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Research Article

On the Differentiability of Weak Solutions of an Abstract Evolution Equation with a Scalar Type Spectral Operator

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For the evolution equation $y'(t) = Ay(t)$ with a scalar type spectral operator A in a Banach space, conditions on A are found that are necessary and sufficient for all weak solutions of the equation on $[0,\infty)$ to be strongly infinite differentiable on $[0,\infty)$ or $[0,\infty)$. Certain effects of smoothness improvement of the weak solutions are analyzed.

1. Introduction

Consider the evolution equation

$$y'(t) = Ay(t) \tag{1.1}$$

with a *scalar type spectral operator* A in a complex Banach space X.

Following [1], we understand by a *weak solution* of equation (1.1) on an interval $[0,T)$ $(0 < T \leq \infty)$ such a vector function $y : [0,T) \to X$ that

(1) $y(\cdot)$ is strongly continuous on $[0,T)$;

(2) for any $g^* \in D(A^*)$,

$$\frac{d}{dt}\langle y(t), g^*\rangle = \langle y(t), A^*g^*\rangle, \quad t \in [0,T). \tag{1.2}$$

($D(\cdot)$ is the domain of an operator, A^* is the operator adjoint to A, and $\langle \cdot,\cdot\rangle$ is the pairing between the space X and its dual X^*).

As was shown [2], Theorem 4.2, the *general weak solution* of (1.1) on $[0,T)$ $(0 < T \le \infty)$ is as follows:

$$y(t) = e^{tA} f, \quad t \in [0,T), \ f \in \bigcap_{0\le t<T} D\left(e^{tA}\right), \tag{1.3}$$

the operator exponentials e^{tA}, $0 \le t < T$, defined in the sense of the *operational calculus* for scalar type spectral operators (see Section 2).

Here, we are to find conditions on A that are necessary and sufficient for all *weak solutions* of (1.1) on $[0,\infty)$ to be strongly infinite differentiable on $[0,\infty)$ or $(0,\infty)$.

This goal attained; all the principal results of paper [3] and the corresponding ones of [4] obtain their natural generalizations.

2. Preliminaries

Henceforth, unless specified otherwise, let A be a *scalar type spectral operator* in a complex Banach space X with a norm $\|\cdot\|$, $E_A(\cdot)$ its *spectral measure* (the *resolution of the identity*), and $\sigma(A)$ the operator's *spectrum*, with the latter being the support for the former.

Observe that, in a Hilbert space, the *scalar type spectral operators* are those similar to the *normal* ones [5].

For scalar type spectral operators, there has been developed an *operational calculus* for Borel measurable functions defined on $\sigma(A)$ [6, 7]. With $F(\cdot)$ being such a function, a new scalar type spectral operator

$$F(A) \stackrel{\text{def}}{=} \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{2.1}$$

is defined as follows:

$$\begin{aligned} F(A)f &\stackrel{\text{def}}{=} \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)), \\ D(F(A)) &\stackrel{\text{def}}{=} \left\{ f \in X \mid \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}, \end{aligned} \tag{2.2}$$

where

$$F_n(\cdot) \stackrel{\text{def}}{=} F(\cdot)\chi_{\{\lambda\in\sigma(A) \mid |F(\lambda)|\le n\}}(\cdot), \quad n = 1,2,\ldots, \tag{2.3}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α) and

$$F_n(A) \stackrel{\text{def}}{=} \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1,2,\ldots, \tag{2.4}$$

are bounded scalar type spectral operators on X defined in the same manner as for *normal operators* (see, e.g., [8, 9]).

In particular,

$$A = \int_{\sigma(A)} \lambda \, dE_A(\lambda). \tag{2.5}$$

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* underlying our discourse are exhaustively delineated in [6, 7]. We will outline here a few noteworthy facts.

Due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded [10], that is, there is an $M > 0$ such that

$$\|E_A(\alpha)\| \leq M, \quad \alpha \in \mathcal{B} \tag{2.6}$$

($\mathcal{B}$ is the σ-algebra of Borel sets in the complex plane $\mathbb{C}$).

Observe that $\|\cdot\|$ is used in (2.6) to designate the norm in the space of bounded linear operators on X. We will adhere to this rather common economy of symbols adopting the same notation for the norm in the dual space X^* as well.

For any $f \in X$ and $g^* \in X^*$, let $v(f, g^*, \cdot)$ be the *total variation* of the complex-valued measure $\langle E_A(\cdot) f, g^* \rangle$ on $\mathcal{B}$.

As we discussed in [4, 11], with $F(\cdot)$ being an arbitrary Borel measurable function on $\mathbb{C}$, for any $f \in D(F(A))$, $g^* \in X^*$, and $\delta \in \mathcal{B}$,

$$\int_\delta |F(\lambda)| dv(f, g^*, \lambda) \leq 4M \|E_A(\delta)F(A)f\| \|g^*\|, \tag{2.7}$$

where $M > 0$ is the constant from (2.6).

In particular, for $F(\lambda) = 1$, $\lambda \in \mathbb{C}$, and $\delta = \mathbb{C}$ (or any Borel set containing $\sigma(A)$), we have

$$v(f, g^*, \delta) \leq 4M \|f\| \|g^*\|, \quad f \in X, \; g^* \in X^*. \tag{2.8}$$

Further, for a Borel measurable nonnegative function $F(\cdot)$ on $\mathbb{C}$, a $\delta \in \mathcal{B}$, and a sequence $\{\Delta_n\}_{n=1}^\infty$ of pairwise disjoint Borel sets in $\mathbb{C}$,

$$\begin{aligned} &\int_\delta F(\lambda) dv\left(E_A\left(\cup_{n=1}^\infty \Delta_n\right) f, g^*, \lambda\right) \\ &\quad = \sum_{n=1}^\infty \int_{\delta \cap \Delta_n} F(\lambda) dv\left(E_A(\Delta_n) f, g^*, \lambda\right), \quad f \in X, \; g^* \in X^*. \end{aligned} \tag{2.9}$$

Indeed, since for the spectral measure [6, 7]

$$E_A(\alpha) E_A(\beta) = E_A(\alpha \cap \beta), \quad \alpha, \beta \in \mathcal{B}, \tag{2.10}$$

for the total variation, we have

$$v(E_A(\beta) f, g^*, \alpha) = v(f, g^*, \alpha \cap \beta), \quad \alpha, \beta \in \mathcal{B}. \tag{2.11}$$

