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*Research Article*

# Approaching the Power Logarithmic and Difference Means by Iterative Algorithms Involving the Power Binomial Mean

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Introducing the notion of cross means we give iterative algorithms involving the power binomial mean and converging to the power logarithmic and difference means. At the end, we address a list of open problems derived from our present work.

## 1. Introduction

Throughout this paper, we understand by mean a binary map $m$ between positive real numbers satisfying the following statements:

(i) $m(a,a) = a$, for all $a > 0$ (normalization axiom);

(ii) $m(a,b) = m(b,a)$, for all $a,b > 0$ (symmetry axiom);

(iii) $m(ta,tb) = tm(a,b)$, for all $a,b,t > 0$ (homogeneity axiom);

(iv) $m(a,b)$ is an increasing function in $a$ (and in $b$) (monotonicity axiom);

(v) $m(a,b)$ is a continuous function of $a$ and $b$ (continuity axiom).

A binary mean is also called mean with two variables. Henceforth, we shortly call mean instead of binary mean. The definition of mean with three or more variables can be stated in a similar manner. A mean (resp., map) with four variables will be called 2-binary mean (resp., 2-binary map). For two means $m_1$ and $m_2$, we write $m_1 \leq m_2$ if and only if

$m_1(a,b) \le m_2(a,b)$ for all $a,b > 0$. Two trivial means are $(a,b) \mapsto \min(a,b)$ and $(a,b) \mapsto \max(a,b)$, and every mean $m$ satisfies

$$\min(a,b) \le m(a,b) \le \max(a,b), \tag{1.1}$$

for all $a,b > 0$. We denote min and max the two trivial means which we call lower and upper means, respectively. The standard examples of means satisfying the above requirements are recalled in the following:

(i) Arithmetic mean, $A(a,b) = (a+b)/2$,

(ii) Geometric mean, $G(a,b) = \sqrt{ab}$,

(iii) Harmonic mean, $H(a,b) = 2ab/(a+b)$,

(iv) Logarithmic mean, $L(a,b) = (a-b)/(\ln a - \ln b)$, $a \neq b$, $L(a,a) = a$,

(v) Identric (or exponential) mean, $I(a,b) = (1/e)(b^b/a^a)^{1/b-a}$, $a \neq b$, $I(a,a) = a$,

(vi) Quadratic mean, $K(a,b) = \sqrt{(a^2+b^2)/2}$.

As well known, these means satisfy the following inequalities:

$$\min \le H \le G \le L \le I \le A \le K \le \max. \tag{1.2}$$

A mean $m$ is called strict mean if $m(a,b)$ is strictly increasing in $a$ (and in $b$). Also, every strict mean $m$ satisfies that $m(a,b) = a \Rightarrow a = b$. It is easy to see that the lower and upper means are not strict, while $A$, $G$, $H$, $L$, $I$, $K$ are strict means.

There are many families of means, called power means, which extend the above standard ones. For instance, let $p$ be a real number; we cite

(i) power binomial mean defined by

$$B_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}. \tag{1.3}$$

It is understood that

$$B_{-\infty} := \lim_{p \to -\infty} B_p = \min, \qquad B_{-1} = H, \qquad B_0 = G, \qquad B_1 = A, \qquad B_\infty = \max. \tag{1.4}$$

Notice that

$$B_{1/2}(a,b) = \frac{1}{2}A(a,b) + \frac{1}{2}G(a,b) := H_e(a,b) \tag{1.5}$$

is called the Heron mean,

(ii) power logarithmic mean given by

$$L_p(a,b) = \left(\frac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right)^{1/p} = \left(\frac{1}{b-a}\int_a^b t^p\,dt\right)^{1/p}, \quad a \neq b, \ L_p(a,a) = a. \tag{1.6}$$

The particular special values of $p$ are understood as

$$L_{-\infty} = \min, \qquad L_{-2} = G, \qquad L_{-1} = L, \qquad L_0 = I, \qquad L_1 = A, \qquad L_{\infty} = \max. \tag{1.7}$$

Further, the following inequalities are well known:

$$B_p \le L_p \le A \quad \text{for } p \le 1, \qquad A \le L_p \le B_p \quad \text{for } p \ge 1, \tag{1.8}$$

(iii) power difference mean defined as follows:

$$D_p(a,b) = \frac{p}{p+1}\frac{a^{p+1} - b^{p+1}}{a^p - b^p}, \quad a \ne b, \ D_p(a,a) = a. \tag{1.9}$$

This includes some of the most familiar means in the sense

$$\begin{gathered} D_{-\infty} = \min, \qquad D_{-2} = H, \qquad D_{-1} = \frac{G^2}{L}, \qquad D_{-1/2} = G, \qquad D_0 = L, \\ D_1 = A, \qquad D_{\infty} = \max. \end{gathered} \tag{1.10}$$

It is not hard to see that the above power means $B_p$, $L_p$, and $D_p$ are strict means for all real numbers $p$ $(-\infty < p < +\infty)$.

