Research Article

Common Fixed-Point Problem for a Family Multivalued Mapping in Banach Space

Zhanfei Zuo

*Department of Mathematics and Computer Science, Chongqing Three Gorges University,
 Wanzhou 404000, China*

Correspondence should be addressed to Zhanfei Zuo, zuozhanfei@163.com

Received 11 December 2010; Revised 15 March 2011; Accepted 17 March 2011

Academic Editor: Vittorio Colao

Copyright © 2011 Zhanfei Zuo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is our purpose in this paper to prove two convergents of viscosity approximation scheme to a common fixed point of a family of multivalued nonexpansive mappings in Banach spaces. Moreover, it is the unique solution in F to a certain variational inequality, where $F := \cap_{n=0}^{\infty} F(T_n)$ stands for the common fixed-point set of the family of multivalued nonexpansive mapping $\{T_n\}$.

1. Introduction

Let X be a Banach space with dual X^* , and let K be a nonempty subset of X . A gauge function is a continuous strictly increasing function $\varphi : R^+ \rightarrow R^+$ such that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The duality mapping $J_\varphi : X \rightarrow X^*$ associated with a gauge function φ is defined by $J_\varphi(x) := \{f \in X^* : \langle x, f \rangle = \|x\| \|f\|, \|f\| = \varphi(\|x\|)\}$, $x \in X$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the particular case $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the normalized duality map. We note that $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$. It is known that if X is smooth, then J_φ is single valued and norm to weak* continuous (see [1]). When $\{x_n\}$ is a sequence in X , then $x_n \rightarrow x$ ($x_n \rightharpoonup x$, $x_n \rightharpoonup^* x$) will denote strong (weak, weak*) convergence of the sequence $\{x_n\}$ to x .

Following Browder [2], we say that a Banach space X has the weakly continuous duality mapping if there exists a gauge function φ for which the duality map J_φ is single valued and weak to weak* sequentially continuous, that is, if $\{x_n\}$ is a sequence in X weakly convergent to a point x , then the sequence $\{J_\varphi(x_n)\}$ converges weak* to $J_\varphi(x)$. It is known that l_p ($1 < p < \infty$) spaces have a weakly continuous duality mapping J_φ with a gauge $\varphi(t) = t^{p-1}$. Setting

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \quad t \geq 0, \quad (1.1)$$

it is easy to see that $\Phi(t)$ is a convex function and $J_\varphi(x) = \partial\Phi(\|x\|)$, for $x \in X$, where ∂ denotes the subdifferential in the sense of convex analysis. We will denote by 2^X the family of all subsets of X , by $CB(X)$ the family of all nonempty closed bounded subsets of X , and by $C(X)$ the family of all nonempty compact subsets of X . A multivalued mapping $T : K \rightarrow 2^X$ is said to be nonexpansive (resp., contractive) if

$$\begin{aligned} H(Tx, Ty) &\leq \|x - y\|, \quad x, y \in K, \\ (\text{resp.}, H(Tx, Ty) &\leq k\|x - y\|, \quad \text{for some } k \in (0, 1)), \end{aligned} \quad (1.2)$$

where $H(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\}, \quad A, B \in CB(X). \quad (1.3)$$

Since Banach's contraction mapping principle was extended nicely to multivalued mappings by Nadler in 1969 (see [3]), many authors have studied the fixed-point theory for multivalued mappings.

In this paper, we construct two viscosity approximation sequences for a family of multivalued nonexpansive mappings in Banach spaces. Let K be a nonempty closed convex subset of Banach space X and let $T_n : K \rightarrow C(K)$, $n = 1, 2, \dots$ be a family of multivalued nonexpansive mapping, $f : K \rightarrow K$ is a contraction mapping with constant $\alpha \in (0, 1)$. Let $\alpha_n \in (0, 1)$, $\beta_n \in (0, 1)$. For any given $x_0 \in K$, let $y_0 \in T_0 x_0$ such that