Whence, due to *nonnegativity* of $F(\cdot)$ [12],

$$\begin{aligned}
&\int_\delta F(\lambda)dv\left(E_A\left(\cup_{n=1}^\infty \Delta_n\right)f, g^*, \lambda\right)\\
&\quad = \int_{\delta\cap\cup_{n=1}^\infty \Delta_n} F(\lambda)dv(f, g^*, \lambda) = \sum_{n=1}^\infty \int_{\delta\cap\Delta_n} F(\lambda)dv(f, g^*, \lambda)\\
&\quad = \sum_{n=1}^\infty \int_{\delta\cap\Delta_n} F(\lambda)dv(E_A(\Delta_n)f, g^*, \lambda).
\end{aligned} \tag{2.12}$$

We shall need the following regions in the complex plane:

$$\begin{aligned}
\mathcal{L}_{b_+} &\stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \max(0, b_+ \ln|\operatorname{Im}\lambda|)\},\\
\mathcal{L}_{b_-,b_+} &\stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq \min(0, -b_- \ln|\operatorname{Im}\lambda|) \quad \text{or} \quad \operatorname{Re}\lambda \geq \max(0, b_+ \ln|\operatorname{Im}\lambda|)\},
\end{aligned} \tag{2.13}$$

where b_+ and b_- are positive constants.

The terms *spectral measure* and *operational calculus* frequently referred to will be abbreviated to *s.m.* and *o.c.*, respectively.

3. Differentiability of a Particular Weak Solution

Proposition 3.1. *Let* $n = 1, 2, \ldots$ *and* I *be a subinterval of an interval* $[0, T)$ $(0 < T \leq \infty)$. *A weak solution* $y(\cdot)$ *of* (1.1) *on* $[0, T)$ *is* n *times strongly differentiable on* I *if and only if*

$$y(t) \in D(A^n), \quad t \in I, \tag{3.1}$$

in which case,

$$y^{(k)}(t) = A^k y(t), \quad k = 1, 2, \ldots, n, \ t \in I. \tag{3.2}$$

Proof. "Only if" part.

Let $n = 1, 2, \ldots$ and suppose that a weak solution $y(\cdot)$ of (1.1) on $[0, T)$ is n times strongly differentiable on a subinterval $I \subseteq [0, T)$.

Then for any $g^* \in D(A^*)$,

$$\langle y'(t), g^*\rangle = \frac{d}{dt}\langle y(t), g^*\rangle = \langle y(t), A^* g^*\rangle, \quad t \in I. \tag{3.3}$$

Hence, by the *closedness* of the operator A (cf. [1]),

$$y(t) \in D(A), \quad y'(t) = Ay(t), \quad t \in I. \tag{3.4}$$

This concludes proving the "only if" part for $n = 1$, in which case, as is to be noted, the subinterval I can shrink to a *single point* $t \in [0, T)$.

Let $n \geq 2$ and let the interval I be not a *singleton*. Then differentiating (3.4) at an arbitrary $t \in I$, we obtain

$$y''(t) = \lim_{\Delta t \to 0} \frac{y'(t + \Delta t) - y'(t)}{\Delta t} = \lim_{\Delta t \to 0} A \frac{y(t + \Delta t) - y(t)}{\Delta t}, \tag{3.5}$$

with the increments Δt being such that $t + \Delta \in I$.

Since

$$\lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = y'(t), \quad t \in I, \tag{3.6}$$

by the closedness of A, we infer that

$$y'(t) \in D(A), \quad y''(t) = Ay'(t), \quad t \in I. \tag{3.7}$$

Considering (3.4) and (3.7),

$$y(t) \in D\left(A^2\right), \quad y''(t) = A^2 y(t), \quad t \in I. \tag{3.8}$$

Continuing inductively in this manner, we arrive at

$$y(t) \in D\left(A^k\right), \quad y^{(k)}(t) = A^k y(t), \quad k = 1, 2, \ldots, n, \ t \in I. \tag{3.9}$$

"If" part.

Let $y(\cdot)$ be a weak solution of (1.1) on an interval $[0, T)$ $(0 < T \leq \infty)$ such that for an $n = 1, 2, \ldots$ and a subinterval $I \subseteq [0, T)$,

$$y(t) \in D(A^n), \quad t \in I. \tag{3.10}$$

As was discussed (see Section 1),

$$y(t) = e^{tA} f, \quad t \in [0, T), \tag{3.11}$$

with some $f \in \bigcap_{0 \leq t < T} D(e^{tA})$.

The fact that $e^{tA} f \in D(A^n)$, $t \in I$, implies by the properties of the *o.c.* and [2], Proposition 3.1, that, for any $g^* \in X^*$,

$$\int_{\sigma(A)} |\lambda|^k e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) < \infty, \quad k = 1, \ldots, n, \ t \in I. \tag{3.12}$$

Given a $k = 1, \ldots, n$ and an arbitrary $t \in I$, let us choose a segment $[a, b] \subset I$ $(a < b)$ so that t is its midpoint if $0 < t < T$ or $a = 0$ if $t = 0$. For increments Δt such that $a \leq t + \Delta t \leq b$ and any $g^* \in X^*$, we have

$$\begin{aligned}
&\left|\left\langle \frac{A^{k-1}y(t+\Delta t) - A^{k-1}y(t)}{\Delta t} - A^k y(t), g^* \right\rangle\right| \quad \text{by the properties of the } \textit{o.c.}; \\
&\quad = \left|\left\langle \int_{\sigma(A)} \left[\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda}\right] dE_A(\lambda) f, g^* \right\rangle\right| \quad \text{by the properties of the } \textit{o.c.}; \\
&\quad = \left| \int_{\sigma(A)} \left[\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda}\right] d\langle E_A(\lambda) f, g^* \rangle \right| \\
&\quad \leq \int_{\sigma(A)} \left|\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda}\right| dv(f, g^*, \lambda) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by the } \textit{Lebesgue Dominated Convergence Theorem}; \\
&\quad \longrightarrow 0 \quad \text{as } \Delta t \longrightarrow 0.
\end{aligned} \tag{3.13}$$

Indeed, for any $k = 1, \ldots, n$ and an arbitrary $\lambda \in \sigma(A)$,

$$\begin{aligned}
&\left|\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda}\right| \\
&\quad \leq \left|\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t}\right| + \left|\lambda^k e^{t\lambda}\right| \quad \text{by the } \textit{Total Change Theorem}; \\
&\quad \leq 2|\lambda|^k \max_{a \leq s \leq b} e^{s \operatorname{Re}\lambda} \leq 2 \begin{cases} |\lambda|^k e^{a \operatorname{Re}\lambda} & \text{if } \operatorname{Re}\lambda < 0, \\ |\lambda|^k e^{b \operatorname{Re}\lambda} & \text{if } \operatorname{Re}\lambda \geq 0, \end{cases} \quad \text{by (3.12), considering that } a, b \in I; \\
&\qquad \in \mathcal{L}^1(\sigma(A), v(f, g^*, \cdot)), \\
&\left|\frac{\lambda^{k-1}e^{(t+\Delta t)\lambda} - \lambda^{k-1}e^{t\lambda}}{\Delta t} - \lambda^k e^{t\lambda}\right| \longrightarrow 0 \quad \text{as } \Delta t \to 0.
\end{aligned} \tag{3.14}$$