The remainder of this paper will be organized as follows: after this section, Section 2 contains some new basic notions and results about a class of means, termed cross means. Section 3 is devoted to introduce two adjacent recursive sequences, depending only of $B_p$ and converging to the power logarithmic mean $L_p$. Section 4 displays briefly an analogue of the above section for the power difference mean $D_p$. Finally, Section 5 is focused to address a list of open problems derived from our present work and put as future research for the interested readers.

## 2. Cross Means

In this section, we will introduce the tensor product of two binary means from which we derive the definition of a class of special means termed cross means.

*Definition 2.1.* Let $m_1$ and $m_2$ be two binary means. The tensor product of $m_1$ and $m_2$ is the 2-binary map, denoted $m_1 \otimes m_2$, defined by

$$\forall a,b,c,d > 0 \quad m_1 \otimes m_2(a,b,c,d) = m_1(m_2(a,b),\ m_2(c,d)). \tag{2.1}$$

It is simple to verify that $m_1 \otimes m_2$ and $m_2 \otimes m_1$ are, in general, different. Further, the map $m_1 \otimes m_2$ satisfies all axioms of a 2-binary mean except the symmetry axiom (ii). That is, the tensor product of two binary means is not, in general, a 2-binary mean. For a mean $m$, we write $m^{\otimes 2} := m \otimes m$. To not lengthen this section, we omit the study of the elementary properties of $m_1 \otimes m_2$ not needed later. However, our goal here is to derive the following definition which will be needed in the sequel.

*Definition 2.2.* A binary mean $m$ will be called cross mean if $m^{\otimes 2}$ is a 2-binary mean, that is,

$$\forall a,b,c,d>0 \quad m^{\otimes 2}(a,b,c,d)=m^{\otimes 2}(a,c,b,d). \tag{2.2}$$

By the symmetry axiom (ii) for $m$, relation (2.2) is equivalent to one of the three following equalities:

$$\begin{gathered}\forall a,b,c,d>0 \quad m^{\otimes 2}(a,b,c,d)=m^{\otimes 2}(b,c,a,d),\\ \forall a,b,c,d>0 \quad m^{\otimes 2}(a,b,c,d)=m^{\otimes 2}(a,d,b,c),\\ \forall a,b,c,d>0 \quad m^{\otimes 2}(a,b,c,d)=m^{\otimes 2}(b,d,a,c).\end{gathered} \tag{2.3}$$

It is not hard to see that the two trivial means min and max are cross means. Other examples of cross means are given in the following.

**Theorem 2.3.** *For all real numbers $p$, the power binomial mean $B_p$ is a cross mean.*

*Proof.* According to Definition 2.2, with the explicit form of $B_p$, the desired result follows from an elementary computation. We left the routine detail here. □

**Corollary 2.4.** *The arithmetic, geometric, and harmonic means are cross means.*

*Proof.* Theorem 2.3 can be formulated as follows:

$$\forall a,b,c,d>0 \quad B_p\big(B_p(a,b),B_p(c,d)\big)=B_p\big(B_p(a,c),B_p(b,d)\big), \tag{2.4}$$

from which, setting $p=1$ and $p=-1$, we obtain the announced result for the arithmetic and harmonic means, respectively. Letting $p \to 0$ in the latter formulae, and using (1.4) with an argument of continuity, we deduce the result for the geometric mean. □

From the above theorem, we immediately deduce that the quadratic and Heron means $K$ and $H_e$ are also cross means. However, the logarithmic and identric means are not cross means. The following counterexample shows this latter situation.

