Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 534391, 15 pages
doi:10.1155/2011/534391

Research Article

On a Result of Levin and Stečkin

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Received 18 August 2010; Accepted 25 January 2011

Academic Editor: Seppo Hassi

The following inequality for $0 < p < 1$ and $a_n \geq 0$ originates from a study of Hardy, Littlewood, and Pólya: $\sum_{n=1}^{\infty}((1/n)\sum_{k=n}^{\infty} a_k)^p \geq c_p \sum_{n=1}^{\infty} a_n^p$. Levin and Stečkin proved the previous inequality with the best constant $c_p = (p/(1-p))^p$ for $0 < p \leq 1/3$. In this paper, we extend the result of Levin and Stečkin to $0 < p \leq 0.346$.

1. Introduction

Let $p > 1$, and l^p be the Banach space of all complex sequences $\mathbf{a} = (a_n)_{n\geq 1}$. The celebrated Hardy's inequality [1, Theorem 326] asserts that for $p > 1$ and any $\mathbf{a} \in l^p$

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n} a_k\right|^p \leq \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty}|a_k|^p. \tag{1.1}$$

As an analogue of Hardy's inequality, Theorem 345 of [1] asserts that the following inequality holds for $0 < p < 1$ and $a_n \geq 0$ with $c_p = p^p$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty} a_k\right)^p \geq c_p \sum_{n=1}^{\infty} a_n^p. \tag{1.2}$$

It is noted in [1] that the constant $c_p = p^p$ may not be best possible, and a better constant was indeed obtained by Levin and Stečkin [2, Theorem 61]. Their result is more general as they proved, among other things, the following inequality [2, Theorem 62], valid for $0 < r \leq p \leq 1/3$ or $1/3 < p < 1, r \leq (1-p)^2/(1+p)$ with $a_n \geq 0$:

$$\sum_{n=1}^{\infty} \frac{1}{n^r} \left(\sum_{k=n}^{\infty} a_k \right)^p \geq \left(\frac{p}{1-r} \right)^p \sum_{n=1}^{\infty} \frac{a_n^p}{n^{r-p}}. \tag{1.3}$$

We note here that the constant $(p/(1-r))^p$ is best possible, as shown in [2] by setting $a_n = n^{-1-(1-r)/p-\epsilon}$ and letting $\epsilon \to 0^+$. This implies inequality (1.2) for $0 < p \leq 1/3$ with the best possible constant $c_p = (p/(1-p))^p$. On the other hand, it is also easy to see that inequality (1.2) fails to hold with $c_p = (p/(1-p))^p$ for $p \geq 1/2$. The point is that in these cases $p/(1-p) \geq 1$ so one can easily construct counterexamples.

A simpler proof of Levin and Stečkin's result (for $0 < r = p \leq 1/3$) is given in [3]. It is also pointed out there that, using a different approach, one may be able to extend their result to p slightly larger than $1/3$; an example is given for $p = 0.34$. The calculation however is more involved, and therefore it is desirable to have a simpler approach. For this, we let q be the number defined by $1/p + 1/q = 1$ and note that by the duality principle (see [4, Lemma 2], but note that our situation is slightly different since we have $0 < p < 1$ with an reversed inequality), the case $0 < r < 1, 0 < p < 1$ of inequality (1.3) is equivalent to the following one for $a_n > 0$:

$$\sum_{n=1}^{\infty} \left(n^{(r-p)/p} \sum_{k=1}^{n} \frac{a_k}{k^{r/p}} \right)^q \leq \left(\frac{p}{1-r} \right)^q \sum_{n=1}^{\infty} a_n^q. \tag{1.4}$$

The above inequality can be regarded as an analogue of a result of Knopp [5, 6], which asserts that Hardy's inequality (1.1) is still valid for $p < 0$ if we assume $a_n > 0$. We may also regard inequality (1.4) as an inequality concerning the factorable matrix with entries $n^{(r-p)/p}k^{-r/p}$ for $k \leq n$ and 0 otherwise. Here we recall that a matrix $A = (a_{nk})$ is factorable if it is a lower triangular matrix with $a_{nk} = a_n b_k$ for $1 \leq k \leq n$. We note that the approach in [7] for the l^p norms of weighted mean matrices can also be easily adopted to treat the l^p norms of factorable matrices, and it is our goal in this paper to use this similar approach to extend the result of Levin and Stečkin. Our main result is the following.

Theorem 1. *Inequality* (1.2) *holds with the best possible constant* $c_p = (p/(1-p))^p$ *for any* $1/3 < p < 1/2$ *satisfying*

$$2^{p/(1-p)} \left(\left(\frac{1-p}{p} \right)^{1/(1-p)} - \frac{1-p}{p} \right) - \left(1 + \frac{3-1/p}{2} \right)^{1/(1-p)} \geq 0. \tag{1.5}$$

In particular, inequality (1.2) *holds for* $0 < p \leq 0.346$.

It readily follows from Theorem 1 and our discussions above that we have the following dual version of Theorem 1.

Corollary 1. *Inequality* (1.4) *holds with* $r = p$ *for any* $1/3 < p < 1/2$ *satisfying* (1.5) *and the constant is best possible. In particular, inequality* (1.4) *holds with* $r = p$ *for* $0 < p \leq 0.346$.

