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Research Article

Almost α-Hyponormal Operators with Weyl Spectrum of Area Zero

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We define the class of almost α-hyponormal operators and prove that for an operator T in this class, $(T^*T)^\alpha - (TT^*)^\alpha$ is trace-class and its trace is zero when $\alpha \in (0,1]$ and the area of the Weyl spectrum is zero.

This note is dedicated to Professor Carl M. Pearcy with the occasion of his 75th birthday.

Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space, and let $L(\mathcal{H})$ denote the algebra of all linear bounded operators on $\mathcal{H}$, and for $1 \le p < \infty$, let $\mathcal{C}_p(\mathcal{H})$ denote the p-Schatten class on $\mathcal{H}$. For $K \in \mathcal{C}_p(\mathcal{H})$, the expression $\|K\|_p := (\sum_{n=1}^{\infty} \mu_n(K)^p\ )^{1/p}$, where $\mu_1(K) \ge \mu_2(K) \ge \cdots$ are the singular values of K, is a norm for $p \ge 1$, and is only a quasinorm for $0 < p < 1$ (it does not satisfy the triangle inequality). Nevertheless, the latter case will be used in what follows.

For $T \in L(\mathcal{H})$, $\sigma(T)$ and $\sigma_w(T)$ will denote the spectrum and the Weyl spectrum, respectively. Recall that Weyl spectrum is the union of the essential spectrum, $\sigma_e(T)$, and all bounded components of $\mathbb{C} \setminus \sigma_e(T)$ associated with nonzero Fredholm index. An operator $T \in L(\mathcal{H})$ is called $(\mathcal{C}_p, \alpha)$-*normal* (notation: $T \in N_p^\alpha(\mathcal{H})$) if $C_T^\alpha := (T^*T)^\alpha - (TT^*)^\alpha$ belongs to $\mathcal{C}_p(\mathcal{H})$, and T is called $(\mathcal{C}_p, \alpha)$-*hyponormal* (notation: $T \in H_p^\alpha(\mathcal{H})$) if C_T^α is the sum of a positive definite operator and an operator in $\mathcal{C}_p(\mathcal{H})$, or equivalently, $(C_T^\alpha)_-$ (the negative part of C_T^α) belongs to $\mathcal{C}_p(\mathcal{H})$, where α is a positive number. This note will be concerned with the particular class $H_1^\alpha(\mathcal{H})$, which by some parallelism with some terminology used in [1], would be appropriate to be referred as *almost α-hyponormal* operators.

Voiculescu's [1] generalization of Berger-Shaw inequality gives an estimate for the trace of C_T^1. The result was extended in [2]. The combination of these results will be stated after recalling some terminology and notation. The *rational cyclic multiplicity* of an operator

T in $L(\mathscr{H})$, denoted by $m(T)$, is the smallest cardinal number m with the property that there are m vectors $x_1, \ldots, x_m$ in $\mathscr{H}$ such that

$$\vee\left\{f(T)x_j \mid 1 \leq j \leq m, f \in \mathrm{Rat}(\sigma(T))\right\} = \mathscr{H}, \tag{1}$$

where $\mathrm{Rat}(\sigma(T))$ is the algebra of complex-valued rational functions with poles off $\sigma(T)$.

For a Borel subset $E \subseteq \mathbb{C}$ and $\alpha > 0$, denote $\mu_\alpha(E) = (\alpha/2)\iint_E \rho^{\alpha-1}\, d\rho d\theta$. In particular, μ_2 is the planar Lebesgue measure.

Theorem A (see [1, 2]). *Suppose $T \in H_1^1(\mathscr{H})$. If there exists $K \in \mathcal{C}_2(\mathscr{H})$ such that either $m(T + K) < \infty$ or $\mu_2(\sigma(T+K)) = 0$, then $T \in N_1^1(\mathscr{H})$. Moreover, when $m(T+K) < \infty$,*

$$\mathrm{tr}\left(C_T^1\right) \leq \frac{m(T+K)}{\pi} \cdot\ \mu_2(\sigma(T+K)), \tag{2}$$

and when $\mu_2(\sigma(T+K)) = 0, \mathrm{tr}(C_T^1) \leq 0,$ and consequently, $\mathrm{tr}(C_T^1) = 0$.