*Example 2.5.* According to the above definitions, simple computations yield the following results:

$$\begin{gathered}L^{\otimes 2}(1,2,3,4)=\frac{\ln(8/3)}{(\ln 2)(\ln(3/4))\ln(\ln 2/\ln(3/4))}\neq\frac{\ln(9/4)}{(\ln 2)(\ln 3)\ln(\ln 3/\ln 2)}=L^{\otimes 2}(1,3,2,4),\\ I^{\otimes 2}(1,2,3,4)=\frac{4}{e}\left(\frac{9}{4}\right)^{5/4}\neq\frac{9}{4e}=I^{\otimes 2}(1,3,2,4).\end{gathered} \tag{2.5}$$

The above example, with (1.7) and (1.10), shows that the power logarithmic and difference means are not always cross means. In the next sections, we will approximate $L_p$ and $D_p$ by iterative processes in terms of the cross mean $B_p$.

## 3. Approximation of the Power Logarithmic Mean $L_p$

As already pointed before, our aim in this section is to approximate the noncross mean $L_p$ by iterative scheme involving the cross mean $B_p$. For all positive real numbers $a$, $b$ and all fixed real numbers $p$, define the following iterative algorithms:

$$\begin{gathered}\Upsilon_{p,n+1}(a,b) = B_p\left(\Upsilon_{p,n}\left(a,\frac{a+b}{2}\right),\Upsilon_{p,n}\left(\frac{a+b}{2},b\right)\right),\\ \Upsilon_{p,0}(a,b) = B_p(a,b),\end{gathered} \tag{3.1}$$

$$\begin{gathered}\Theta_{p,n+1}(a,b) = B_p\left(\Theta_{p,n}\left(a,\frac{a+b}{2}\right),\Theta_{p,n}\left(\frac{a+b}{2},b\right)\right),\\ \Theta_{p,0}(a,b) = A(a,b) := \frac{a+b}{2}.\end{gathered} \tag{3.2}$$

By a mathematical induction, it is easy to see that $\Upsilon_{p,n}$ and $\Theta_{p,n}$ are means for all $n \geq 0$. In particular, the symmetry axiom for $\Upsilon_{p,n}$ and $\Theta_{p,n}$ holds, that is,

$$\forall n \geq 0 \quad \Upsilon_{p,n}(a,b) = \Upsilon_{p,n}(b,a), \quad \Theta_{p,n}(a,b) = \Theta_{p,n}(b,a). \tag{3.3}$$

In terms of tensor product, the above recursive relation defining the sequence $(\Upsilon_{p,n}(a,b))_n$ can be written as follows:

$$\Upsilon_{p,n+1}(a,b) = B_p \otimes \Upsilon_{p,n}\left(a,\frac{a+b}{2},\frac{a+b}{2},b\right), \tag{3.4}$$

with analogous form for $(\Theta_{p,n}(a,b))_n$. However, for the sake of simplicity for the reader we omit these tensor writings and we use the recursive forms (3.1) and (3.2) throughout the following.

In what follows, we will study the convergence of the above algorithms. We start with the next result giving a link between the two sequences $(\Upsilon_{p,n}(a,b))_n$ and $(\Theta_{p,n}(a,b))_n$.

**Proposition 3.1.** *With the above, the sequences $(\Upsilon_{p,n}(a,b))_n$ and $(\Theta_{p,n}(a,b))_n$ satisfy the following relationship:*

$$\Upsilon_{p,n+1}(a,b) = B_p\left(\Upsilon_{p,n}(a,b),\Theta_{p,n}(a,b)\right), \tag{3.5}$$

*for all $a,b > 0$ and every $n \geq 0$.*

*Proof.* For $n = 0$, relations (3.1) give

$$\Upsilon_{p,1}(a,b) = B_p\left(B_p\left(a,\frac{a+b}{2}\right),B_p\left(\frac{a+b}{2},b\right)\right). \tag{3.6}$$

According to Theorem 2.3, with the symmetry axiom of $B_p$, we obtain

$$\Upsilon_{p,1}(a,b) = B_p\left(B_p(a,b), B_p\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right) = B_p\left(B_p(a,b), \frac{a+b}{2}\right). \tag{3.7}$$

This, with (3.1) and (3.2), yields

$$\Upsilon_{p,1}(a,b) = B_p(\Upsilon_{p,0}(a,b), \Theta_{p,0}(a,b)). \tag{3.8}$$

By a mathematical induction, the desired result follows with the same arguments as previously mentioned. The proof is complete. □