An alternative proof of Theorem 1 is given in Section 3, via an approach using the duality principle. In Section 4, we will study some inequalities which can be regarded as generalizations of (1.2). Motivations for considerations for such inequalities come both from their integral analogues as well as from their counterparts in the l^p spaces. As an example, we consider the following inequality for $0 < p < 1,\ 0 < \alpha < 1/p$:

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{\alpha k^{\alpha-1}}{n^{\alpha}}a_k\right)^p \geq \left(\frac{\alpha p}{1-\alpha p}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.6}$$

As in the case of (1.2), the above inequality does not hold for all $0 < p < 1, 0 < \alpha < 1/p$. In Section 4, we will however prove a result concerning the validity of (1.6) that can be regarded as an analogue to that of Levin and Stečkin's concerning the validity of (1.2).

Inequality (1.6) is motivated partially by integral analogues of (1.2), as we will explain in Section 4. It is also motivated by the following inequality for $p > 1,\ \alpha p > 1,\ a_n \geq 0$:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\alpha k^{\alpha-1}}{n^{\alpha}}a_k\right)^p \leq \left(\frac{\alpha p}{\alpha p-1}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.7}$$

The above inequality is in turn motivated by the following inequality:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{k^{\alpha-1}}{\sum_{i=1}^{n}i^{\alpha-1}}a_k\right)^p \leq \left(\frac{\alpha p}{\alpha p-1}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.8}$$

Inequality (1.8) was first suggested by Bennett [8, pages 40-41]; see [9] and the references therein for recent progress on this. We point out here that it is easy to see that inequality (1.7) implies (1.8) when $\alpha > 1$; hence, it is interesting to know that, for which values of α's, inequality (1.7) is valid. We first note that, on setting $a_1 = 1$ and $a_n = 0$ for $n \geq 2$ in (1.7) that it is impossible for it to hold when α is large for fixed p. On the other hand, when $\alpha = 1$, inequality (1.7) becomes Hardy's inequality, and hence one may expect it to hold for α close to 1, and we will establish such a result in Section 5.

2. Proof of Theorem 1

First we need a lemma.

Lemma 1. *The following inequality holds for* $0 \leq y \leq 1$ *and* $1/2 < t < 1$*:*

$$\left(1+\frac{y}{2t}\right)^{1+t} - (1+y)^{-t}\left(1+\frac{(2t-1)y}{2t}\right)^{1+t} - \frac{y}{t} \geq 0. \tag{2.1}$$

Proof. We set $x = y/2t$ so that $0 \leq x \leq 1$, and we recast the above inequality as

$$f(x,t) := (1+x)^{1+t} - (1+2tx)^{-t}(1+(2t-1)x)^{1+t} - 2x \geq 0. \tag{2.2}$$

Direct calculation shows that $f(0,t) = (\partial f/\partial x)(0,t) = 0$ and

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x,t) &= t(1+t)(1+x)^{t-1}\left(1-(1+2tx)^{-t-2}(1+(2t-1)x)^{t-1}(1+x)^{1-t}\right)\\ &:= t(1+t)(1+x)^{t-1}g(x,t).\end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned}\frac{\partial g}{\partial x}(x,t) &= (1+2tx)^{-t-3}(1+(2t-1)x)^{t-2}(1+x)^{-t}\\ &\quad\times\left(2(4t-1)+4t(4t-1)x+2t(2t-1)(t+2)x^2\right)\geq 0.\end{aligned} \tag{2.4}$$

As $g(0,t) = 0$, it follows that $g(x,t) \geq 0$ for $0 \leq x \leq 1$ which in turn implies the assertion of the lemma. □

We now describe a general approach towards establishing inequality (1.3) for $0 < r < 1$, $0 < p < 1$. A modification from the approach in Section 3 of [3] shows that, in order for (1.3) to hold for any given p with $c_{p,r}(= (p/(1-r))^p)$, it suffices to find a sequence $\mathbf{w}$ of positive terms for each $0 < r < 1$ and $0 < p < 1$, such that for any integer $n \geq 1$

$$n^{(p-r)/(1-p)}(w_1+\cdots+w_n)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}\left(\frac{w_n^{-1/(1-p)}}{n^{r/(1-p)}}-\frac{w_{n+1}^{-1/(1-p)}}{(n+1)^{r/(1-p)}}\right). \tag{2.5}$$

We note here that if we study the equivalent inequality (1.4) instead, then we can also obtain the above inequality from inequality (2.2) of [3], on setting $\Lambda_n = n^{-(r-p)/p}$, $\lambda_n = n^{-r/p}$ there. For the moment, we assume that $c_{p,r}$ is an arbitrary fixed positive number, and, on setting $b_n^{p-1} = w_n/w_{n+1}$, we can recast the above inequality as

$$\left(\sum_{k=1}^{n}\prod_{i=k}^{n}b_i^{p-1}\right)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}n^{(r-p)/(1-p)}\left(\frac{b_n}{n^{r/(1-p)}}-\frac{1}{(n+1)^{r/(1-p)}}\right). \tag{2.6}$$