$$x_1 = \alpha_0 f(x_0) + (1 - \alpha_0) y_0. \quad (1.4)$$

From Nadler Theorem (see [3]), we can choose $y_1 \in T_1 x_1$ such that

$$\|y_0 - y_1\| \leq H(T_0 x_0, T_1 x_1). \quad (1.5)$$

Inductively, we can get the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad \forall n \in N, \quad (1.6)$$

where, for each $n \in N$, $y_n \in T_n x_n$ such that

$$\|y_{n+1} - y_n\| \leq H(T_{n+1} x_{n+1}, T_n x_n). \quad (1.7)$$

Similarly, we also have the following multivalued version of the modified Mann iteration:

$$x_{n+1} = \beta_n f(x_n) + \alpha_n x_n + (1 - \alpha_n - \beta_n) y_n, \quad (1.8)$$

and $y_n \in T_n x_n$ such that $\|y_{n+1} - y_n\| \leq H(T_{n+1} x_{n+1}, T_n x_n)$. Then, $\{x_n\}$ is said to satisfy Condition (A') if for any subsequence $x_{n_k} \rightarrow x$ and $d(x_{n+1}, T_n(x_n)) \rightarrow 0$ implies that $x \in F$,

where $F := \cap_{i=0}^{\infty} F(T_n) \neq \emptyset$ is the common fixed-point set of the family of multivalued mapping $\{T_n\}$. We give an example of a family of multivalued nonexpansive mappings with Condition (A') as follows.

Example 1.1. Take $X = \mathbb{R}$ and $T_n = T$ (for all $n \geq 0$), where T is defined by

$$T(x) = \begin{cases} \{0\}, & x \leq 1, \\ \left\{x - \frac{1}{2}, \frac{1}{2} - x\right\}, & \text{otherwise.} \end{cases} \quad (1.9)$$

Let $f : \mathbb{R} \rightarrow \{0\}$ and $\alpha_n = 1/n$, $n \geq 2$, then $F = \{0\}$ and the iteration (1.6), reduced to

$$x_{n+1} = \left(1 - \frac{1}{n+2}\right)y_n, \quad \forall n \geq 0, \quad (1.10)$$

where $y_n \in Tx_n$, and it satisfies Condition (A'). In fact, if $x_0 \leq 1$, then (for all $n \in \mathbb{N}$, $n > 0$) $x_n = 0$ and Condition (A') is automatically satisfied. If $x_0 > 1$, then there exists an integer $p \geq 2$, such that

$$x_0 \in \left(\frac{p(p+1)}{4} - \frac{1}{2}, \frac{(p+1)(p+2)}{4} - \frac{1}{2}\right], \quad x_{p-1} = \frac{1}{p} \left(x_0 - \frac{p(p-1)}{4}\right). \quad (1.11)$$

Then, $y_p \in Tx_{p-1} = \{0\}$; hence, $x_n = 0$ (for all $n \geq p$), from which we deduce that Condition (A') is satisfied.

2. Preliminaries

Let $K \subset X$ be a closed convex and Q a mapping of X onto K , then Q is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in X$ and $t \geq 0$. A mapping Q of X into X is said to be a retraction if $Q^2 = Q$. A subset K of X is said to be a sunny nonexpansive retract of X if there exists a sunny nonexpansive retraction of X onto K , and it is said to be a nonexpansive retract of X . If $X = H$, the metric projection P is a sunny nonexpansive retraction from H to any closed convex subset of H . The following Lemmas will be useful in this paper.

Lemma 2.1 (see [4]). *Let K be a nonempty convex subset of a smooth Banach space X , let $J : X \rightarrow X^*$ be the (normalized) duality mapping of X , and let $Q : X \rightarrow K$ be a retraction, then the following are equivalent:*

- (1) $\langle x - Px, j(y - Px) \rangle \leq 0$ for all $x \in X$ and $y \in K$,
- (2) Q is both sunny and nonexpansive.

We note that Lemma 2.1 still holds if the normalized duality map J is replaced with the general duality map J_φ , where φ is a gauge function.