**Proposition 3.2.** *Assume that $p \le 1$, then, the following inequalities:*

$$B_p(a,b) \le \cdots \le \Upsilon_{p,n-1}(a,b) \le \Upsilon_{p,n}(a,b) \le \Theta_{p,n}(a,b) \le \Theta_{p,n-1}(a,b) \le \cdots \le A(a,b) \tag{3.9}$$

*hold for all $a, b > 0$ and every $n \ge 0$.*

*If $p \ge 1$, the above inequalities are reversed, with equalities for $p = 1$.*

*Proof.* Let $p \le 1$. The map $x \mapsto x^{1/p}$ is convex on $]0, +\infty[$ and so $B_p(a,b) \le A(a,b)$, that is, $\Upsilon_{p,0}(a,b) \le \Theta_{p,0}(a,b)$ for all $a, b > 0$. Using (3.1) and (3.2), we easily show by mathematical induction that, for all $a, b > 0$,

$$\Upsilon_{p,n}(a,b) \le \Theta_{p,n}(a,b), \tag{3.10}$$

for every $n \ge 0$. This, with Proposition 3.1 and the monotonicity axiom of $B_p$, implies that

$$\Upsilon_{p,n+1}(a,b) \ge B_p(\Upsilon_{p,n}(a,b), \Upsilon_{p,n}(a,b)) = \Upsilon_{p,n}(a,b), \tag{3.11}$$

for each $n \ge 0$, that is, $(\Upsilon_{p,n}(a,b))_n$ is an increasing sequence. Now, let us show the decrease monotonicity of $(\Theta_{p,n}(a,b))_n$. By (3.2), we have

$$\Theta_{p,1}(a,b) = B_p\left(A\left(a, \frac{a+b}{2}\right), A\left(\frac{a+b}{2}, b\right)\right), \tag{3.12}$$

which, with $B_p(a,b) \le A(a,b)$, yields

$$\Theta_{p,1}(a,b) \le A\left(A\left(a, \frac{a+b}{2}\right), A\left(\frac{a+b}{2}, b\right)\right), \tag{3.13}$$

and, with the fact that $A$ is a cross mean, we obtain

$$\Theta_{p,1}(a,b) \le A\left(A(a,b), A\left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right) = A(A(a,b), A(a,b)) = A(a,b) = \Theta_{p,0}(a,b), \tag{3.14}$$

for all $a, b > 0$. This, with (3.2) and a simple mathematical induction, gives the decrease monotonicity of $(\Theta_{p,n}(a,b))_n$. The proof of inequalities (3.9) is complete. For $p \geq 1$, the map $x \mapsto x^{1/p}$, $(p \neq 0)$, is concave and all inequalities in the above case are reversed. The proof is completed. □

**Theorem 3.3.** *The sequences $(\Upsilon_{p,n}(a,b))_n$ and $(\Theta_{p,n}(a,b))_n$ both converge to the same limit $L_p(a,b)$, power logarithmic mean of $a$ and $b$, with the following estimations:*

$$\forall n \geq 0 \quad B_p(a,b) \leq \cdots \leq \Upsilon_{p,n}(a,b) \leq L_p(a,b) \leq \Theta_{p,n}(a,b) \leq \cdots \leq A(a,b) \tag{3.15}$$

*if $p \leq 1$, with reversed inequalities if $p \geq 1$ and equalities if $p = 1$.*

*Proof.* By Proposition 3.2 the sequences $(\Upsilon_{p,n}(a,b))_n$ and $(\Theta_{p,n}(a,b))_n$ are monotone and bounded then they converge. Calling $m_p(a,b)$ and $M_p(a,b)$ their limits, respectively, we deduce from Proposition 3.1, with an argument of continuity, that

$$m_p(a,b) = B_p\big(m_p(a,b), M_p(a,b)\big). \tag{3.16}$$

This, with the fact that $B_p$ is a strict mean for all real numbers $p$, yields $m_p(a,b) = M_p(a,b)$, that is, $(\Upsilon_{p,n}(a,b))_n$ and $(\Theta_{p,n}(a,b))_n$ converge with the same limit. Let us prove that this common limit is exactly $L_p(a,b)$. It is sufficient to show that $L_p(a,b)$ is an intermediary mean between the two means $\Upsilon_{p,n}(a,b)$ and $\Theta_{p,n}(a,b)$, for all $n \geq 0$. First, using the integral explicit form of $L_p(a,b)$, it is easy to verify the following relationship:

$$L_p(a,b) = B_p\left(L_p\left(a, \frac{a+b}{2}\right), L_p\left(\frac{a+b}{2}, b\right)\right). \tag{3.17}$$