The choice of b_n in Section 3 of [3] suggests that, for optimal choices of the b_n's, we may have asymptotically $b_n \sim 1 + c/n$ as $n \to +\infty$ for some positive constant c (depending on p). This observation implies that $n^{1/(1-p)}$ times the right-hand side expression above should be asymptotically a constant. To take the advantage of possible contributions of higher-order terms, we now further recast the above inequality as

$$\left(\frac{1}{n+a}\sum_{k=1}^{n}\prod_{i=k}^{n}b_i^{p-1}\right)^{-1/(1-p)} \leq c_{p,r}^{-1/(1-p)}n^{(r-p)/(1-p)}(n+a)^{1/(1-p)}\left(\frac{b_n}{n^{r/(1-p)}}-\frac{1}{(n+1)^{r/(1-p)}}\right), \tag{2.7}$$

where a is a constant (may depend on p) to be chosen later. It will also be clear from our arguments below that the choice of a will not affect the asymptotic behavior of b_n to the first order of magnitude. We now choose b_n so that

$$n^{(r-p)/(1-p)}(n+a)^{1/(1-p)}\left(\frac{b_n}{n^{r/(1-p)}}-\frac{1}{(n+1)^{r/(1-p)}}\right)=c_{p,r}^{-\alpha/(1-p)}, \tag{2.8}$$

where α is a parameter to be chosen later. This implies that

$$b_n = c_{p,r}^{-\alpha/(1-p)}\frac{n^{p/(1-p)}}{(n+a)^{1/(1-p)}}+\frac{n^{r/(1-p)}}{(n+1)^{r/(1-p)}}. \tag{2.9}$$

For the so-chosen b_n's, inequality (2.7) becomes

$$\sum_{k=1}^{n}\prod_{i=k}^{n}b_i^{p-1} \geq (n+a)c_{p,r}^{1+\alpha}. \tag{2.10}$$

We first assume that the above inequality holds for $n=1$. Then induction shows that it holds for all n as long as

$$b_n^{1-p} \leq \frac{n+a+c_{p,r}^{-(1+\alpha)}-1}{n+a}. \tag{2.11}$$

Taking into account the value of b_n, the above becomes (for $0 \leq y \leq 1$ with $y = 1/n$)

$$\left(1+\left(a+c_{p,r}^{-(1+\alpha)}-1\right)y\right)^{1/(1-p)}-(1+y)^{-r/(1-p)}(1+ay)^{1/(1-p)}-c_{p,r}^{-\alpha/(1-p)}y \geq 0. \tag{2.12}$$

The first-order term of the Taylor expansion of the left-hand side expression above implies that it is necessary to have

$$c_{p,r}^{-(1+\alpha)}-(1-p)c_{p,r}^{-\alpha/(1-p)}+r-1 \geq 0. \tag{2.13}$$

For fixed $c_{p,r}$, the left-hand side expression above is maximized when $\alpha = 1/p - 1$ with value $pc_{p,r}^{-1/p}+r-1$. This suggests for us to take $c_{p,r} = (p/(1-r))^p$. From now on we fix $c_{p,r} = (p/(1-r))^p$ and note that in this case (2.12) becomes

$$\left(1+\left(a+\frac{1-r}{p}-1\right)y\right)^{1/(1-p)}-(1+y)^{-r/(1-p)}(1+ay)^{1/(1-p)}-\frac{1-r}{p}y \geq 0. \tag{2.14}$$

We note that the choice of $a = 0$ in (2.14) with $r = p$ reduces to that considered in Section 3 of [3] (in which case the case $n = 1$ of (2.10) is also included in (2.14)). Moreover, with $a = 0$ in the above inequality and following the treatment in Section 3 of [3], one is able to improve some cases of the previously mentioned result of Levin and Stečkin concerning the validity of (1.3). We will postpone the discussion of this to the next section and focus now on the proof of Theorem 1. Since the cases $0 < p \le 1/3$ of the assertion of the theorem are known, we may assume $1/3 < p < 1/2$ from now on. In this case we set $r = p$ in (2.14), and Taylor expansion shows that it is necessary to have $a \ge (3 - 1/p)/2$ in order for inequality (2.14) to hold. We now take $a = (3 - 1/p)/2$ and write $t = p/(1 - p)$ to see that inequality (2.14) is reduced to (2.1) and Lemma 1 now implies that inequality (2.14) holds in this case. Inequality (1.2) with the best possible constant $c_p = (p/(1 - p))^p$ thus follows for any $1/3 < p < 1/2$ as long as the case $n = 1$ of (2.10) is satisfied, which is just inequality (1.5), and this proves the first assertion of Theorem 1.

For the second assertion, we note that inequality (1.5) can be rewritten as

$$\frac{2^t}{t}\left(t^{-t} - 1\right) \ge (1 + a)^{1/(1-p)}, \tag{2.15}$$

where t is defined as above. Note that $1/2 < t < 1$ for $1/3 < p < 1/2$ and both $2^t/t$ and $t^{-t} - 1$ are decreasing functions of t. It follows that the left-hand side expression of (2.15) is a decreasing function of p. Note also that for fixed a, the right-hand side expression of (2.15) is an increasing function of $p < 1$. As $a = (3-1/p)/2$ in our case, it follows that one just needs to check the above inequality for $p = 0.346$ and the assertion of the theorem now follows easily.

We remark here that, in the proof of Theorem 1, instead of choosing b_n to satisfy (2.8) (with $r = p$ and $c_{p,p} = (p/(1 - p))^p$ there), we can choose b_n for $n \ge 2$ so that

$$(n + c)^{1/(1-p)}\left(\frac{1}{n^{p/(1-p)}} - \frac{1}{(n + 1)^{p/(1-p)} b_n}\right) = \frac{(1 - p)}{p}. \tag{2.16}$$

Moreover, note that we can also rewrite (2.7) for $n \ge 2$ as (with a replaced by c and $r = p$, $c_{p,p} = (p/(1 - p))^p$)

$$\left(\frac{1}{n + c}\left(\sum_{k=1}^{n-1}\prod_{i=k}^{n-1} b_i^{p-1} + 1\right)\right)^{-1/(1-p)} \le \left(\frac{1 - p}{p}\right)^{p/(1-p)} (n + c)^{1/(1-p)}\left(\frac{1}{n^{p/(1-p)}} - \frac{1}{(n + 1)^{p/(1-p)} b_n}\right). \tag{2.17}$$

If we further choose b_1 so that

$$1 = \left(\frac{1 - p}{p}\right)^{p/(1-p)}\left(1 - \frac{1}{2^{p/(1-p)} b_1}\right), \tag{2.18}$$

then, repeating the same process as in the proof of Theorem 1, we find that the induction part (with $c = (1/p - 1)/2$ here) leads back to inequality (2.14) (with $r = p$ and $a = (3 - 1/p)/2$ there) while the initial case (corresponding to $n = 2$ here) is just (2.15), so this approach gives another proof of Theorem 1.