In fact, it was observed in [2] that the inequality can be improved by replacing $m(T+K)$ with $\tau(T+K)$, where

$$\tau(S) := \liminf\ [\mathrm{rank}(I-P)SP], \tag{3}$$

and the lim inf is taken over all sequences of finite-rank orthogonal projections such that $P \rightarrow I$ in the strong operator topology.

Corollary B (see [2]). *Let $T \in H_1^1(\mathscr{H})$ such that $\mu_2(\sigma_w(T)) = 0$. Then $T \in N_1^1(\mathscr{H})$ and $\mathrm{tr}(C_T^1) = 0$.*

On the other hand, Berger-Shaw inequality was extended to operators in $H_1^\alpha(\mathscr{H})$ using similar circle of ideas used in [1]. This was done in [3] for the case $\alpha \in [(1/2), 1]$ and later on in [4] for the case $\alpha \in (0, (1/2)]$.

Theorem C (see [3, 4]). *Let $0 < \alpha \leq 1$, and let $T \in H_1^\alpha(\mathscr{H})$ and $K \in \mathcal{C}_{2\alpha}(\mathscr{H})$ with $m(T+K) < \infty$. Then $T \in N_1^\alpha(\mathscr{H})$ and*

$$\mathrm{tr}(C_T^\alpha) \leq \frac{m(T+K)}{\pi} \cdot \mu_{2\alpha}(\sigma(T+K)). \tag{4}$$

The case in which $m(T+K) = \infty$ and $\mu_{2\alpha}(\sigma(T+K)) = 0$ was not discussed in [4] or [3]. It is the goal of this note to make some progress towards this case. We have the following.

Theorem 1. *Let $\alpha \in (0,1)$ and let $T \in H_1^\alpha(\mathscr{H})$ and $K \in \mathcal{C}_\alpha(\mathscr{H})$ with $\mu_{2\alpha}(\sigma(T+K)) = 0$. Then $T \in N_1^\alpha(\mathscr{H})$ and $\mathrm{tr}(C_T^\alpha) = 0$.*

Remark. It would have been desirable that Theorem 1 be proved with the hypothesis that $K \in \mathcal{C}_{2\alpha}(\mathscr{H})$.

Before we prove Theorem 1, we extract a similar consequence to Corollary B.

Corollary 2. *Let* $\alpha \in (0,1]$ *and let* $T \in H_1^\alpha(\mathscr{H})$ *such that* $\mu_2(\sigma_w(T)) = 0$. *Then* $T \in N_1^\alpha(\mathscr{H})$ *and* $\mathrm{tr}\ (C_T^\alpha) = 0$.

Proof. If $\alpha = 1$, then conclusion holds according to Corollary B. Let $\alpha \in (0,1)$. First, a careful inspection of the proof of a result of Stampfli [5] leads to the following. For $T \in L(\mathscr{H})$ and $\alpha > 0$, there exists $K_\alpha \in \mathcal{C}_\alpha(\mathscr{H})$ such that $\sigma(T + K_\alpha) \setminus \sigma_w(T)$ consists of a countable set which clusters only on $\sigma_w(T)$. Therefore $\mu_2(\sigma(T + K_\alpha)) = 0$ and thus Theorem 1 applies. □

The proof of Theorem 1 makes use of the following three inequalities.

Proposition D (Hansen's inequality [6]). *If* $A, B \in L(\mathscr{H})$, $A \geq 0$, $||B|| \leq 1$, *and* $\alpha \in (0,1]$, *then* $B^* A^\alpha B \leq (B^* A B)^\alpha$.

Proposition E (Lowner's inequality [7]). *If* $A, B \in L(\mathscr{H})$, $A \geq B \geq 0$, *and* $\alpha \in (0,1]$, *then* $A^\alpha \geq B^\alpha$.

The following is a consequence of Theorem 3.4 of [8].

Proposition F (Jocic's inequality [8]). *Let* $A, B \in L(\mathscr{H})$, $A,\ B \geq 0$, $\alpha \in (0,1]$, *and* $1 \leq p < \infty$. *If* $A - B \in \mathcal{C}_{\alpha p}(\mathscr{H})$, *then* $A^\alpha - B^\alpha \in \mathcal{C}_p(\mathscr{H})$ *and* $||B^\alpha - A^\alpha||_p \leq ||\ |B - A|^\alpha\ ||_p$.