For $n = 0$, inequalities (1.8) imply that $L_p(a,b)$ is between $\Upsilon_{p,0}(a,b) := B_p(a,b)$ and $\Theta_{p,0}(a,b) := A(a,b)$. Assuming that $p \leq 1$ and using (1.8) again with the recursive relations (3.1) and (3.2), we easily prove with a simple mathematical induction that

$$\Upsilon_{p,n}(a,b) \leq L_p(a,b) \leq \Theta_{p,n}(a,b), \tag{3.18}$$

for all $a, b > 0$ and every $n \geq 0$. Letting $n \to +\infty$ in inequalities (3.18), we deduce that

$$m_p(a,b) \leq L_p(a,b) \leq M_p(a,b), \tag{3.19}$$

with reversed inequalities if $p \geq 1$. This, with the fact that $m_p(a,b) = M_p(a,b)$, yields the desired results. The proof of the theorem is complete. □

We notice that inequalities (3.15) give some iterative refinements of (1.8). Further, the above theorem has many consequences as recited in the two following corollaries.

**Corollary 3.4.** *The sequences $(\Upsilon_{0,n}(a,b))_n$ and $(\Theta_{0,n}(a,b))_n$ converge to the same limit $I(a,b)$, identric mean of $a$ and $b$, with the following relationship:*

$$I(a,b) = G(I(a, A(a,b)), I(A(a,b), b)). \tag{3.20}$$

*Proof.* Setting $p = 0$ in the above theorem, with the sake of convenience,

$$I(a,b) = \lim_{p \to 0} L_p(a,b), \tag{3.21}$$

we obtain the desired result. □

We notice that for $p = 0$, the inequalities (3.15) imply that $G(a,b) \le I(a,b) \le A(a,b)$, which is the known arithmetic-identric-geometric mean inequality, and the relationship (3.20) can be directly verified from the explicit form of $I(a,b)$.

Now, setting $p = -1$ in the previous theorem, we immediately deduce the following result whose proof is similar to that of the above corollary.

**Corollary 3.5.** *The sequences $(\Upsilon_{-1,n}(a,b))_n$ and $(\Theta_{-1,n}(a,b))_n$ both converge to the same limit $L(a,b)$, logarithmic mean of $a$ and $b$, with the relationship*

$$L(a,b) = H(L(a, A(a,b)), L(A(a,b), b)). \tag{3.22}$$

We left to the reader the routine task of formulating, from the above corollaries with (3.1) and (3.2), the relevant iterative algorithms converging, respectively, to the identric and logarithmic means of $a$ and $b$.

## 4. Approximation of the Power Difference Mean $D_p$

We preserve the same notations as in the previous sections. The present section is devoted to approximate the noncross mean $D_p$ in terms of the cross mean $B_p$. For this, we define the following schemes:

$$\begin{gathered}
\Phi_{p,n+1}(a,b) = A\big(\Phi_{p,n}\big(a, B_p(a,b)\big), \Phi_{p,n}\big(B_p(a,b), b\big)\big), \\
\Phi_{p,0}(a,b) = A(a,b), \\
\Psi_{p,n+1}(a,b) = A\big(\Psi_{p,n}\big(a, B_p(a,b)\big), \Psi_{p,n}\big(B_p(a,b), b\big)\big), \\
\Psi_{p,0}(a,b) = B_p(a,b).
\end{gathered} \tag{4.1}$$

Similarly to the above section, $\Phi_{p,n}$ and $\Psi_{p,n}$ are binary means for all $n \ge 0$, and

$$\Phi_{p,n+1}(a,b) = A \otimes \Phi_{p,n}\big(a, B_p(a,b), B_p(a,b), b\big), \tag{4.2}$$

with analogous relation for $(\Psi_{p,n}(a,b))_n$. The study of the convergence of the sequences $(\Phi_{p,n}(a,b))_n$ and $(\Psi_{p,n}(a,b))_n$, together with related properties and common limit, is similar to that of the above section, and we omit the routine details for the reader as an interesting exercise. The main results of this section are summarized in the following statement.