We end this section by pointing out the relation between the treatment in Sections 3 and 4 in [3] on inequality (1.2). We note that it is shown in Section 3 of [3] that, for any $N \geq 1$ and any positive sequence $\mathbf{w}$, we have

$$\sum_{n=1}^{N} a_n^p \leq \sum_{n=1}^{N} w_n \left(\sum_{k=n}^{N} \left(\sum_{i=1}^{k} w_i \right)^{-1/(1-p)} \right)^{1-p} \left(\sum_{k=n}^{N} a_k \right)^p. \tag{2.19}$$

We now use $w_n = \sum_{k=1}^{n} w_k - \sum_{k=1}^{n-1} w_k$ and set (with $\nu_N = 0$)

$$\nu_n = \frac{\sum_{k=n+1}^{N} \left(\sum_{i=1}^{k} w_i \right)^{1/(p-1)}}{\left(\sum_{i=1}^{n} w_i \right)^{1/(p-1)}} \tag{2.20}$$

to see that inequality (2.19) leads to (with $\nu_0 = 0$)

$$\sum_{n=1}^{N} a_n^p \leq \sum_{n=1}^{N} \left((1+\nu_n)^{1-p} - \nu_{n-1}^{1-p} \right) \left(\sum_{k=n}^{N} a_k \right)^p. \tag{2.21}$$

The above inequality is essentially what is used in Section 4 of [3].

3. An Alternative Proof of Theorem 1

In this section we give an alternative proof of Theorem 1, using the following.

Lemma 2 (see, Lemma 2.4 [10]). *Let $\{\lambda_i\}_{i\geq 1}^{\infty}$, $\{a_i\}_{i\geq 1}^{\infty}$ be two sequences of positive real numbers, and let $S_n = \sum_{i=1}^{n} \lambda_i a_i$. Let $0 \neq p < 1$ be fixed and let $\{\mu_i\}_{i\geq 1}^{\infty}$, $\{\eta_i\}_{i\geq 1}^{\infty}$ be two positive sequences of real numbers such that $\mu_i \leq \eta_i$ for $0 < p < 1$ and $\mu_i \geq \eta_i$ for $p < 0$, then for $n \geq 2$*

$$\sum_{i=2}^{n-1} \left(\mu_i - \left(\mu_{i+1}^q - \eta_{i+1}^q \right)^{1/q} \right) S_i^{1/p} + \mu_n S_n^{1/p} \leq \left(\mu_2^q - \eta_2^q \right)^{1/q} \lambda_1^{1/p} a_1^{1/p} + \sum_{i=2}^{n} \eta_i \lambda_i^{1/p} a_i^{1/p}. \tag{3.1}$$

Following the treatment in Section 4 of [3], on first setting $\eta_i = \lambda_i^{-1/p}$, then a change of variables: $\mu_i \mapsto \mu_i \eta_i$ followed by setting $\mu_i^q - 1 = \nu_i$ and lastly a further change of variable: $p \mapsto 1/p$, we can transform inequality (3.1) to the following inequality (with $\nu_1 = 0$ here):

$$\sum_{i=1}^{n-1} \left(\frac{(1+\nu_i)^{1-p}}{\lambda_i^p} - \frac{\nu_{i+1}^{1-p}}{\lambda_{i+1}^p} \right) S_i^p + \frac{(1+\nu_n)^{1-p}}{\lambda_n^p} S_n^p \leq \sum_{i=1}^{n} a_i^p. \tag{3.2}$$

Here the ν_i's are arbitrary nonnegative real numbers for $2 \leq i \leq n$. On setting ν_{n+1} to be any non-negative real number, we deduce immediately from the above inequality the following:

$$\sum_{i=1}^{n}\left(\frac{(1+\nu_i)^{1-p}}{\lambda_i^p}-\frac{\nu_{i+1}^{1-p}}{\lambda_{i+1}^p}\right)S_i^p \leq \sum_{i=1}^{n} a_i^p. \tag{3.3}$$

Now we consider establishing inequality (1.3) for $0 < r < 1$, $1/3 < p < 1/2$ in general, and, as has been pointed out in Section 1, we know this is equivalent to establishing inequality (1.4). Now, in order to establish inequality (1.4), it suffices to consider the cases of (1.4) with the infinite summations replaced by any finite summations, say from 1 to $N \geq 1$ there. We now set $n = N$, $p = q$, $\lambda_i = i^{-r/p}$ in inequality (3.3) to recast it as (with $\nu_1 = 0$, $S_n = \sum_{k=1}^{n} k^{-r/p} a_k$ here)

$$\sum_{n=1}^{N}\left(\frac{(1+\nu_n)^{1/(1-p)}}{n^{r/(1-p)}}-\frac{\nu_{n+1}^{1/(1-p)}}{(n+1)^{r/(1-p)}}\right)S_n^q \leq \sum_{n=1}^{N} a_n^q. \tag{3.4}$$