Therefore, for any $g^* \in X^*$,

$$\frac{d^k}{dt^k}\langle y(t), g^* \rangle = \left\langle A^k y(t), g^* \right\rangle, \quad k = 1, \ldots, n, \ t \in I. \tag{3.15}$$

For $\Delta_m := \{\lambda \in \mathbb{C} \mid |\lambda| \le m\}$, $m = 1, 2, \ldots$, let us fix an arbitrary $k = 1, \ldots, n$ and consider the sequence of vector functions

$$y_m(t) := E_A(\Delta_m) A^k y(t) = E_A(\Delta_m) A^k e^{tA} f, \quad m = 1, 2, \ldots, \ t \in I. \tag{3.16}$$

Since $f \in D(A^k e^{tA})$, $t \in I$, by the properties of the *o.c.*,

$$E_A(\Delta_m) A^k e^{tA} f = [A E_A(\Delta_m)]^k e^{tAE_A(\Delta_m)} f, \quad m = 1, 2, \ldots, \ t \in I. \tag{3.17}$$

Due to the boundedness of Δ_m, $m = 1, 2, \ldots$, by the properties of the *o.c.*, $AE_A(\Delta_m)$, $m = 1, 2, \ldots$, is a bounded linear operator on X ($\|AE_A(\Delta_m)\| \le 4Mm$, $m = 1, 2, \ldots$, [7]). Hence, the vector functions

$$y_m(t) = E_A(\Delta_m) A^k e^{tA} f, \quad m = 1, 2, \ldots, \ t \in I, \tag{3.18}$$

are strongly continuous on I.

For an arbitrary segment $[a, b] \subseteq I$, we have

$$\begin{aligned}
&\sup_{a\le t\le b} \left\| A^k e^{tA} f - E_A(\Delta_m) A^k e^{tA} f \right\| \quad \text{by the properties of the } \textit{o.c.}; \\
&\quad = \sup_{a\le t\le b} \left\| \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} \lambda^k e^{t\lambda} dE_A(\lambda) f \right\| \quad \text{as follows from the } \textit{Hahn-Banach Theorem}; \\
&\quad = \sup_{a\le t\le b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left| \left\langle \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} \lambda^k e^{t\lambda} dE_A(\lambda) f, g^* \right\rangle \right| \quad \text{by the properties of the } \textit{o.c.}; \\
&\quad = \sup_{a\le t\le b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left| \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} \lambda^k e^{t\lambda} d\langle E_A(\lambda) f, g^* \rangle \right| \\
&\quad \le \sup_{a\le t\le b} \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} |\lambda|^k e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\quad \le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sup_{a\le t\le b} \left[\int_{\{\lambda\in\sigma(A) \mid |\lambda|>m,\ \operatorname{Re}\lambda\le 0\}} |\lambda|^k e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \right. \\
&\qquad\qquad \left. + \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m,\ \operatorname{Re}\lambda> 0\}} |\lambda|^k e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m,\ \operatorname{Re}\lambda\leq 0\}} |\lambda|^k e^{a\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad+\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m,\ \operatorname{Re}\lambda> 0\}} |\lambda|^m e^{b\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} |\lambda|^k e^{a\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad+\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|>m\}} |\lambda|^k e^{b\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \quad \text{by (2.7);}\\
&\leq 4M\Bigg[\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left\|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>m\})A^k e^{aA}f\right\|\|g^*\|\\
&\qquad+\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \left\|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>m\})A^k e^{bA}f\right\|\|g^*\|\Bigg]\\
&= 4M\left[\left\|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>m\})A^k e^{aA}f\right\| + \left\|E_A(\{\lambda\in\sigma(A) \mid |\lambda|>m\})A^k e^{bA}f\right\|\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by the strong continuity of the } s.m.;\\
&\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.
\end{aligned}
\tag{3.19}
$$

Hence, for any $k = 1,\ldots,n$, the vector function $A^k e^{tA}f$, $t\in I$, is strongly continuous on I, being the uniform limit of the sequence of strongly continuous on I vector functions $\{y_m(\cdot)\}_{m=1}^{\infty}$ on any segment $[a,b]\subseteq I$.

Let us fix an arbitrary $a\in I$ and integrate (3.15) for $k=1$ between a and an arbitrary $t\in I$. Considering the strong continuity of $Ae^{tA}f$, $t\in I$, we have

$$
\left\langle e^{tA}f - e^{aA}f, g^*\right\rangle = \left\langle \int_a^t Ae^{sA}f\,ds, g^*\right\rangle, \quad g^*\in X^*. \tag{3.20}
$$

Whence, as follows from the *Hahn-Banach Theorem*,

$$
e^{tA}f - e^{aA}f = \int_a^t Ae^{sA}f\,ds, \quad t\in I. \tag{3.21}
$$

By the strong continuity of $Ae^{tA}f$, $t\in I$,

$$
\frac{d}{dt}e^{tA}f = Ae^{tA}f, \quad t\in I. \tag{3.22}
$$

Consequently, by (3.15), for $n=2$,

$$
\frac{d}{dt}\left\langle \frac{d}{dt}e^{tA}f, g^*\right\rangle = \left\langle A^2e^{tA}f, g^*\right\rangle, \quad t\in I. \tag{3.23}
$$

Whence, analogously,

$$\frac{d^2}{dt^2}e^{tA}f = A^2e^{tA}f, \quad t \in I. \tag{3.24}$$

Continuing inductively in this manner, we infer that, for any $k = 1, \ldots, n$,

$$\frac{d^k}{dt^k}e^{tA}f = A^ke^{tA}f, \quad t \in I. \tag{3.25}$$

□

Corollary 3.2. *Let I be a subinterval of an interval $[0, T)$ $(0 < T \leq \infty)$. A weak solution $y(\cdot)$ of* (1.1) *on $[0, T)$ is strongly infinite differentiable on I if and only if*

$$y(t) \in C^\infty(A) \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} D(A^n), \quad t \in I, \tag{3.26}$$

in which case,

$$y^{(n)}(t) = A^n y(t), \quad n = 1, 2, \ldots, \ t \in I. \tag{3.27}$$

Thus, we have obtained generalizations of Proposition 4.1 and Corollary 4.1 of [3], respectively.

