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Research Article On Degenerate Parabolic Equations

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The paper deals with the existence of solutions of some generalized Stefan-type equation in the framework of Orlicz spaces.

1. Introduction

In this paper, we deal with the following boundary value problems

$$\frac{\partial u}{\partial t} + A(\theta(u)) = f, \quad \text{in } Q,$$

$$u = 0, \quad \text{on } \partial Q = \partial \Omega \times (0, T),$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega,$$
(P)

where

$$A(u) = -\operatorname{div}(a(\cdot, t, \nabla u)), \tag{1.1}$$

 $Q = \Omega \times [0, T], T > 0$, and Ω is a bounded domain of \mathbb{R}^N , with the segment property, f is a smooth function, $u_0 \in L^2(\Omega)$, θ is a positive real function increasing but not necessarily strictly increasing, $\theta(0) = 0$, and $\theta(u_0) \in L^2(\Omega)$. $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., measurable with respect to x in Ω for every (t, ξ) in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to ξ in \mathbb{R}^N for almost every x in Ω) such that for all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$a(x,t,\xi)\xi \ge \alpha B(|\xi|),\tag{1.2}$$

$$[a(x,t,\xi) - a(x,t,\xi^*)][\xi - \xi^*] > 0, \tag{1.3}$$

$$|a(x,t,\xi)| \le c(x,t) + k_1 \overline{B}^{-1} B(k_2|\xi|).$$
(1.4)

There exist an N-function M such that

$$B(\theta(t)) \ll M(t), \tag{1.5}$$

where c(x, t) belongs to $E_{\overline{B}}(Q), c \ge 0$ and $k_i(i = 1, 2)$ to \mathbb{R}^+ , and α to \mathbb{R}^+_* .

Some examples of such operator are in particular the case where

$$a(\cdot, \nabla \theta(u)) = \frac{B(|\nabla \theta(u)|)}{|\nabla \theta(u)|^2} \nabla \theta(u), \qquad (1.6)$$

where *B* is an *N*-function.

Many physical models in hydrology, infiltration through porous media, heat transport, metallurgy, and so forth lead to the nonlinear equations (systems) of the form

$$\partial_t u = \nabla \psi \big(\nabla \beta(u) \big), \tag{E}$$

where β, ψ are monotone, $\psi(s) s$ is even and convex for $s \ge s_o > 0$, $|\psi(s)| \to \infty$, $|\beta(s)| \to \infty$ for $|s| \to \infty$ (for the details see [1]). Jäger and Kačur treated the porous medium systems where β is strictly monotone in [2] and Stefan-type problems where β is only monotone. For the last model, there exists a large number of references. Among them, let us mention the earlier works [3–5] for a variational approach and [6] for semigroup.

In [7], a different approach was introduced to study the porous and Stefan problems. The enthalpy formulation and the variational technique are used. Nonstandard semidiscretization in time is used, and Newton-like iterations are applied to solve the corresponding elliptic problems.

Due to the possible jumps of θ , problem (*P*) enters the class of Stefan problems. In the present paper, we are interested in the parabolic problem with regular data. It is similar in many respects to the so-called porous media equation. However, the equation we consider has a more general structure than that in the references above.

Two main difficulties appear in the study of existence of solutions of problem (*P*). The first one comes from the diffusion terms in (*P*) since they do not depend on *u* but on $\theta(u)$, and, moreover, at the same time, (*P*) poses big problems, since in general we have not information on *u* but on $\theta(u)$. For the last reason, the authors in [8] define a new notion of weak solution to overcome this problem.

In the above cited references, the authors have shown the existence of a weak solution when the function $a(x,t,\xi)$ was assumed to satisfy a polynomial growth condition with respect to ∇u . When trying to relax this restriction on the function $a(\cdot,\xi)$, we are led to replace the space $L^p(0,T;W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_B$ built from an Orlicz space L_B instead of L^p , where the *N*-function *B* which defines L_B is related to the actual growth of the Carathéodory's function. International Journal of Mathematics and Mathematical Sciences

Our goal in this paper is, on the one hand, to give a generalization of (E) in the case of one equation in the framework of Leray-Lions operator in Orlicz-Sobolev spaces. on the second hand, we prove the existence of solutions in the BV(Q) space.