Lemma 2.2 (see [5]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \beta_n, \quad n \geq 0, \quad (2.1)$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\beta_n\}$ is a real number sequence such that

$$(i) \sum_{n=0}^{\infty} \gamma_n = \infty,$$

$$(ii) \text{ either } \limsup_{n \rightarrow \infty} (\beta_n / \gamma_n) \leq 0 \text{ or } \sum_{n=0}^{\infty} |\beta_n| < \infty,$$

then $\{a_n\}$ converges to zero, as $n \rightarrow \infty$.

Lemma 2.3 (see [1]). Let X be a real Banach space, then for all $x, y \in X$, one gets that

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_{\varphi}(x + y) \rangle, \quad \forall j_{\varphi} \in J_{\varphi}. \quad (2.2)$$

Lemma 2.4 (see [6]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n) y_n, \quad n \geq 0, \quad (2.3)$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$ such that

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1. \quad (2.4)$$

Assume that $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

3. Main Results

Theorem 3.1. Let X be a reflexive Banach space with weakly sequentially continuous duality mapping J_{φ} for some gauge φ , let K be a nonempty closed convex subset of X , and let $T_n : K \rightarrow C(K)$, $n = 0, 1, 2, \dots$, be a family of multivalued nonexpansive mappings with $F \neq \emptyset$ which is sunny nonexpansive retract of K with Q a nonexpansive retraction. Furthermore, $T_n(p) = \{p\}$ for any fixed-point $p \in F$, $\{x_n\}$ is defined by (1.6), and $\alpha_n \in (0, 1)$ satisfies the following conditions:

$$(1) \alpha_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(2) \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$(3) \{x_n\} \text{ satisfies Condition } (A').$$

Then, $\{x_n\}$ converges strongly to a common fixed-point $\bar{x} = Q(f(\bar{x}))$ of a family T_n , $n = 0, 1, 2, \dots$, as $n \rightarrow \infty$. Moreover, $\bar{x}$ is the unique solution in F to the variational inequality

$$\langle f(\bar{x}) - \bar{x}, j_{\varphi}(y - \bar{x}) \rangle \leq 0, \quad \forall y \in F. \quad (3.1)$$

Proof. First, we show the uniqueness of the solution to the variational inequality (3.1) in X . In fact, let $\bar{y} \in F$ be another solution of (3.1) in F , then we have

$$\langle f(\bar{x}) - \bar{x}, j_{\varphi}(\bar{y} - \bar{x}) \rangle \leq 0, \quad \langle f(\bar{y}) - \bar{y}, j_{\varphi}(\bar{x} - \bar{y}) \rangle \leq 0. \quad (3.2)$$

From (3.2), we have that

$$(1 - \alpha)\varphi(\|\bar{x} - \bar{y}\|)\|\bar{x} - \bar{y}\| \leq 0. \quad (3.3)$$

We must have $\bar{x} = \bar{y}$ and the uniqueness is proved. Let $p \in F$, then, from iteration (1.6), we obtain that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \|x_{n+1} - \alpha_n f(p) - (1 - \alpha_n)p\| + \|\alpha_n f(p) + (1 - \alpha_n)p - p\| \\
 &= \|\alpha_n(f(x_n) - f(p)) + (1 - \alpha_n)(y_n - p)\| + \alpha_n\|f(p) - p\| \\
 &\leq \alpha_n\alpha\|x_n - p\| + (1 - \alpha_n)H(T_n x_n, T_n p) + \alpha_n\|f(p) - p\| \\
 &\leq (1 - (1 - \alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\|.
 \end{aligned} \tag{3.4}$$

Using an induction, we obtain $\|x_n - p\| \leq \max\{\|x_0 - p\|, (1/(1 - \alpha))\|f(p) - p\|\}$, for all integers n , thus, $\{x_n\}$ is bounded and so are $\{T_n x_n\}$ and $\{f(x_n)\}$. This implies that

$$d(x_{n+1}, T_n(x_n)) \leq \|x_{n+1} - y_n\| = \alpha_n\|f(x_n) - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.5}$$