4. Differentiability of Weak Solutions

Theorem 4.1. *Every weak solution of* (1.1) *on $[0, \infty)$ is strongly infinite differentiable on $[0, \infty)$ if and only if there is a $b_+ > 0$ such that the set $\sigma(A) \setminus \mathcal{L}_{b_+}$ is bounded.*

Proof. "If" part.

Let $b_+ > 0$ be such that the set $\sigma(A) \setminus \mathcal{L}_{b_+}$ is bounded and $y(\cdot)$ a weak solution of (1.1) on $[0, \infty)$. Then (see Section 1)

$$y(t) = e^{tA}f, \quad 0 \leq t < \infty, \tag{4.1}$$

with some $f \in \bigcap_{0 \leq t < \infty} D(e^{tA})$.

For any $n = 1, 2, \ldots$, $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} &\int_{\sigma(A)} |\lambda|^n e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \\ &\quad = \int_{\sigma(A) \setminus \mathcal{L}_{b_+}} |\lambda|^n e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) + \int_{\sigma(A) \cap \mathcal{L}_{b_+}} |\lambda|^n e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) < \infty. \end{aligned} \tag{4.2}$$

Indeed, the former integral is finite due to the boundedness of the set $\sigma(A) \setminus \mathcal{L}_{b_+}$, the finiteness of the measure $v(f, g^*, \cdot)$ and the continuity of the integrated function on $\mathbb{C}$.

For the latter one, we have

$$\begin{aligned}
&\int_{\sigma(A)\cap\mathcal{L}_{b_+}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\le \int_{\sigma(A)\cap\mathcal{L}_{b_+}} [|\operatorname{Re}\lambda|+|\operatorname{Im}\lambda|]^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{for } \lambda\in\sigma(\mathrm{A})\cap\mathcal{L}_{b_+}, \operatorname{Re}\lambda\ge 0,\ |\operatorname{Im}\lambda|\le e^{b_+^{-1}\operatorname{Re}\lambda};\\
&\le \int_{\sigma(A)\cap\mathcal{L}_{b_+}} \left[\operatorname{Re}\lambda + e^{b_+^{-1}\operatorname{Re}\lambda}\right]^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{since } x\le e^x, \quad x\ge 0;\\
&\le \int_{\sigma(A)\cap\mathcal{L}_{b_+}} \left[b_+ e^{b_+^{-1}\operatorname{Re}\lambda} + e^{b_+^{-1}\operatorname{Re}\lambda}\right]^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&= [b_+ + 1]^n \int_{\sigma(A)} e^{[nb_+^{-1}+t]\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{since } f\in\bigcap_{0\le t<\infty} D\left(e^{tA}\right), \text{ by [2], Proposition 3.1;}\\
&<\infty.
\end{aligned} \tag{4.3}$$

Further, for any $n=1,2,\ldots$ and $t\ge 0$,

$$\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad\quad + \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.
\end{aligned} \tag{4.4}$$

Indeed, since, due to the boundedness of the set $\sigma(A)\setminus\mathcal{L}_{b_+}$, the set $\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda} > m\}$ is void for all sufficiently large m's,

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } m\to\infty. \tag{4.5}$$

In addition to this

$$\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{analogously to (4.3);}\\
&\leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} [b_+ + 1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} e^{[nb_+^{-1}+t]\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad \text{since } f \in \bigcap_{0\leq t<\infty} D\left(e^{tA}\right), \text{ by (2.7);}\\
&\leq [b_+ + 1]^n \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M\left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right)\right.\\
&\qquad \left. e^{(nb_+^{-1}+t)A} f\right\| \|g^*\|\\
&= 4M[b_+ + 1]^n \left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right) e^{(nb_+^{-1}+t)A} f\right\|\\
&\qquad \text{by the strong continuity of the } s.m.;\\
&\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.
\end{aligned} \tag{4.6}$$

By the properties of the *o.c.* and [2], Proposition 3.1, (4.2) and (4.4) imply that

$$y(t) = e^{tA} f \in C^\infty(A), \quad 0\leq t<\infty. \tag{4.7}$$

Then, by Corollary 3.2, $y(\cdot)$ is strongly infinite differentiable on $[0,\infty)$.
"Only if" part.
We will prove this part by contrapositive.
Assume that, for any $b_+ > 0$, the set $\sigma(A) \setminus \mathcal{L}_{b_+}$ is unbounded.
In particular, for any $n = 1, 2, \ldots$, unbounded is the set $\sigma(A) \setminus \mathcal{L}_{(2n)^{-1}}$.
Hence, we can choose a sequence of points in the complex plane $\{\lambda_n\}_{n=1}^\infty$ as follows:

$$\begin{gathered}
\lambda_n \in \sigma(A), \quad n = 1, 2, \ldots,\\
\operatorname{Re}\lambda_n < \max\left(0, (2n)^{-1}\ln|\operatorname{Im}\lambda|\right), \quad n = 1, 2, \ldots,\\
\lambda_0 := 0, |\lambda_n| > \max\left[n^4, |\lambda_{n-1}|\right], \quad n = 1, 2, \ldots.
\end{gathered} \tag{4.8}$$

The latter implies in particular that the points λ_n are distinct ($\lambda_i \neq \lambda_j$, $i\neq j$).
Since, for any $n = 1, 2, \ldots$, the set

$$\left\{\lambda\in\mathbb{C} \mid \operatorname{Re}\lambda < \max\left(0, (2n)^{-1}\ln|\operatorname{Im}\lambda|\right),\ |\lambda| > \max\left[n^4, |\lambda_{n-1}|\right]\right\} \tag{4.9}$$

is *open* in $\mathbb{C}$, there exists an $\varepsilon_n > 0$ such that this set along with the point λ_n contains the *open disk* of radius ε_n centered at λ_n

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}. \tag{4.10}$$

Then, for any $\lambda \in \Delta_n$, $n = 1, 2, \ldots$,

$$\operatorname{Re}\lambda < \max\Big(0, (2n)^{-1}\ln|\operatorname{Im}\lambda|\Big), \quad |\lambda| > \max\Big[n^4, |\lambda_{n-1}|\Big]. \tag{4.11}$$

Further, since the points λ_n, $n = 1, 2, \ldots$, are distinct, we can regard the radii of the disks, ε_n, $n = 1, 2, \ldots$, to be small enough so that

$$\begin{gathered} 0 < \varepsilon_n < \frac{1}{n}, \quad n = 1, 2, \ldots, \\ \Delta_i \cap \Delta_j = \emptyset, \quad i \neq j \ (\text{i.e., the disks are } \textit{pairwise disjoint}). \end{gathered} \tag{4.12}$$

Whence, by the properties of the *s.m.*,

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j. \tag{4.13}$$

Observe also, that the subspaces $E_A(\Delta_n)X$, $n = 1, 2, \ldots$, are nontrivial since $\Delta_n \cap \sigma(A) \neq \emptyset$, with Δ_n being an open set.