2. Preliminaries

Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an *N*-function, that is, *M* is continuous, convex, with M(t) > 0 for t > 0, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to \infty$ as $t \to \infty$. The *N*-function \overline{M} conjugate to *M* is defined by $\overline{M}(t) = \sup\{st - M(s) : s > 0\}$.

Let *P* and *Q* be two *N*-functions. $P \ll Q$ means that *P* grows essentially less rapidly than *Q*, that is, for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty.$$
(2.1)

Let Ω be an open subset of \mathbb{R}^N . The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_{\Omega} M(u(x)) dx < +\infty$ (resp., $\int_{\Omega} M(u(x)/\lambda) dx < +\infty$ for some $\lambda > 0$).

Note that $L_M(\Omega)$ is a Banach space under the norm $||u||_{M,\Omega} = \inf\{\lambda > 0 : \int_{\Omega} M(u(x)/\lambda) dx \leq 1\}$ and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. In general, $E_M(\Omega) \neq L_M(\Omega)$ and the dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x) dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

We say that u_n converges to u for the modular convergence in $L_M(\Omega)$ if, for some $\lambda > 0$, $\int_{\Omega} M((u_n - u)/\lambda) dx \to 0$. This implies convergence for $\sigma(L_M, L_{\overline{M}})$.

The inhomogeneous Orlicz-Sobolev spaces are defined as follows: $W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^{\alpha}u \in L_M(Q) \text{ for all } |\alpha| \leq 1\}$. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there are α -order derivatives, $|\alpha| \leq 1$. We define the space $W_0^{1,x}L_M(Q) = \overline{\mathfrak{D}(Q)}^{\sigma(\Pi L_M,\Pi L_{\overline{M}})}$. (For more details, see [9].)

For k > 0, we define the truncation at height $k, T_k : \mathbf{R} \to \mathbf{R}$ by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k \frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

$$(2.2)$$

3. Main Result

Before giving our main result, we give the following lemma which will be used.

Lemma 3.1 (see [10]). Under the hypothesis (1.2)–(1.4), $\theta(s) = s$, the problem (P) admits at least one solution u in the following sense:

$$u \in W_0^{1,x} L_B(Q) \cap L^2(Q),$$

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + \int_Q a(\cdot, \nabla u) \nabla v = \int_Q f v \, dx dt,$$
(3.1)

for all $v \in W_0^{1,x}L_B(Q) \cap L^2(Q)$ and for v = u.

Theorem 3.2. Under the hypothesis (1.2)-(1.5), the problem (P_0) admits at least one solution u in the following sense:

$$u \in BV_{loc}(Q), \qquad \theta(u) \in W_0^{1,x} L_B(Q) \cap L^2(Q),$$

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle + \int_Q a(\cdot, \nabla \theta(u)) \nabla v = \int_Q f v \, dx dt,$$
(3.2)

for all $v \in W_0^{1,x}L_B(Q)$.

Proof.

Step 1 (approximation and a priori estimate). Consider the approximate problem:

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(a(\cdot, \nabla \theta(u_n))) - \frac{1}{n} \Delta_M(u_n) = f, \quad \text{in } Q,$$

$$u_n(x, 0) = u_{0n}(x), \quad \text{in } \Omega,$$
(3.3)

where $-\Delta_M(u) = -\operatorname{div}((M(|\nabla u_n|)/|\nabla u_n|^2)\nabla u_n)$ is the *M*-Laplacian operator and (u_{0n}) is a smooth sequence converging strongly to u_0 in $L^2(Q)$.

The approximate problem has a regular solution u_n and in particular $u_n \in W_0^{1,x}L_M(Q)$ (by Lemma 3.1).

Let $\Theta(s) = \int_0^s \theta(t) dt$.

Let $v = \theta(u_n)\chi_{(0,\tau)}$ as test function, one has

$$\int_{\Omega} \Theta(u_n(\tau)) dx + \alpha \int_{Q_\tau} B(|\nabla \theta(u_n)|) + \leq \int_{Q_\tau} f\theta(u_n) + \int_{\Omega} \Theta(u_n(0)) dx, \tag{3.4}$$

then, $(\theta(u_n))_n$ bounded in $W_0^{1,x}L_B(Q)$.