Proof of Theorem 1. Let $\alpha \in (0,1)$, $T \in H_1^\alpha(\mathscr{H})$, and $K \in \mathcal{C}_\alpha(\mathscr{H})$ with $\mu_{2\alpha}(\sigma(T + K)) = 0$, and assume $m(T + K) = \infty$, otherwise Theorem C implies $T \in N_1^\alpha(\mathscr{H})$.

Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathscr{H}$ and let

$$\mathscr{H}_n = \vee\{r(T + K)e_j \mid\ j = 1, \ldots, n, r \in \mathrm{Rat}(\sigma(T + K))\}. \tag{5}$$

Assume that with respect to the decomposition $\mathscr{H} = \mathscr{H}_n \oplus \mathscr{H}_n^\perp$, operators T and K are written as

$$T = \begin{pmatrix} T_{1n} & T_{2n} \\ T_{3n} & T_{4n} \end{pmatrix}, \qquad K = \begin{pmatrix} K_{1n} & K_{2n} \\ K_{3n} & K_{4n} \end{pmatrix}. \tag{6}$$

Since $\mathscr{H}_n$ is a rationally invariant subspace for $T + K$, we have $T_{3n} + K_{3n} = 0$, and thus $T_{3n} = -K_{3n} \in \mathcal{C}_\alpha(\mathscr{H}_n) \subseteq \mathcal{C}_{2\alpha}(\mathscr{H}_n)$, and $\sigma(T_{1n} + K_{1n}) \subseteq \sigma(T + K)$, which implies $\mu_{2\alpha}(\sigma(T_{1n} + K_{1n})) = 0$.

Let P_n be the orthogonal projection onto $\mathscr{H}_n$, and thus $P_n \uparrow I$ strongly. We will prove next that $T_{1n} \in H_1^\alpha(\mathscr{H}_n)$ by first establishing that

$$P_n C_T^\alpha P_n - C_{T_{1n}}^\alpha = -Q_n' + K_n', \tag{7a}$$

where $Q_n' \in L(\mathscr{H}_n)$ is positive semidefinite and $K_n' \in \mathcal{C}_1(\mathscr{H}_n)$.

Assuming that equality (7a) was already proved and writing $C_T^\alpha = Q + K$ with $Q \geq 0$ and $K \in \mathcal{C}_1(\mathscr{H})$, then we have

$$C_{T_{1n}}^\alpha = P_n Q P_n + P_n K P_n + Q_n' - K_n', \tag{7b}$$

that is, $C_{T_{1n}}^\alpha$ is the sum of $P_n Q P_n + Q_n'$, which is a positive semidefinite operator, and of $P_n K P_n - K_n'$, which is a trace-class operator.

Indeed, the expression $P_n C_T^\alpha P_n - C_{T_{1n}}^\alpha$ can be written as $D_1 - D_2$, where

$$D_1 = P_n (T^* T)^\alpha P_n - \left(T_{1n}^* T_{1n}\right)^\alpha,$$
$$D_2 = P_n (T T^*)^\alpha P_n - \left(T_{1n} T_{1n}^*\right)^\alpha. \tag{8}$$

We can write $D_1 = -Q_n'' + K_n''$, where

$$Q_n'' = \left[(P_n T^* T P_n)^\alpha - P_n (T^* T)^\alpha P_n\right], \tag{9}$$

which according to Hansen's inequality is a positive semidefinite operator, and

$$K_n'' = \left[(P_n T^* T P_n)^\alpha - (P_n T^* P_n T P_n)^\alpha\right], \tag{10}$$

which according to Jocic's inequality is a trace-class operator that satisfies

$$\begin{aligned} \left\|K_n''\right\|_1 &\le \left\||((P_n T^* T P_n - P_n T^* P_n T P_n))|^\alpha\right\|_1 = \left\|(T_{3n}^* T_{3n})^\alpha\right\|_1 \\ &= \left\|T_{3n}^* T_{3n}\right\|_\alpha^\alpha \le \left\|T_{3n}^*\right\|^\alpha \cdot \|T_{3n}\|_\alpha^\alpha \le \|T\|^\alpha \cdot \|T_{3n}\|_\alpha^\alpha. \end{aligned} \tag{11}$$