Comparing the above inequality with (1.4), we see that inequality (1.4) holds as long as we can find non-negative ν_n's (with $\nu_1 = 0$) such that

$$\frac{(1+\nu_n)^{1/(1-p)}}{n^{r/(1-p)}}-\frac{\nu_{n+1}^{1/(1-p)}}{(n+1)^{r/(1-p)}} \geq n^{(p-r)/(1-p)}\left(\frac{p}{1-r}\right)^{p/(1-p)}. \tag{3.5}$$

Now, on setting for $n \geq 2$,

$$\nu_n = \frac{n+a-1}{(1-r)/p}, \tag{3.6}$$

and $y = 1/n$, we see easily that inequality (3.5) can be transformed into inequality (2.14). In the case of $r = p$, we further set $a = (3-1/p)/2$ to see that the validity of (2.14) established for this case in Section 2 ensures the validity of (3.5) for $n \geq 2$. Moreover, with the above chosen ν_2 with $r = p$ and $a = (3-1/p)/2$, the $n = 1$ case of (3.5) is easily seen to be equivalent to inequality (1.5), and hence this provides an alternative proof of Theorem 1.

4. A Generalization of Theorem 1

Let $0 < p < 1$, $\alpha < 1/p$, and let $f(x)$ be a non-negative function. We note the following identity:

$$\int_0^\infty \left(\frac{1}{x^\alpha}\int_x^\infty f(t)t^{\alpha-1}dt\right)^p dx = \left(\frac{p}{1-\alpha p}\right)\int_0^\infty \left(\frac{1}{x^\alpha}\int_x^\infty f(t)t^{\alpha-1}dt\right)^{p-1} f(x)dx. \tag{4.1}$$

In the above expression, we assume, f is taken so that all the integrals converge. The case of $\alpha = 1$ is given in the proof of Theorem 337 of [1], and the general case is obtained by some changes of variables. As in the proof of Theorem 337 of [1], we then deduce the following inequality (with the same assumptions as above):

$$\int_0^\infty \left(\frac{1}{x^\alpha} \int_x^\infty f(t) t^{\alpha-1} dt \right)^p dx \geq \left(\frac{p}{1-\alpha p} \right)^p \int_0^\infty f^p(x) dx. \tag{4.2}$$

The above inequality can also be deduced from Theorem 347 of [1] (see also [11, equation (2.4)]). Following the way how Theorem 338 is deduced from Theorem 337 of [1], we deduce similarly from (4.1) the following inequality for $0 < p < 1,\ 0 < \alpha < 1/p$, and $a_n \geq 0$:

$$\sum_{n=1}^{\infty}{}' \left(\frac{1}{n^\alpha} \sum_{k=n}^{\infty} \left((k+1)^\alpha - k^\alpha \right) a_k \right)^p \geq \left(\frac{\alpha p}{1-\alpha p} \right) \sum_{n=1}^{\infty} \left(\frac{1}{n^\alpha} \sum_{k=n}^{\infty} \left((k+1)^\alpha - k^\alpha \right) a_k \right)^{p-1} a_n. \tag{4.3}$$

The dash over the summation on the left-hand side expression above (and in what follows) means that the term corresponding to $n = 1$ is to be multiplied by $1 + 1/(1-\alpha p)$. It's easy to see here the constant is best possible (on taking $a_n = n^{-1/p-\epsilon}$ and letting $\epsilon \to 0^+$). By Hölder's inequality, the above inequality readily implies the following inequality:

$$\sum_{n=1}^{\infty}{}' \left(\frac{1}{n^\alpha} \sum_{k=n}^{\infty} \left((k+1)^\alpha - k^\alpha \right) a_k \right)^p \geq \left(\frac{\alpha p}{1-\alpha p} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{4.4}$$

We are thus motivated to consider the above inequality with the dash sign removed, and this can be regarded as an analogue of inequality (1.2) with $c_p = (p/(1-p))^p$, which corresponds to the case $\alpha = 1$ here. As in the case of (1.2), such an inequality does not hold for all α and p satisfying $0 < p < 1$ and $0 < \alpha < 1/p$. However, on setting $a_n = n^{-1/p-\epsilon}$ and letting $\epsilon \to 0^+$, one sees easily that if such an inequality holds for certain α and p, then the constant is best possible. More generally, we can consider the following inequality:

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sum_{i=1}^{n} L_\beta^{\alpha-1}(i, i-1)} \sum_{k=n}^{\infty} L_\beta^{\alpha-1}(k \pm 1, k) a_k \right)^p \geq \left(\frac{\alpha p}{1-\alpha p} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{4.5}$$

where the function $L_r(a,b)$ for $a > 0, b > 0, a \neq b$, and $r \neq 0, 1$ (the only case we will concern here) is defined as $L_r^{r-1}(a,b) = (a^r - b^r)/(r(a-b))$. It is known [12, Lemma 2.1] that the function $r \mapsto L_r(a,b)$ is strictly increasing on $\mathbb{R}$. Here we restrict our attention to the plus sign in (4.5) for the case $\beta > 0,\ \max(1,\beta) \leq \alpha$ and to the minus sign in (4.5) for the case $0 < \alpha < 1$ and $\beta \geq \alpha$. Our remark above implies that in either case (note that $L_\beta(1,0)$ is meaningful)

$$\sum_{i=1}^{n} L_\beta^{\alpha-1}(i, i-1) \leq \sum_{i=1}^{n} L_\alpha^{\alpha-1}(i, i-1) = \frac{n^\alpha}{\alpha}. \tag{4.6}$$