Next, we will show that

$$\limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \leq 0. \tag{3.6}$$

Since X is reflexive and $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup q$ such that

$$\limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle = \limsup_{k \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n_k} - \bar{x}) \rangle. \tag{3.7}$$

From (3.5) and $\{x_n\}$ satisfying Condition (A'), we obtain that $q \in F$. On the other hand, we notice that the assumption that the duality mapping J_φ is weakly continuous implies that X is smooth; from Lemma 2.1, we have

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle &= \limsup_{k \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n_k} - \bar{x}) \rangle \\
 &= \langle f(\bar{x}) - \bar{x}, j_\varphi(q - \bar{x}) \rangle \\
 &= \langle Q(\bar{x}) - \bar{x}, j_\varphi(q - \bar{x}) \rangle \leq 0.
 \end{aligned} \tag{3.8}$$

Finally, we will show that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. From iteration (1.6) and Lemma 2.3, we get that

$$\begin{aligned}
 \Phi(\|x_{n+1} - \bar{x}\|) &\leq \Phi(\|\alpha_n(f(x_n) - f(\bar{x})) + (1 - \alpha_n)(y_n - \bar{x})\|) + \alpha_n \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \\
 &\leq \Phi(\alpha_n\alpha\|x_n - \bar{x}\| + (1 - \alpha_n)H(T_n x_n, T_n \bar{x})) + \alpha_n \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \\
 &\leq (1 - \alpha_n(1 - \alpha))\Phi(\|x_n - \bar{x}\|) + \alpha_n \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle.
 \end{aligned} \tag{3.9}$$

Lemma 2.2 gives that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Moreover, $\bar{x}$ satisfying the variational inequality follows from the property of Q . $\square$

Let $f \equiv u \in K$ in iteration (1.6) be a constant mapping, then $\bar{x} = Qu$. In fact, we have the following corollary.

Corollary 3.2. *Let $\{x_n\}$ and T_n be as in Theorem 3.1, $f \equiv u \in K$, then $\{x_n\}$ converges strongly to a common fixed-point $\bar{x} = Q(u)$ of a family T_n , $n = 0, 1, 2, \dots$, as $n \rightarrow \infty$. Moreover, $\bar{x}$ is the unique solution in F to the variational inequality*

$$\langle u - Q(u), j_\varphi(y - Q(u)) \rangle \leq 0, \quad \forall y \in F. \quad (3.10)$$

If $X = H$, then the condition that F is a sunny nonexpansive retract of K in Theorem 3.1 is not necessary, and one has the following Corollary.

Corollary 3.3. *Let H be a Hilbert space with weakly sequentially continuous duality mapping J_φ for some gauge φ , and let $\{x_n\}$ and T_n be as in Theorem 3.1, then $\{x_n\}$ converges strongly to a common fixed-point $\bar{x} = P_F f(\bar{x})$ of a family of T_n , $n = 0, 1, 2, \dots$, where P_F is the metric projection from K onto F .*

Proof. It is well known that H is reflexive; by Propositions 2.3 and 2.6(ii) of [7], we get that F is closed and convex, and hence the projection mapping P_F is sunny nonexpansive retraction mapping, and the result follows from Theorem 3.1. $\square$

Corollary 3.4. *Let X be a real smooth Banach space, let K be a nonempty compact subset of X , and let T_n and $\{x_n\}$ be as in Theorem 3.1, then $\{x_n\}$ converges strongly to a common fixed-point $\bar{x} = Q(f(\bar{x}))$ of a family of T_n , $n = 0, 1, 2, \dots$, as $n \rightarrow \infty$. Moreover, $\bar{x}$ is the unique solution in F to the variational inequality*

$$\langle f(\bar{x}) - \bar{x}, j_\varphi(y - \bar{x}) \rangle \leq 0, \quad \forall y \in F. \quad (3.11)$$