**Theorem 4.1.** *With the above, the following assertions are met.*

(i) *For all $n \geq 0$, $a, b > 0$, and $p$ real number,*

$$\Phi_{p,n+1}(a,b) = A\big(\Phi_{p,n}(a,b), \Psi_{p,n}(a,b)\big). \tag{4.3}$$

(ii) *For $p \geq 1$, the inequalities*

$$A(a,b) \leq \cdots \leq \Phi_{p,n-1}(a,b) \leq \Phi_{p,n}(a,b) \leq \Psi_{p,n}(a,b) \leq \Psi_{p,n-1}(a,b) \leq \cdots \leq B_p(a,b) \tag{4.4}$$

*hold and, if $p \leq 1$ the above inequalities are reversed, with equalities for $p = 1$.*

(iii) *The sequences $(\Phi_{p,n}(a,b))_n$ and $(\Psi_{p,n}(a,b))_n$ both converge to the same limit $D_p(a,b)$, power difference mean of $a$ and $b$, with the following relationship:*

$$D_p(a,b) = A\big(D_p(a, B_p(a,b)), D_p(B_p(a,b), b)\big). \tag{4.5}$$

For $p = 0$ (in the sense $p \to 0$), we recall that (see (1.4) and (1.10)) $B_0(a,b) = G(a,b) = \sqrt{ab}$ and $D_0(a,b) = L(a,b)$. In this case, the above sequences become, respectively,

$$\begin{gathered}\Phi_{0,n+1}(a,b) = \frac{\Phi_{0,n}\left(a, \sqrt{ab}\right) + \Phi_{0,n}\left(\sqrt{ab}, b\right)}{2}, \\ \Phi_{0,0}(a,b) = A(a,b) := \frac{a+b}{2},\end{gathered} \tag{4.6}$$

$$\begin{gathered}\Psi_{0,n+1}(a,b) = \frac{\Psi_{0,n}\left(a, \sqrt{ab}\right) + \Psi_{0,n}\left(\sqrt{ab}, b\right)}{2}, \\ \Psi_{0,0}(a,b) = G(a,b) := \sqrt{ab}.\end{gathered} \tag{4.7}$$

With this, we may state the next result.

**Corollary 4.2.** *The sequences $(\Phi_{0,n}(a,b))_n$ and $(\Psi_{0,n}(a,b))_n$ defined by (4.6) and (4.7) both converge to the same limit $L(a,b)$ logarithmic mean of $a$ and $b$, with the following formulae:*

$$L(a,b) = \prod_{n=1}^{\infty} \left( \frac{a^{1/2^n} + b^{1/2^n}}{2} \right). \tag{4.8}$$

*Proof.* The first part of the corollary follows from the above theorem with the fact that $D_0(a,b) = L(a,b)$. Let us prove the second part. Since $\Phi_{0,n}$ is a mean for all $n \geq 0$, the homogeneity axiom with (4.6) yields

$$\Phi_{0,n+1}(a,b) = \frac{\sqrt{a} + \sqrt{b}}{2} \Phi_{0,n}\left(\sqrt{a}, \sqrt{b}\right), \tag{4.9}$$

for all $n \geq 0$, with similar recursive relation for $(\Psi_{0,n})_n$. By mathematical induction, with $\Phi_{0,0}(a,b) = (a+b)/2$ and $\Psi_{0,0}(a,b) = \sqrt{ab}$, we easily deduce that

$$\begin{aligned}\Phi_{0,n}(a,b) &= \left(\frac{a^{1/2^n} + b^{1/2^n}}{2}\right)\prod_{i=1}^{n}\left(\frac{a^{1/2^i} + b^{1/2^i}}{2}\right),\\ \Psi_{0,n}(a,b) &= (ab)^{1/2^{n+1}}\prod_{i=1}^{n}\left(\frac{a^{1/2^i} + b^{1/2^i}}{2}\right),\end{aligned} \tag{4.10}$$

for every $n \geq 0$. This, when combined with the first part, gives the desired result so completes the proof. □

The explicit formulae of $L(a,b)$, in terms of infinite product, obtained in the above corollary is not obvious to establish directly. However, for $p = 0$, inequalities (4.4) give $G(a,b) \leq L(a,b) \leq A(a,b)$ the known arithmetic-logarithmic-geometric mean inequality, while relationship (4.5) implies that

$$L(a,b) = A(L(a,G(a,b)), L(G(a,b),b)), \tag{4.11}$$

which can be directly verified from the explicit form of $L(a,b)$.

We end this section by stating the following remark showing the interest of the above algorithms and the generality of our approach.

*Remark 4.3.* In the two above sections, we have obtained the following.