By choosing a unit vector $e_n \in E_A(\Delta_n)X$ ($\|e_n\| = 1$), $n = 1, 2, \ldots$, we obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i, \quad i, j = 1, 2, \ldots \tag{4.14}$$

(δ_{ij} is the *Kronecker delta symbol*).

As is readily verified, (4.14) implies that the vectors $\{e_1, e_2, \ldots\}$ are linearly independent.

Moreover, there is an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}\big(e_n, \operatorname{span}(\{e_k \mid k = 1, 2, \ldots,\ k \neq n\})\big) \geq \varepsilon, \quad n = 1, 2, \ldots. \tag{4.15}$$

Indeed, the opposite implies the existence of a subsequence $\{d_{n(k)}\}_{k=1}^{\infty}$ such that

$$d_{n(k)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.16}$$

Then, for any $k = 1, 2, \ldots$, by selecting a vector $f_{n(k)} \in \operatorname{span}(\{e_j \mid j = 1, 2, \ldots,\ j \neq n(k)\})$ such that $\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/k$, considering (4.14) and (2.6), we arrive at the contradiction:

$$1 = \left\|e_{n(k)}\right\| = \left\|E_A\left(\Delta_{n(k)}\right)\left(e_{n(k)} - f_{n(k)}\right)\right\| \leq M\left\|e_{n(k)} - f_{n(k)}\right\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.17}$$

As follows from the *Hahn-Banach Theorem*, for any $n = 1, 2, \ldots$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad \left\langle e_i, e_j^* \right\rangle = \delta_{ij} d_i, \quad i, j = 1, 2, \ldots. \tag{4.18}$$

Concerning the sequence of the real parts, $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either bounded or unbounded above.

Suppose that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is bounded above, that is, there is such an $\omega > 0$ that

$$\operatorname{Re}\lambda_n \leq \omega, \quad n = 1, 2, \ldots. \tag{4.19}$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n. \tag{4.20}$$

By (4.14),

$$E_A\left(\cup_{n=1}^{\infty} \Delta_n\right) f = f, \quad E_A(\Delta_n) f = \frac{1}{n^2} e_n, \quad n = 1, 2, \ldots. \tag{4.21}$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \quad \text{by (4.21);} \\
&\quad = \int_{\sigma(A)} e^{t \operatorname{Re}\lambda} dv\left(E_A\left(\cup_{n=1}^{\infty} \Delta_n\right) f, g^*, \lambda\right) \quad \text{by (2.9);} \\
&\quad = \sum_{n=1}^{\infty} \int_{\Delta_n} e^{t \operatorname{Re}\lambda} dv\left(E_A(\Delta_n) f, g^*, \lambda\right) \quad \text{by (4.21);} \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\Delta_n} e^{t \operatorname{Re}\lambda} dv(e_n, g^*, \lambda) \\
&\qquad\qquad \text{for } \lambda \in \Delta_n, \text{by (4.12), and (4.19)}, \operatorname{Re}\lambda = \operatorname{Re}\lambda_n + (\operatorname{Re}\lambda - \operatorname{Re}\lambda_n) \\
&\qquad\qquad\qquad\qquad \leq \operatorname{Re}\lambda_n + |\lambda - \lambda_n| \leq \omega + \varepsilon_n \leq \omega + 1; \\
&\quad \leq e^{t(\omega+1)} \sum_{n=1}^{\infty} \frac{1}{n^2} v(e_n, g^*, \Delta_n) \quad \text{by (2.8);} \\
&\quad \leq e^{t(\omega+1)} \sum_{n=1}^{\infty} \frac{1}{n^2} 4M \|e_n\| \|g^*\| = 4M e^{t(\omega+1)} \|g^*\| \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\end{aligned} \tag{4.22}$$

Similarly, for any $t \geq 0$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
&\quad \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{t(\omega+1)} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\Delta_n \cap \{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1\, dv(e_n, g^*, \lambda) \quad \text{by (2.9) and (4.21);} \\
&\quad = e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 v(f, g^*, \lambda) \quad \text{by (2.7);} \\
&\quad \leq e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\right\}\right) f \right\| \|g^*\| \\
&\quad = 4M e^{t(\omega+1)} \left\| E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\right\}\right) f \right\| \quad \text{by the strong continuity of the } s.m.; \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.23}$$

By [2], Proposition 3.1, (4.22) and (4.23) imply that $f \in \bigcap_{0 \leq t < \infty} D(e^{tA})$.

Therefore (see Section 1), $y(t) := e^{tA} f$, $0 \leq t < \infty$, is a weak solution of (1.1) on $[0, \infty)$.

Let

$$h^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^* \in X^*, \tag{4.24}$$

(cf. (4.18)).

Considering (4.18) and (4.15), we have

$$\langle e_n, h^* \rangle = \frac{d_n}{n^2} \geq \frac{\varepsilon}{n^2}, \quad n = 1, 2, \ldots. \tag{4.25}$$

Similarly to (4.22),

$$\begin{aligned}
\int_{\sigma(A)} |\lambda| dv(f, h^*, \lambda) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\Delta_n} |\lambda| dv(e_n, h^*, \lambda) \quad \text{for } \lambda \in \Delta_n, \text{ by (4.11), } |\lambda| \geq n^4; \\
&\geq \sum_{n=1}^{\infty} n^2 v(e_n, h^*, \Delta_n) \geq \sum_{n=1}^{\infty} n^2 |\langle E_A(\Delta_n) e_n, h^* \rangle| \quad \text{by (4.14) and (4.25);} \\
&\geq \sum_{n=1}^{\infty} n^2 \frac{\varepsilon}{n^2} = \infty.
\end{aligned} \tag{4.26}$$

Whence, by [2], Proposition 3.1, $y(0) = f \notin D(A)$.

Consequently, by Proposition 3.1, the weak solution $y(t) = e^{tA}f$, $0 \le t < \infty$, of (1.1) on $[0, \infty)$ is not strongly differentiable at 0.

Now, assume that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is unbounded above.

Therefore, there is a subsequence $\{\operatorname{Re}\lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re}\lambda_{n(k)} \ge k, \quad k = 1, 2, \dots. \tag{4.27}$$

Then the vector

$$f := \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e_{n(k)} \tag{4.28}$$

is well defined.