There exist a measurable function v and a subsequence, also denoted (u_n) , such that,

$$\theta(u_n) \rightarrow v$$
, a.e in Q and weakly in $W_0^{1,x} L_B(Q)$. (3.5)

Let us consider the C^2 function defined by

$$\eta_k(s) = \begin{cases} s \quad |s| \le \frac{k}{2}, \\ k \operatorname{sign}(s) \quad |s| \ge k. \end{cases}$$
(3.6)

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Multiplying the approximating equation by $\eta'_k(u_n)$, we get

$$\frac{\partial \eta_k(u_n)}{\partial t} - \operatorname{div}\left(a(\cdot, \nabla \theta(u_n))\eta'_k(u_n)\right) + a(\cdot, \nabla \theta(u_n))\eta''_k(u_n),$$

$$-\frac{1}{n}\operatorname{div}\left(\frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n\eta'_k(u_n)\right) + \frac{1}{n}\frac{M(|\nabla u_n|)}{|\nabla u_n|^2}\nabla u_n\eta''_k(u_n) = f\eta'_k(u_n)$$
(3.7)

in the distributions sense. We deduce, then, $\eta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$ and $\partial \eta_k(u_n)/\partial t$ in $W^{-1,x}L_{\overline{M}}(Q) + L^2(Q)$. Then, $\eta_k(u_n)$ is compact in $L^1(Q)$.

Following the same way as in [11], we obtain $\theta(u_n) \rightharpoonup \theta(u)$, weakly in $W_0^{1,x} L_B(Q)$ for $\sigma(\Pi L_B, \Pi E_{\overline{B}})$, strongly in $L^1(Q)$ and a.e in Q.

Step 2 (passage to the limit). Let set $b(\cdot, \nabla u) = (M(|\nabla u|)/|\nabla u|^2)\nabla u$. Let $v \in W_0^{1,x}L_B(Q)$, one has

$$\left\langle \frac{\partial u_n}{\partial t}, \theta(u_n) - v \right\rangle + \int_Q a(\cdot, \nabla \theta(u_n)) \nabla(\theta(u_n) - v) + \int_Q \frac{1}{n} b(\cdot, \nabla u_n) \nabla(\theta(u_n) - v) dx$$

$$= \int_Q f(\theta(u_n) - v) dx dt.$$
(3.8)

By using the following decomposition:

$$a(\cdot, \nabla \theta(u_n)) \nabla (\theta(u_n) - v) = a(\cdot, \nabla \theta(u_n)) - a(\cdot, \theta(\nabla v)) \nabla (\theta(u_n) - v) + a(\cdot, \theta(\nabla v)) \nabla (\theta(u_n) - v),$$

$$b(\cdot, \nabla u_n) \nabla (\theta(u_n) - v) = b(\cdot, \nabla u_n) - b(\cdot, \nabla v) \nabla (\theta(u_n) - v) + b(\cdot, \nabla v) \nabla (\theta(u_n) - v), \nabla (\theta(u_n) - v) = (\theta'(u_n) \nabla u_n - \nabla v) = \theta'(u_n) \nabla u_n - \theta'(u_n) \nabla v + \theta'(u_n) \nabla v - \nabla v,$$
(3.9)

and by the monotonicity of the operator defined by *a* and *b*, we obtain

$$\left\langle \frac{\partial u_n}{\partial t}, \theta(u_n) - v \right\rangle + \int_{Q} a(\cdot, \theta(\nabla v)) (\nabla \theta(u_n) - \nabla v) + \int_{Q} \frac{1}{n} b(\cdot, \nabla v) (\nabla \theta(u_n) - \nabla v) + \int_{Q} \frac{1}{n} (b(\cdot, \nabla u_n) - b(\cdot, \nabla v)) (\theta'(u_n) - 1) \nabla v \leq \int_{Q} f(\theta(u_n) - v),$$

$$(3.10)$$

by passage to the limit with a standard argument as in [10, 11], and using the above convergence of $\theta(u_n)$, we have

$$\left\langle \frac{\partial u}{\partial t}, \theta(u) - v \right\rangle + \int_{Q} a(\cdot, \nabla \theta(v)) (\nabla \theta(u) - \nabla v) dx \le \int_{Q} f(\theta(u) - v).$$
(3.11)

Taking now $v = \theta(u) - t\psi$, with $\psi \in W_0^{1,x}L_B(Q)$ and $t \in (-1, 1)$, we deduce that u is solution of the problem (1.2).

