $A_{\lambda}(u_2) < 0$.*

Proof. First note that using the auxiliary function $L(t) = |t|^p B(u)$ and the assumption (\mathcal{B}_3) , it is easy to see that $B(t_1 u_1^+) = -1$ for some $t_1 < 1$. Thus, the set $\{u \in X; B(u) = -1\}$ is nonempty. Also, for this t_1 , we have

$$A_{\lambda}(t_1 u_1^+) = |t_1|^p (\lambda_1^+(\alpha) - \lambda) \int_{\Omega} a(x) |u_1^+|^p dx < 0, \quad (4.11)$$

for $\lambda > \lambda_1^+(\alpha)$.

Again, we assume that the result is not true. Then, there exists $\varepsilon_k \rightarrow 0$ such that for $\lambda_k = \lambda_1^+(\alpha) + \varepsilon_k$, problem (1.1) has no solution.

Let $\{u_n^k\}$ be a minimizing sequence of this problem, that is, $B(u_n^k) = -1$, $A_{\lambda_k}(u_n^k) \rightarrow \mathcal{N}_{\lambda_k} < 0$.

Assume that $\{u_n^k\}$ is bounded. Using similar argument as in the proof of Lemma 4.1, we find a solution of problem (1.1) which is a contradiction. Let us assume $\{u_n^k\}$ is unbounded. Again, we consider

$$v_n^k = \frac{u_n^k}{\|u_n^k\|}, \quad \text{with } \|u_n^k\| \rightarrow \infty, \quad (4.12)$$

$\{v_n^k\}$ is bounded, and so $v_n^k \rightarrow v_0^k$ in X for some $v_0^k \in X$. Thus, $B(v_n^k) = \frac{u_n^k}{\|u_n^k\|^Y} = \frac{-1}{\|u_n^k\|^Y} \rightarrow 0$.

Letting $n \rightarrow \infty$, we arrive at $B(v_0^k) = 0$. It follows easily that $A_{\lambda_k}(u_n^k) = \|u_n^k\|^p A_{\lambda_k}(v_n^k) \leq 0$, and so $A_{\lambda_k}(v_0^k) \leq 0$. As might be expected, we arrive at a contradiction with assumption (B_3) , by using similar steps as in the proof of Theorem 1.3. This completes the proof. \square

Proof of Theorem 1.4. Let $\eta = \min\{\varepsilon, \delta\}$, $\lambda^* = \lambda_1^+(\alpha) + \eta$ and $w_1 = t(u_1)u_1$ and $w_2 = t(u_2)u_2$, where $t(u_i)$ is defined by (2.16).

It is easy to see that $A_\lambda(w_1) > 0$ and $A_\lambda(w_2) \leq 0$, and so $w_1 \neq w_2$. Thus, w_1 and w_2 are two distinct non negative weak solutions to (1.2). Other properties of w_1 and w_2 follow in the same way as in Section 2. \square

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