Proof. Following the method of the proof of Theorem 3.1, we get that

$$d(x_{n+1}, T_n(x_n)) \leq \|x_{n+1} - y_n\| = \alpha_n \|f(x_n) - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.12)$$

Next, we will show that

$$\limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \leq 0. \quad (3.13)$$

Since K is compact and $\{x_n\}$ is bounded, we can assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow q \in K$,

$$\limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle = \lim_{k \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n_k} - \bar{x}) \rangle. \quad (3.14)$$

From (3.12) and $\{x_n\}$ satisfying Condition (A'), we obtain that $q \in F$. On the other hand, from the fact that X is smooth, the duality being norm to weak* continuous, and the standard characterization of retraction on F , we obtain that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle &= \lim_{k \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n_k} - \bar{x}) \rangle \\ &= \langle f(\bar{x}) - \bar{x}, j_\varphi(q - \bar{x}) \rangle \\ &= \langle Q(\bar{x}) - \bar{x}, j_\varphi(q - \bar{x}) \rangle \leq 0. \end{aligned} \quad (3.15)$$

Now, following the method of the proof of Theorem 3.1, we get the required result. $\square$

Theorem 3.5. *Let X be a reflexive Banach space with weakly sequentially continuous duality mapping J_φ for some gauge φ , let K be a nonempty closed convex subset of X , and let $T_n : K \rightarrow C(K)$, $n = 0, 1, 2, \dots$, be a family of multivalued nonexpansive mappings with $F \neq \emptyset$ which is sunny nonexpansive retract of K with Q a nonexpansive retraction. $H(T_{n+1}x, T_n y) \leq \|x - y\|$ for arbitrary $n \in \mathbb{N}$. Furthermore, $T_n(p) = \{p\}$ for any fixed-point $p \in F$. $\{x_n\}$ is defined by (1.8) and α_n, β_n satisfy the following conditions:*

- (i) $\beta_n \rightarrow 0$ as $n \rightarrow \infty$,
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,
- (iii) $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$.

If $\{x_n\}$ satisfies Condition (A'), then $\{x_n\}$ converges strongly to a common fixed-point $\bar{x} = Q(f(\bar{x}))$ of a family of T_n , $n = 0, 1, 2, \dots$, as $n \rightarrow \infty$. Moreover, $\bar{x}$ is the unique solution in F to the variational inequality

$$\langle f(\bar{x}) - \bar{x}, j_\varphi(y - \bar{x}) \rangle \leq 0, \quad \forall y \in F. \quad (3.16)$$

Proof. We first show that the sequence $\{x_n\}$ defined by (1.8) is bounded. In fact, take $p \in F$, noting that $T_n(p) = \{p\}$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= (1 - \alpha_n - \beta_n) \|y_n - p\| + \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - p\| \\ &= (1 - \alpha_n - \beta_n) \|y_n - T_n p\| + \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - p\| \\ &\leq (1 - \alpha_n - \beta_n) H(T_n x_n, T_n p) + \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - p\| \\ &\leq (1 - \alpha_n - \beta_n) \|x_n - p\| + \alpha_n \|x_n - p\| + \beta_n \|f(x_n) - p\| \\ &\leq (1 - \beta_n) \|x_n - p\| + \beta_n (\alpha \|x_n - p\| + \|f(p) - p\|) \\ &\leq (1 + (\alpha - 1)\beta_n) \|x_n - p\| + \beta_n (1 - \alpha) \frac{\|f(p) - p\|}{1 - \alpha}. \end{aligned} \quad (3.17)$$

It follows from induction that

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \alpha} \right\}, \quad (3.18)$$

so are $\{y_n\}$ and $\{f(x_n)\}$. Thus, we have that

$$\lim_{n \rightarrow \infty} \beta_n \|f(x_n) - y_n\| = 0. \quad (3.19)$$

Next, we show that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, T_n(x_n)) = 0. \quad (3.20)$$