As we also have $L_\beta^{\alpha-1}(k \pm 1, k) \geq k^{\alpha-1}$, we see that the validity of (4.5) follows from that of (1.6). We therefore focus on (1.6) from now on, and we proceed as in Section 3 of [3] to see that in order for inequality (1.6) to hold, it suffices to find a sequence $\mathbf{w}$ of positive terms for each $0 < p < 1$, such that for any integer $n \geq 1$

$$\left(\sum_{k=1}^{n} w_k\right)^{1/(p-1)} \leq \left(\frac{\alpha p}{1-\alpha p}\right)^{p/(p-1)} \left(\alpha n^{\alpha-1}\right)^{p/(1-p)} \left(\frac{w_n^{1/(p-1)}}{n^{\alpha p/(1-p)}} - \frac{w_{n+1}^{1/(p-1)}}{(n+1)^{\alpha p/(1-p)}}\right). \tag{4.7}$$

We now choose $\mathbf{w}$ inductively by setting $w_1 = 1$, and for $n \geq 1$

$$w_{n+1} = \frac{n + 1/p - \alpha - 1}{n} w_n. \tag{4.8}$$

The above relation implies that

$$\sum_{k=1}^{n} w_k = \frac{n + 1/p - \alpha - 1}{1/p - \alpha} w_n. \tag{4.9}$$

We now assume $0 < p < 1/2$ and note that, for the so-chosen $\mathbf{w}$, inequality (4.7) follows (with $x = 1/n$) from $f(x) \geq 0$ for $0 \leq x \leq 1$, where

$$f(x) = \left(1 + \left(\frac{1}{p} - \alpha - 1\right)x\right)^{1/(1-p)} - (1+x)^{-\alpha p/(1-p)} - \frac{1-\alpha p}{p} x. \tag{4.10}$$

As $f(0) = f'(0) = 0$, it suffices to show $f''(x) \geq 0$, which is equivalent to showing $g(x) \geq 0$, where

$$g(x) = \left(\frac{(1/p - \alpha - 1)^2}{\alpha((\alpha-1)p+1)}\right)^{(1-p)/(1-2p)} (1+x)^{(2+(\alpha-2)p)/(1-2p)} - \left(1 + \left(\frac{1}{p} - \alpha - 1\right)x\right). \tag{4.11}$$

Now

$$\begin{aligned} g'(x) &= \left(\frac{(1/p - \alpha - 1)^2}{\alpha((\alpha-1)p+1)}\right)^{(1-p)/(1-2p)} \left(\frac{2 + (\alpha-2)p}{1-2p}\right)(1+x)^{(2+(\alpha-2)p)/(1-2p)-1} - \left(\frac{1}{p} - \alpha - 1\right) \\ &\geq \left(\frac{(1/p - \alpha - 1)^2}{\alpha((\alpha-1)p+1)}\right)^{(1-p)/(1-2p)} \left(\frac{2 + (\alpha-2)p}{1-2p}\right) - \left(\frac{1}{p} - \alpha - 1\right) := h(\alpha, p). \end{aligned} \tag{4.12}$$

Suppose now $\alpha \geq 1$, then, when $1/p \geq (\alpha+2)(\alpha+1)/2$, we have $1/p \geq \alpha(\alpha-1)p+2\alpha+1$ since $p < 1/2$ so that both inequalities $1/p-\alpha-1 \geq 1$ and $1/p-\alpha-1 \geq \alpha((\alpha-1)p+1)$ are satisfied. In this case we have

$$h(\alpha,p) \geq \left(\frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)}\right)^{(1-p)/(1-2p)} - \left(\frac{1}{p}-\alpha-1\right) \geq \frac{(1/p-\alpha-1)^2}{\alpha((\alpha-1)p+1)} - \left(\frac{1}{p}-\alpha-1\right) \geq 0. \tag{4.13}$$

It follows that $g'(x) \geq 0$ and as $g(0) \geq 0$, and we conclude that $g(x) \geq 0$, and hence $f(x) \geq 0$. Similar discussion leads to the same conclusion for $0 < \alpha < 1$ when $p \leq 1/(\alpha+2)$. We now summarize our discussions above in the following.

Theorem 2. *Let $0 < p < 1/2$ and $0 < \alpha < 1/p$. Let $h(\alpha,p)$ be defined as in* (4.12). *Inequality* (1.6) *holds for α, p satisfying $h(\alpha,p) \geq 0$. In particular, when $\alpha \geq 1$, inequality* (1.6) *holds for $0 < p \leq 2/((\alpha+2)(\alpha+1))$. When $0 < \alpha \leq 1$, inequality* (1.6) *holds for $0 < p \leq 1/(\alpha+2)$.*

Corollary 2. *Let $0 < p < 1/2$ and $0 < \alpha < 1/p$. Let $h(\alpha,p)$ be defined as in* (4.12). *When $\beta > 0, \max(1,\beta) \leq \alpha$, inequality* (4.5) *holds (where one takes the plus sign) for α, p satisfying $h(\alpha,p) \geq 0$. In particular, inequality* (4.5) *holds for $0 < p \leq 2/((\alpha+2)(\alpha+1))$. When $0 < \alpha < 1, \beta \geq \alpha$, inequality* (4.5) *holds (where one takes the minus sign) for α, p satisfying $h(\alpha,p) \geq 0$. In particular, inequality* (4.5) *holds for $0 < p \leq 1/(\alpha+2)$.*