(i) The logarithmic mean $L(a,b)$ of $a$ and $b$, containing logarithm, has been approached by two iterative algorithms and explicit formulae involving only the elementary operations sum, product, inverse, and square root of positive real numbers. Such algorithms are simple and practical in the theoretical context as in the numerical purpose.

(ii) The identric mean $I(a,b)$ of $a$ and $b$, having a transcendent expression, is here approached by algorithms of algebraic type, that is, containing only the sum, product, and square root of positive real numbers. Such algorithms are useful for the theoretical study and simple for the numerical computation.

## 5. Motivation and Some Open Problems

As we have already seen, the power binomial mean $B_p$ is a cross mean while the power logarithmic and difference means $L_p$ and $D_p$ are not always cross means. Approximations of $L_p$ and $D_p$ by simple iterative algorithms involving $B_p$ have been discussed. In particular, relationships (3.20), (3.22), and (4.11) are derived from the related algorithms and appear to be new in their brief forms. From this, we may naturally arise the following.

*Problem 1.* (1) Determine the set of all real numbers $p$, such that $L_p$ (resp., $D_p$) is a cross mean.

(2) How to obtain the relationships (3.20), (3.22), and (4.11) under a general point of view.

There are many other scalar means which we have not recalled above. For instance, let $r$, $s$ be two given real numbers and $a, b > 0$, we recall the following.

(i) The Stolarsky mean $E_{r,s}(a,b)$ of order $(r,s)$ of $a$ and $b$ is given by, [1, 2],

$$E_{r,s}(a,b) = \left(\frac{r}{s}\frac{b^s - a^s}{b^r - a^r}\right)^{1/(s-r)}, \qquad E_{r,s}(a,a) = a. \tag{5.1}$$

This includes some of the most familiar cases in the sense

$$E_{r,r}(a,b) = \exp\left(-\frac{1}{r} + \frac{a^r \ln a - b^r \ln b}{a^r - b^r}\right), \qquad E_{r,0}(a,b) = \left(\frac{1}{r}\frac{b^r - a^r}{\ln b - \ln a}\right)^{1/r} \tag{5.2}$$

if $r \neq 0$, with $E_{0,0}(a,b) = G(a,b)$.

(ii) The Gini mean $G_{r,s}(a,b)$ of order $(r,s)$ of $a$ and $b$ is defined by, [3],

$$G_{r,s}(a,b) = \left(\frac{a^s + b^s}{a^r + b^r}\right)^{1/(s-r)}. \tag{5.3}$$

The mean $E_{r,s}$ extends the power binomial, logarithmic, and difference means by virtue of the following relations:

$$E_{p,2p} = B_p, \qquad E_{1,p+1} = L_p, \qquad E_{p,p+1} = D_p, \tag{5.4}$$

for all real numbers $p$. So, $E_{r,s}$ is not always a cross mean. Our second open problem can be recited as follows.

*Problem 2.* (1) Determine the set of all couples $(r,s)$ of real numbers, such that $E_{r,s}$ is a cross mean.

(2) Is it possible to approximate $E_{r,s}$ by an iterative algorithm involving only the power binomial cross mean?

Analogue of the second point of the above problem for the Gini mean $G_{r,s}$ is without any greatest interest since $G_{r,s}$ can be explicitly written in terms of the power binomial cross mean as follows:

$$G_{r,s}(a,b) = \left(\frac{B_s^s}{B_r^r}\right)^{1/(s-r)}. \tag{5.5}$$

Clearly, $G_{0,p} = B_p$ for all real numbers $p$. However, it is not hard to verify by a counterexample that the Gini mean $G_{r,s}$ is not always a cross mean. So, to determine the set of all couples $(r,s)$ such that $G_{r,s}$ is a cross mean is not obvious and appears to be interesting.

Summarizing the above, we have seen that (at least) the means considered in the present paper turn, explicitly or approximately, out of the power binomial cross mean. This needs an interpretation in a general point of view allowing us to put the following general problem.

*Problem 3.* Prove or disprove that every mean m can be, explicitly or approximately, defined in terms of cross means.

The extension of scalar means from the case that the variables are positive real numbers to the case that the variables are positive operators (resp., convex functionals) has extensive several developments and interesting applications, see [4, 5] and the related references cited therein. So, it is natural to put the following.

*Problem 4.* What should be the reasonable analogues of the above notions and results for means with operator (resp., functional) variables?

## References

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