Concerning operator D_2, we can write $D_2 = Q_n''' + K_n'''$, where

$$Q_n''' = P_n (T T^*)^\alpha P_n - P_n (T P_n T^*)^\alpha P_n, \tag{12}$$

which according to Lowner's inequality is a positive semidefinite operator, and

$$K_n''' = P_n (T P_n T^*)^\alpha P_n - (P_n T P_n T^* P_n)^\alpha = P_n\left[(T P_n T^*)^\alpha - (P_n T P_n T^* P_n)^\alpha\right] P_n, \tag{13}$$

which is also a trace-class operator since

$$\begin{aligned} T P_n T^* - P_n T P_n T^* P_n &= (T P_n T^* - T P_n T^* P_n) + (T P_n T^* P_n - P_n T P_n T^* P_n) \\ &= T P_n T^* (I - P_n) + (I - P_n) T P_n T^* P_n \\ &= T T_{3n}^* + T_{3n} T^* P_n \in \mathcal{C}_\alpha(\mathscr{H}), \end{aligned} \tag{14}$$

and according to Jocic's inequality

$$\begin{aligned} \left\|K_n'''\right\|_1 &\le \left\|(T P_n T^*)^\alpha - (P_n T P_n T^* P_n)^\alpha\right\|_1 \le \left\|\left|\ T T_{3n}^* + T_{3n} T^* P_n\right|^\alpha\right\|_1 \\ &= \left\|T T_{3n}^* + T_{3n} T^* P_n\right\|_\alpha^\alpha \le C\left(\left\|T T_{3n}^*\right\|_\alpha^\alpha + \|T_{3n} T^* P_n\|_\alpha^\alpha\right) \\ &\le C\ \|T\|^\alpha\left(\left\|T_{3n}^*\right\|_\alpha^\alpha + \|T_{3n}\|_\alpha^\alpha\right) = 2C\ \|T\|^\alpha \|T_{3n}\|_\alpha^\alpha. \end{aligned} \tag{15}$$

Therefore,

$$D_2 = Q_n''' + K_n''', \text{ with } Q_n''' \geq 0,\ K_n''' \in C_1(H), \tag{16}$$

and consequently, $D_1 - D_2 = (-Q_n'' + K_n'') - (Q_n''' + K_n''') = -(Q_n'' + Q_n''') + (K_n'' - K_n''')$, where $Q_n'' + Q_n''' =: Q_n'$ is positive semidefinite and $K_n'' - K_n''' =: K_n'$ is trace-class, which establishes equality (7a).

According to (7b), $T_{1n} \in H_1^{\alpha}(\mathscr{H}_n)$, and since $m(T_{1n} + K_{1n}) \leq n$ and $\sigma(T_{1n} + K_{1n}) \subseteq \sigma(T + K)$, Theorem C implies that $\operatorname{tr}(C_{T_{1n}}^{\alpha}) \leq 0$, and furthermore, by replacing T_{1n} with T_{1n}^*, $\operatorname{tr}(C_{T_{1n}}^{\alpha}) = 0$. Furthermore, equality (7a) implies

$$P_n C_T^{\alpha} P_n \leq C_{T_{1n}}^{\alpha} + K_n', \tag{17}$$

which further implies

$$\operatorname{tr}\left(P_n C_T^{\alpha} P_n\right) \leq \operatorname{tr}\left(K_n'\right). \tag{18}$$

Similar utilization of Lowner's and Hansen's inequalities implies that K_n'' and $-K_n'''$ are positive semidefinite, and thus so is $K_n' = K_n'' - K_n'''$. Therefore

$$\operatorname{tr}\left(K_n'\right) \leq \left\|\left(K_n''\right)\right\|_1 + \left\|\left(K_n'''\right)\right\|_1 \leq (1 + 2C)\|T\|^{\alpha}\|T_{3n}\|_{\alpha}^{\alpha}. \tag{19}$$

Since $T_{3n} = -K_{3n} \in \mathcal{C}_p(\mathscr{H}_n)$ and $K_{3n} \to 0$ weakly and both $|T_{3n}|$ and $|T_{3n}^*| \leq \|T\|\ I$, we have $\|T_{3n}\|_{\alpha} \to 0$, and thus $\operatorname{tr}(C_T^{\alpha}) \leq 0$. Replacing T with T^* we conclude that $\operatorname{tr}(C_T^{\alpha}) = 0$. □

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