We note here a special case of the above corollary: the case $0 < \alpha < 1$ and $\beta \to +\infty$ leads to the following inequality, valid for $0 < p \leq 1/(\alpha+2)$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sum_{i=1}^{n} i^{\alpha-1}}\sum_{k=n}^{\infty}k^{\alpha-1}a_k\right)^p \geq \left(\frac{\alpha p}{1-\alpha p}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{4.14}$$

We further note here that if we set $r = \alpha p$ and $a = 0$ in inequality (2.14), then it is reduced to $f(x) \geq 0$ for $f(x)$ defined as in (4.10). Since the case $0 < r < p \leq 1/3$ is known, we need only to be concerned about the case $\alpha \geq 1$ here and we now have the following improvement of the result of Levin and Stečkin [2, Theorem 62].

Corollary 3. *Let $0 < p < 1/2$ and $1 \leq \alpha < 1/p$. Let $h(\alpha,p)$ be defined as in* (4.12). *Inequality* (1.3) *holds for $r = \alpha p$ for α, p satisfying $h(\alpha,p) \geq 0$. In particular, inequality* (1.3) *holds for $r = \alpha p$ for α, p satisfying $0 < p \leq 2/((\alpha+2)(\alpha+1))$.*

Just as Theorem 1 and Corollary 1 are dual versions to each other, our results above can also be stated in terms of their dual versions, and we will leave the formulation of the corresponding ones to the reader.

5. Some Results on l^p Norms of Factorable Matrices

In this section we first state some results concerning the l^p norms of factorable matrices. In order to compare our result to that of weighted mean matrices, we consider the following type of inequalities:

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_n}a_k\right)^p \le U_p\sum_{n=1}^{\infty}a_n^p, \tag{5.1}$$

where $p > 1, U_p$ is a constant depending on p. Here we assume that the two positive sequences (λ_n) and (Λ_n) are independent (in particular, unlike in the weighted mean matrices case, we do not have $\Lambda_n = \sum_{k=1}^{n}\lambda_k$ in general). We begin with the following result concerning the bound for U_p.

Theorem 3. *Let $1 < p < \infty$ be fixed in* (5.1)*. Let a be a constant such that $\Lambda_n + a\lambda_n > 0$ for all $n \ge 1$. Let $0 < L < p$ be a positive constant, and let*

$$b_n = \left(\frac{p-L}{p}\right)\left(1+a\frac{\lambda_n}{\Lambda_n}\right)^{p-1}\frac{\lambda_n}{\Lambda_n} + \frac{\lambda_n}{\lambda_{n+1}}. \tag{5.2}$$

If, for any integer $n \ge 1$, one has

$$\sum_{k=1}^{n}\lambda_k\prod_{i=k}^{n}b_i^{1/(p-1)} \le \frac{p}{p-L}(\Lambda_n + a\lambda_n), \tag{5.3}$$

then inequality (5.1) *holds with $U_p \le (p/(p-L))^p$.*

We point out that the proof of the above theorem is analogue to that of Theorem 3.1 of [7], except that, instead of choosing b_n to satisfy the equation (3.4) in [7], we choose b_n so that

$$\left(\frac{b_n}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\Lambda_n^p = \left(\frac{p-L}{p}\right)(\Lambda_n + a\lambda_n)^{p-1}. \tag{5.4}$$

We will leave the details to the reader, and we point out that, as in the case of weighted mean matrices in [7], we deduce from Theorem 3 the following.

Corollary 4. *Let $1 < p < \infty$ be fixed in* (5.1)*. Let a be a constant such that $\Lambda_n + a\lambda_n > 0$ for all $n \ge 1$. Let $0 < L < p$ be a positive constant such that the following inequality is satisfied for all $n \ge 1$ (with $\Lambda_0 = \lambda_0 = 0$):*

$$\left(\frac{p-L}{p}\right)\left(1+a\frac{\lambda_n}{\Lambda_n}\right)^{p-1} + \frac{\Lambda_n}{\lambda_{n+1}} \le \frac{\Lambda_n}{\lambda_n}\left(1+a\frac{\lambda_n}{\Lambda_n}\right)^{p-1}\left(\left(1-\frac{L}{p}\right)\frac{\lambda_n}{\Lambda_n} + \frac{\Lambda_{n-1}}{\Lambda_n} + a\frac{\lambda_{n-1}}{\Lambda_n}\right)^{1-p}. \tag{5.5}$$

Then inequality (5.1) *holds with $U_p \le (p/(p-L))^p$.*

We now apply the previous corollary to the special case of (5.1) with $\lambda_n = \alpha n^{\alpha-1}$, $\Lambda_n = n^\alpha$ for some $\alpha > 1$. On taking $L = 1/\alpha$ and $a = 0$ in Corollary 4 and setting $y = 1/n$, we see that inequality (1.7) holds as long as we can show for $0 \le y \le 1$

$$\left(\left(1-\frac{1}{p\alpha}\right)\alpha y + (1-y)^{\alpha}\right)^{p-1}\left(\left(1-\frac{1}{p\alpha}\right)\alpha y + (1+y)^{1-\alpha}\right) \le 1. \tag{5.6}$$