By (4.14),

$$\begin{gathered} E_A\left(\cup_{k=1}^{\infty}\Delta_{n(k)}\right)f = f, \\ E_A\left(\Delta_{n(k)}\right)f = e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e_{n(k)}, \quad k = 1, 2, \dots. \end{gathered} \tag{4.29}$$

For any $t \ge 0$ and an arbitrary $g^* \in X^*$, similar to (4.22),

$$\begin{aligned} \int_{\sigma(A)} e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) &= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} e^{t\operatorname{Re}\lambda} dv(e_{n(k)}, g^*, \lambda) \\ &\qquad \text{for } \lambda \in \Delta_{n(k)}, \text{ by (4.12)}, \operatorname{Re}\lambda = \operatorname{Re}\lambda_{n(k)} + \left(\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}\right) \\ &\qquad\qquad \le \operatorname{Re}\lambda_{n(k)} + \left|\lambda - \lambda_{n(k)}\right| \le \operatorname{Re}\lambda_{n(k)} + 1; \\ &\le e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} v\left(e_{n(k)}, g^*, \Delta_{n(k)}\right) \quad \text{by (2.8);} \\ &\le e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} 4M \left\|e_{n(k)}\right\| \left\|g^*\right\| \\ &= 4Me^t \left\|g^*\right\| \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \quad \text{by the } \textit{Comparison Test;} \\ &< \infty. \end{aligned} \tag{4.30}$$

Indeed, for all sufficiently large k's, $n(k) - t \ge 1$ and by (4.27),

$$e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \le e^{-k}. \tag{4.31}$$

Analogously, for an arbitrary $t \geq 0$,

$$\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \leq \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} e^t \sum_{k=1}^{\infty} e^{-[n(k)-t]\operatorname{Re}\lambda_{n(k)}} \\
&\int_{\Delta_{n(k)}\cap\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(e_{n(k)},g^*,\lambda\right) = e^t \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}} e^{-(n(k)/2)\operatorname{Re}\lambda_{(k)}} \\
&\int_{\Delta_{n(k)}\cap\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(e_{n(k)},g^*,\lambda\right) \\
&\qquad\qquad \text{due to (4.27), there is an L} > 0 \text{ such that } e^{-[(n(k)/2)-t]\operatorname{Re}\lambda_{n(k)}} \leq L, \quad k=1,2,\ldots; \\
&\quad \leq Le^t \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}} \\
&\int_{\Delta_{n(k)}\cap\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(e_{n(k)},g^*,\lambda\right) \quad \text{for } h := \sum_{k=1}^{\infty} e^{-(n(k)/2)\operatorname{Re}\lambda_{n(k)}} e_{n(k)}, \text{by (4.14) and (2.9);} \\
&\quad = Le^t \int_{\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda}>n\}} 1\, dv\left(h,g^*,\lambda\right) \quad \text{by (2.7);} \\
&\quad \leq Le^t \sup_{\{g^*\in \mid \|g^*\|=1\}} 4M \left\| E_A\left(\left\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\right\}\right) h \right\| \|g^*\| \\
&\quad = 4LMe^t \left\| E_A\left(\left\{\lambda\in\sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\right\}\right) h \right\| \quad \text{by the strong continuity of the } s.m.; \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.32}$$

From (4.30) and (4.32), by [2], Proposition 3.1, we infer that $f \in \bigcap_{0\leq t<\infty} D(e^{tA})$. Therefore (see Section 1), $y(t) := e^{tA}f$, $0 \leq t < \infty$, is a weak solution of (1.1) on $[0,\infty)$. For any $\lambda \in \Delta_{n(k)}$, $k = 1,2,\ldots$, by (4.11), (4.12), and (4.27),

$$\begin{aligned}
\operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} - \left(\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right) \\
&\geq \operatorname{Re}\lambda_{n(k)} - \left|\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right| \\
&\geq \operatorname{Re}\lambda_{n(k)} - \varepsilon_{n(k)} \geq \operatorname{Re}\lambda_{n(k)} - \frac{1}{n(k)} \geq k-1 \geq 0, \\
&\operatorname{Re}\lambda < \max\left(0, (2n(k))^{-1}\ln|\operatorname{Im}\lambda|\right).
\end{aligned} \tag{4.33}$$

Hence, for $\lambda \in \Delta_{n(k)}$, $k = 1,2,\ldots$,

$$|\lambda| \geq |\operatorname{Im}\lambda| \geq e^{2n(k)\operatorname{Re}\lambda} \geq e^{2n(k)(\operatorname{Re}\lambda_{n(k)}-1/n(k))}. \tag{4.34}$$

Using this estimate, we obtain

$$\begin{aligned}
\int_{\sigma(A)} |\lambda| dv(f, h^*, \lambda) &= \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} |\lambda| dv(e_{n(k)}, h^*, \lambda) \\
&\geq \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{2n(k)(\operatorname{Re}\lambda_{n(k)} - 1/n(k))} v(e_{n(k)}, h^*, \Delta_{n(k)}) \\
&= e^{-2} \sum_{k=1}^{\infty} e^{n(k)\operatorname{Re}\lambda_{n(k)}} \left| \langle E_A(\Delta_{n(k)}) e_{n(k)}, h^* \rangle \right| \quad \text{by (4.14), (4.25), and (4.27);} \\
&\geq e^{-2} \varepsilon \sum_{k=1}^{\infty} \frac{e^{n(k)}}{n(k)^2} = \infty.
\end{aligned} \tag{4.35}$$

Whence, by [2], Proposition 3.1, $y(0) = f \notin D(A)$.

Therefore, by Proposition 3.1, the weak solution $y(t) = e^{tA} f$, $0 \leq t < \infty$, of (1.1) on $[0, \infty)$ is not strongly differentiable at 0.

With every possibility concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ considered, we infer that the opposite to the assumption that, for a certain $b_+ > 0$, the set $\sigma(A) \setminus \mathcal{L}_{b_+}$ is bounded, allows to single out a weak solution of (1.1) on $[0, \infty)$ that is not strongly differentiable at 0, much less strongly infinite differentiable on $[0, \infty)$.

Thus, the "only if" part has been proved by contrapositive. □

Theorem 5.1 of [3] has been generalized.

Theorem 4.2. *Every weak solution of* (1.1) *on* $[0, \infty)$ *is strongly infinite differentiable on* $(0, \infty)$ *if and only if there is a* $b_+ > 0$ *such that, for any* $b_- > 0$, *the set* $\sigma(A) \setminus \mathcal{L}_{b_-, b_+}$ *is bounded.*

Proof. "If" part.

Let $b_+ > 0$ be such that, for an arbitrary $b_- > 0$, the set $\sigma(A) \setminus \mathcal{L}_{b_-, b_+}$ is bounded.

Let $y(\cdot)$ be a weak solution of (1.1) on $[0, \infty)$. Then (see Section 1)

$$y(t) = e^{tA} f, \quad 0 \leq t < \infty, \tag{4.36}$$

with some $f \in \bigcap_{0 \leq t < \infty} D(e^{tA})$.