Step 3 ($u \in BV_{loc}(Q)$). Let *K* be a compact in *Q*, and let $\varphi \in D(Q)$ with $K \subset supp(\varphi)$ such that

$$\varphi = 1, \quad \text{on } K, \left| \nabla \varphi \right| \le 1.$$
 (3.12)

Using φ as test function in (3.7), we get

$$\begin{split} \int_{Q} \frac{\partial \eta_{k}(u_{n})}{\partial t} \varphi dx dt + \int_{Q} a(\cdot, \nabla \theta(u_{n})) \nabla \varphi \cdot \eta_{k}'(u_{n}) dx dt + \int_{Q} a(\cdot, \nabla \theta(u_{n})) \varphi \cdot \eta_{k}''(u_{n}) dx dt \\ &+ \frac{1}{n} \int_{Q} \frac{M(|\nabla u_{n}|)}{|\nabla u_{n}|^{2}} \nabla u_{n} \nabla \varphi \cdot \eta_{k}'(u_{n}) dx dt + \frac{1}{n} \int_{Q} \frac{M(|\nabla u_{n}|)}{|\nabla u_{n}|^{2}} \nabla u_{n} \varphi \cdot \eta_{k}''(u_{n}) dx dt \\ &= I_{1} + I_{2} + I_{3} + I_{4} + I_{5} \\ &= \int_{Q} f \eta_{k}'(u_{n}) \varphi dx dt. \end{split}$$
(3.13)

The terms I_2 , I_3 , I_4 , I_5 are bounded, so

$$\int_{K} \left| \frac{\partial \eta_{k}(u_{n})}{\partial t} \right| dx dt \leq C.$$
(3.14)

Letting *k* tend to infinity, we have

$$\int_{K} \left| \frac{\partial u_{n}}{\partial t}(x,t) \right| dx dt \le C.$$
(3.15)

We deal now with the following estimation which ends the proof. For all compact $K \subset Q$,

$$\int_{K} |Du_n(x,t)| dx dt \le C.$$
(3.16)

Indeed, we differentiate the approximate problem with respect to x_i , we multiply the obtained equation by $\eta'_k(\partial_{x_i}u_n)$, and one has the following equality in the distributions sense

$$\frac{\partial(\partial_{x_i}u_n)}{\partial t}\eta'_k(\partial_{x_i}u_n) - \operatorname{div}(\partial_{x_i}a(\cdot,\nabla\theta(u_n)))\eta'_k(\partial_{x_i}u_n) - \frac{1}{n}\partial_{x_i}\Delta_M(u_n)\eta'_k(\partial_{x_i}u_n)
= \partial_{x_i}f\eta'_k(\partial_{x_i}u_n),$$
(3.17)

which is equivalent to

$$\frac{\partial \eta_k(\partial_{x_i}u_n)}{\partial t} - \operatorname{div}(\partial_{x_i}a(\cdot,\nabla\theta(u_n)))\eta'_k(\partial_{x_i}u_n) - \frac{1}{n}\operatorname{div}(\partial_{x_i}b(\cdot,\nabla u_n))\eta'_k(\partial_{x_i}u_n)$$

$$= \partial_{x_i}f\eta'_k(\partial_{x_i}u_n).$$
(3.18)

We recall that η_k , η'_k , and η''_k are bounded on **R**, $(\theta(u_n))$ is bounded in $W_0^{1,x}L_B(Q)$, and $(a(\cdot, \nabla \theta(u_n)))$ is bounded in $L_{\overline{B}}(Q)$.

Using now the test function φ (defined below), we obtain, as for (3.14),

$$\int_{K} \left| \frac{\partial \eta_{k}}{\partial t} \left(\frac{\partial u_{n}}{\partial x_{i}} \right) \right| dx dt \leq C.$$
(3.19)

With the same way as above, we conclude the result, $u \in BV_{loc}(Q)$.

Remark 3.3. As in Theorem 3.2, one can prove the same result in the case where we replace the initial condition in the problem (*P*) by $\theta(u(x, 0)) = \theta(u_0(x))$ and $\theta(u_0(x)) \in L^2(Q)$.

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