Let $\lambda_n = \beta_n / (1 - \alpha_n)$ and $z_n = \lambda_n f(x_n) + (1 - \lambda_n)y_n$, then

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)z_n. \quad (3.21)$$

Therefore, we have for some appropriate constant $M > 0$ that the following inequality:

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|\lambda_{n+1}f(x_{n+1}) + (1 - \lambda_{n+1})y_{n+1} - (\lambda_n f(x_n) + (1 - \lambda_n)y_n)\| \\ &\leq |\lambda_{n+1} - \lambda_n| \|f(x_{n+1}) - f(x_n)\| + \|y_{n+1} - y_n\| + \lambda_n \|y_n\| + \lambda_{n+1} \|y_{n+1}\| \\ &\leq |\lambda_{n+1} - \lambda_n| \|f(x_{n+1}) - f(x_n)\| + H(T_{n+1}x_{n+1}, T_n x_n) + (\lambda_n + \lambda_{n+1})M \\ &\leq |\lambda_{n+1} - \lambda_n| \|f(x_{n+1}) - f(x_n)\| + \|x_{n+1} - x_n\| + (\lambda_n + \lambda_{n+1})M \end{aligned} \quad (3.22)$$

holds. Thus, $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq \lim_{n \rightarrow \infty} (|\lambda_{n+1} - \lambda_n| \|f(x_{n+1}) - f(x_n)\| + (\lambda_n + \lambda_{n+1})M) = 0$. By Lemma 2.4, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_n\| &= 0, \\ \|x_n - y_n\| &\leq \|x_n - z_n\| + \|z_n - y_n\| = \|x_n - z_n\| + \lambda_n \|f(x_n) - y_n\| \longrightarrow 0. \end{aligned} \quad (3.23)$$

Therefore, we have

$$d(x_{n+1}, T_n(x_n)) \leq \|x_{n+1} - y_n\| \leq \beta_n \|f(x_n) - y_n\| + \alpha_n \|x_n - y_n\| \longrightarrow 0. \quad (3.24)$$

Using (3.20) and $\{x_n\}$ satisfying Condition (A'), we can use the same argumentation as Theorem 3.1 proves that $\bar{x} \in F$ and

$$\limsup_{n \rightarrow \infty} \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \leq 0. \quad (3.25)$$

Finally, we show that $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. In fact, from iteration (1.8) and Lemma 2.3, we have

$$\begin{aligned}
 \Phi(\|x_{n+1} - \bar{x}\|) &= \Phi(\|\beta_n f(x_n) + \alpha_n x_n + (1 - \alpha_n - \beta_n)y_n - \bar{x}\|) \\
 &= \Phi(\|\alpha_n(x_n - \bar{x}) + (1 - \alpha_n - \beta_n)(y_n - \bar{x}) + \beta_n(f(x_n) - f(\bar{x})) + \beta_n(f(\bar{x}) - \bar{x})\|) \\
 &\leq \Phi(\|\alpha_n(x_n - \bar{x})\| + (1 - \alpha_n - \beta_n)H(T_n x_n, T_n \bar{x}) + \alpha\beta_n\|x_n - \bar{x}\|) \\
 &\quad + \beta_n \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle \\
 &\leq [1 - (1 - \alpha)\beta_n] \Phi(\|x_n - \bar{x}\|) + \beta_n \langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle.
 \end{aligned} \tag{3.26}$$

From (ii) and (3.25), it then follows that

$$\sum_{n=0}^{\infty} (1 - \alpha)\beta_n = \infty, \quad \limsup_n \frac{\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x}) \rangle}{1 - \alpha} \leq 0. \tag{3.27}$$

Apply Lemma 2.2 to conclude that $x_n \rightarrow \bar{x}$. □

References

- [1] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.
- [2] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.
- [3] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
- [4] R. E. Bruck Jr., "Nonexpansive projections on subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.
- [5] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
- [6] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 1, pp. 103–123, 2005.
- [7] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Mathematics of Operations Research*, vol. 26, no. 2, pp. 248–264, 2001.