For any $n = 1, 2, \ldots, t > 0$, and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\quad = \int_{\sigma(A) \setminus \mathcal{L}_{b_-, b_+}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) + \int_{\{\lambda \in \sigma(A) \cap \mathcal{L}_{b_-, b_+} | \operatorname{Re}\lambda \geq 0\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\qquad + \int_{\{\lambda \in \sigma(A) \cap \mathcal{L}_{b_-, b_+} | \operatorname{Re}\lambda < 0\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) < \infty.
\end{aligned} \tag{4.37}$$

The first integral in this sum is finite due the boundedness of the set $\sigma(A) \setminus \mathcal{L}_{b_-, b_+}$, the finiteness of the measure $v(f, g^*, \cdot)$, and the continuity of the integrated function on $\mathbb{C}$.

The finiteness of the second integral is proved in absolutely the same manner as for the corresponding integral in the proof of the "if" part of Theorem 4.1.

Finally for the third one, considering that $b_- > 0$ is arbitrary and setting $b_- := nt^{-1}$, we have

$$\begin{aligned}
&\int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} |\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad\le \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} [|\operatorname{Re}\lambda| + |\operatorname{Im}\lambda|]^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \quad \text{for } \lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } \operatorname{Re}\lambda<0,\ |\operatorname{Im}\lambda|\le e^{b_-^{-1}[-\operatorname{Re}\lambda]};\\
&\quad\le \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} \left[-\operatorname{Re}\lambda + e^{b_-^{-1}[-\operatorname{Re}\lambda]}\right]^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \quad \text{since } x\le e^x,\ x\ge 0;\\
&\quad\le \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} \left[b_- e^{b_-^{-1}[-\operatorname{Re}\lambda]} + e^{b_-^{-1}[-\operatorname{Re}\lambda]}\right]^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad= [b_-+1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} e^{[nb_-^{-1}-t][-\operatorname{Re}\lambda]}\,dv(f,g^*,\lambda) \quad \text{recall that } b_- := nt^{-1};\\
&\quad= [b_-+1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid \operatorname{Re}\lambda<0\}} 1\,dv(f,g^*,\lambda) \le [b_-+1]^n v(f,g^*,\sigma(A)) \quad \text{by (2.8)};\\
&\quad\le 4M[b_-+1]^n \|f\|\|g^*\| < \infty.
\end{aligned} \tag{4.38}$$

Further, for any $n = 1, 2, \ldots$ and $t > 0$,

$$\begin{aligned}
&\sup_{\{g^*\in X^*\mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad\le \sup_{\{g^*\in X^*\mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_-,b_+}\mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\qquad+ \sup_{\{g^*\in X^*\mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+}\mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda)\\
&\quad\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.
\end{aligned} \tag{4.39}$$

Indeed, since, due to the boundedness of the set $\sigma(A)\setminus\mathcal{L}_{b_-,b_+}$, the set $\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_-,b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda} > m\}$ is void for all sufficiently large m's,

$$\sup_{\{g^*\in X^*\mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\setminus\mathcal{L}_{b_-,b_+}\mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda}\,dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } m\longrightarrow\infty. \tag{4.40}$$

Similarly to the "if" part of Theorem 4.1 and (4.38), we have

$$
\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} |\lambda|^n e^{t\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\quad\le \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} [b_+ +1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} e^{[nb_+^{-1}+t]\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad+ \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} [b_- +1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda< 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}}\\
&\qquad e^{[nb_-^{-1}-t][-\operatorname{Re}\lambda]} dv(f,g^*,\lambda) \quad \text{with } b_- := nt^{-1};\\
&\quad= \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} [b_+ +1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} e^{[nb_+^{-1}+t]\operatorname{Re}\lambda} dv(f,g^*,\lambda)\\
&\qquad+ \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} [b_- +1]^n \int_{\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda< 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\}} 1 dv(f,g^*,\lambda)\\
&\qquad\qquad \text{since } f\in \bigcap_{0\le t<\infty} D\left(e^{tA}\right), \text{ by (2.7);}\\
&\quad\le [b_+ +1]^n \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M \left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right)\right.\\
&\qquad\qquad \left. e^{[nb_+^{-1}+t]A} f\right\| \|g^*\|\\
&\qquad+ [b_- +1]^n \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} 4M \left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda< 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right) f\right\| \|g^*\|\\
&\quad= 4M[b_+ +1]^n \left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda\ge 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right) e^{[nb_+^{-1}+t]A} f\right\|\\
&\qquad+ 4M[b_- +1]^n \left\| E_A\left(\left\{\lambda\in\sigma(A)\cap\mathcal{L}_{b_-,b_+} \mid \operatorname{Re}\lambda< 0,\ |\lambda|^n e^{t\operatorname{Re}\lambda}>m\right\}\right) f\ \right\|\\
&\qquad\qquad \text{by the strong continuity of the } s.m.;\\
&\quad\longrightarrow 0 \quad \text{as } m\longrightarrow\infty.
\end{aligned}
\tag{4.41}
$$

By the properties of the *o.c.* and [2], Proposition 3.1, (4.37) and (4.39) imply that

$$
y(t) = e^{tA} f \in C^\infty(A), \quad 0<t<\infty. \tag{4.42}
$$

Whence, by Corollary 3.2, $y(\cdot)$ is strongly infinite differentiable on $(0,\infty)$.

"Only if" part.

As well as in Theorem 4.1, we will prove this part by contrapositive.

Thus, we assume that, for any $b_+>0$, there is such a $b_->0$ that the set $\sigma(A)\setminus\mathcal{L}_{b_-,b_+}$ is unbounded.

Let us show that this assumption can even be strengthened. To wit: there is such a $b_->0$ that, for any $b_+>0$, the set $\sigma(A)\setminus\mathcal{L}_{b_-,b_+}$ is unbounded.

Indeed, there are two possibilities:

(1) for a certain $b_- > 0$, the set $\{\lambda \in \sigma(A) \mid -b_- \ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \leq 0\}$ is unbounded;

(2) for any $b_- > 0$, the set $\{\lambda \in \sigma(A) \mid -b_- \ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \leq 0\}$ is bounded.

In the first case, the set $\sigma(A) \setminus \mathcal{L}_{b_-,b_+}$ is unbounded with the very $b_- > 0$, for which the set $\{\lambda \in \sigma(A) \mid -b_- \ln|\operatorname{Im}\lambda| < \operatorname{Re}\lambda \leq 0\}$ is unbounded, and an arbitrary $b_+ > 0$.