We note first that, as $(1-1/p\alpha)\alpha y + (1-y)^\alpha \le (1-1/p\alpha)\alpha y + (1+y)^{1-\alpha}$, we need to have $(1-1/p\alpha)\alpha y + (1-y)^\alpha \le 1$ in order for the above inequality to hold. Taking $y = 1$ shows that it is necessary to have $\alpha \le 1 + 1/p$. In particular, we may assume $1 < \alpha \le 2$ from now on, and it then follows from Taylor expansion that, in order for (5.6) to hold, it suffices to show that

$$\left(1-\frac{1}{p}y + \frac{\alpha(\alpha-1)}{2}y^2\right)^{p-1}\left(1+\left(1-\frac{1}{p}\right)y + \frac{\alpha(\alpha-1)}{2}y^2\right) \le 1. \tag{5.7}$$

We first assume $1 < p \le 2$, and in this case we use

$$\left(1-\frac{1}{p}y + \frac{\alpha(\alpha-1)}{2}y^2\right)^{p-1} \le 1 + (p-1)\left(-\frac{1}{p}y + \frac{\alpha(\alpha-1)}{2}y^2\right) \tag{5.8}$$

to see that (5.7) follows from

$$h_{1,\alpha,p}(y) := \frac{\alpha(\alpha-1)p}{2} - \left(1-\frac{1}{p}\right)^2 + \frac{\alpha(\alpha-1)(p-1)}{2p}(p-2)y + \frac{\alpha^2(\alpha-1)^2}{4}(p-1)y^2 \le 0. \tag{5.9}$$

We now denote $\alpha_1(p) > 1$ as the unique number satisfying $h_{1,\alpha_1,p}(0) = 0$ and $\alpha_2(p) > 1$ the unique number satisfying $h_{1,\alpha_2,p}(1) = 0$ and let $\alpha_0(p) = \min(\alpha_1(p), \alpha_2(p))$. It is easy to see that both $\alpha_1(p)$ and $\alpha_2(p)$ are $\le 1 + 1/p$ and that, for $1 < \alpha \le \alpha_0$, we have $h_{1,\alpha,p}(y) \le 0$ for $0 \le y \le 1$.

Now suppose that $p > 2$, then we recast (5.7) as

$$1+\left(1-\frac{1}{p}\right)y + \frac{\alpha(\alpha-1)}{2}y^2 \le \left(1-\frac{1}{p}y + \frac{\alpha(\alpha-1)}{2}y^2\right)^{1-p}. \tag{5.10}$$

In order for the above inequality to hold for all $0 \le y \le 1$, we must have $\alpha(\alpha-1)y^2/2 \le y/p$. Therefore, we may from now on assume $\alpha(\alpha-1) \le 2/p$. Applying Taylor expansion again, we see that (5.10) follows from the following inequality:

$$\begin{aligned}1+\left(1-\frac{1}{p}\right)y + \frac{\alpha(\alpha-1)}{2}y^2 \le{}& 1 + (1-p)\left(-\frac{1}{p}y + \frac{\alpha(\alpha-1)}{2}y^2\right)\\ &+ \frac{p(p-1)\left(-(1/p)y + (\alpha(\alpha-1)/2)y^2\right)^2}{2}.\end{aligned} \tag{5.11}$$

We can recast the above inequality as

$$h_{2,\alpha,p}(y) := \frac{\alpha(\alpha-1)p}{2} - \frac{1-1/p}{2} + \frac{(p-1)\alpha(\alpha-1)y}{2} - \frac{p(p-1)\alpha^2(\alpha-1)^2y^2}{8} \le 0. \tag{5.12}$$

We now denote $\alpha_0(p) > 1$ as the unique number satisfying $\alpha(\alpha-1) \le 2/p$ and $h_{2,\alpha_0,p}(1) = 0$. It is easy to see that, for $1 < \alpha \le \alpha_0$, we have $h_{2,\alpha,p}(y) \le 0$ for $0 \le y \le 1$. We now summarize our result in the following.

Theorem 4. *Let $p > 1$ be fixed, and let $\alpha_0(p)$ be defined as above, then inequality* (1.7) *holds for* $1 < \alpha \le \alpha_0(p)$.

As we have explained in Section 1, the study of (1.7) is motivated by (1.8). As (1.7) implies (1.8) and the constant $(\alpha p/(\alpha p-1))^p$ there is best possible (see [9]), we see that the constant $(\alpha p/(\alpha p-1))^p$ in (1.7) is also best possible. More generally, we note that inequality (4.7) in [9] proposes to determine the best possible constant $U_p(\alpha,\beta)$ in the following inequality ($\mathbf{a} \in l^p,\ p > 1,\ \beta \ge \alpha \ge 1$):

$$\sum_{n=1}^{\infty}\left|\frac{1}{\sum_{k=1}^{n} L_{\beta}^{\alpha-1}(k,k-1)}\sum_{i=1}^{n} L_{\beta}^{\alpha-1}(i,i-1)a_i\right|^p \le U_p(\alpha,\beta)\sum_{n=1}^{\infty}|a_n|^p. \tag{5.13}$$

We easily deduce from Theorem 4 the following.

Corollary 5. *Keep the notations in the statement of* Theorem 4. *For fixed $p > 1$ and $1 < \alpha \le \alpha_0(p)$, inequality* (5.13) *holds with $U_p(\alpha,\beta) = (\alpha p/(\alpha p-1))^p$ for any $\beta \ge \alpha$.*

Acknowledgments

During this work, the author was supported by postdoctoral research fellowships at Nanyang Technological University (NTU). The author is also grateful to the referee for his/her helpful comments and suggestions.

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