In the second case, based on the premise we infer that, for any $b_+ > 0$, the set $\{\lambda \in \sigma(A) \mid 0 < \operatorname{Re}\lambda < b_+ \ln|\operatorname{Im}\lambda|\}$ is unbounded. Then so is the set $\sigma(A) \setminus \mathcal{L}_{b_-,b_+}$ for any $b_- > 0$ and $b_+ > 0$.

Thus, let us fix a $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{L}_{b_-,b_+}$ is unbounded for an arbitrary $b_+ > 0$.

In particular, for any $n = 1, 2, \ldots$, the set $\sigma(A) \setminus \mathcal{L}_{b_-,(2n)^{-1}}$ is unbounded.

Hence, we can select a sequence of points in the complex plane $\{\lambda_n\}_{n=1}^{\infty}$ in the following manner:

$$\begin{gathered}\lambda_n \in \sigma(A), \quad n = 1, 2, \ldots, \\ \min(0, -b_- \ln|\operatorname{Im}\lambda|) < \operatorname{Re}\lambda_n < \max\Big(0, (2n)^{-1} \ln|\operatorname{Im}\lambda|\Big) \quad n = 1, 2, \ldots, \\ \lambda_0 := 0, \ |\lambda_n| > \max\Big[n^4, |\lambda_{n-1}|\Big], \quad n = 1, 2, \ldots.\end{gathered} \tag{4.43}$$

The latter, in particular, implies that the points λ_n are distinct.

Since, for any $n = 1, 2, \ldots$, the set

$$\Big\{\lambda \in \mathbb{C} \mid \min(0, -b_- \ln|\operatorname{Im}\lambda|) < \operatorname{Re}\lambda < \max\Big(0, (2n)^{-1} \ln|\operatorname{Im}\lambda|\Big), |\lambda| > \max\Big[n^4, |\lambda_{n-1}|\Big]\Big\} \tag{4.44}$$

is open in $\mathbb{C}$, there exists such an $\varepsilon_n > 0$ that this set, along with the point λ_n, contains the *open disk* of radius ε_n centered at λ_n

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}. \tag{4.45}$$

Hence, for any $\lambda \in \Delta_n$, $n = 1, 2, \ldots$,

$$\begin{gathered}\min(0, -b_- \ln|\operatorname{Im}\lambda|) < \operatorname{Re}\lambda < \max\Big(0, (2n)^{-1} \ln|\operatorname{Im}\lambda|\Big), \\ |\lambda| > \max\Big[n^4, |\lambda_{n-1}|\Big].\end{gathered} \tag{4.46}$$

Since the points λ_n are distinct, we can regard the radii of the disks, ε_n, to be small enough so that

$$0 < \varepsilon_n < \frac{1}{n}, \quad n = 1, 2, \ldots, \quad \Delta_i \cap \Delta_j = \emptyset, \quad i \neq j. \tag{4.47}$$

By the properties of the *s.m.*, the latter implies

$$E_A(\Delta_i) E_A(\Delta_j) = 0, \quad i \neq j. \tag{4.48}$$

Observe that the subspaces $E_A(\Delta_n)X$, $n = 1, 2, \ldots$, are nontrivial since $\Delta_n \cap \sigma(A) \neq \emptyset$, Δ_n being an open set.

By choosing a unit vector $e_n \in E_A(\Delta_n)X$ ($\|e_n\| = 1$), $n = 1, 2, \ldots$, we obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i. \tag{4.49}$$

In the same manner as in the proof of Theorem 4.1, one can show that there is an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}\big(e_n,\ \operatorname{span}(\{e_k \mid k = 1, 2, \ldots,\ k \neq n\})\big) \geq \varepsilon, \quad n = 1, 2, \ldots. \tag{4.50}$$

As follows from the *Hahn-Banach Theorem*, for any $n = 1, 2, \ldots$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \qquad \left\langle e_i, e_j^* \right\rangle = \delta_{ij}d_i. \tag{4.51}$$

Concerning the sequence of the real parts, $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either bounded or unbounded.

First, assume that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is bounded, that is, there is an $\omega > 0$ such that

$$|\operatorname{Re}\lambda_n| \leq \omega, \quad n = 1, 2, \ldots. \tag{4.52}$$

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n. \tag{4.53}$$

By (4.49),

$$E_A\big(\cup_{n=1}^{\infty}\Delta_n\big)f = f, \quad E_A(\Delta_n)f = \frac{1}{n^2}e_n, \quad n = 1, 2, \ldots. \tag{4.54}$$

In absolutely the same fashion as it was done in the case of bounded above sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ in the proof of the "only if" part of Theorem 4.1, it is shown that $f \in \bigcap_{0 \leq t < \infty} D(e^{tA})$.

Therefore (see Section 1), $y(t) := e^{tA}f$, $0 \leq t < \infty$, is a weak solution of (1.1) on $[0, \infty)$.

Let

$$h^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^* \in X^* \tag{4.55}$$

(cf. (4.51)).

Taking (4.50) and (4.51) into account, we have

$$\langle e_n, h^* \rangle = \frac{d_n}{n^2} \geq \frac{\varepsilon}{n^2}, \quad n = 1, 2, \ldots. \tag{4.56}$$

Thus,

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda| e^{\operatorname{Re}\lambda} dv(f, h^*, \lambda) \quad \text{considering (4.54), by (2.9);} \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\Delta_n} |\lambda| e^{\operatorname{Re}\lambda} dv(e_n, h^*, \lambda) \quad \text{for } \lambda \in \Delta_n, \text{ by (4.46), } |\lambda| \geq n^4; \\
&\qquad\qquad\qquad\qquad \text{for } \lambda \in \Delta_n, \text{ by (4.47) and (4.52), } \operatorname{Re}\lambda = \operatorname{Re}\lambda_n - (\operatorname{Re}\lambda_n - \operatorname{Re}\lambda) \\
&\quad \geq \operatorname{Re}\lambda_n - |\operatorname{Re}\lambda_n - \operatorname{Re}\lambda| \geq -\omega - \varepsilon_n \geq -\omega - 1; \\
&\quad \geq e^{-(\omega+1)} \sum_{n=1}^{\infty} n^2 v(e_n, h^*, \Delta_n) \geq e^{-(\omega+1)} \sum_{n=1}^{\infty} n^2 |\langle E_A(\Delta_n) e_n, h^* \rangle| \quad \text{by (4.49) and (4.56);} \\
&\quad \geq e^{-(\omega+1)} \sum_{n=1}^{\infty} n^2 \frac{\varepsilon}{n^2} = \infty.
\end{aligned} \tag{4.57}$$

Whence, by the properties of the *o.c.* and [2], Proposition 3.1, $y(1) = e^A f \notin D(A)$.

Consequently, by Proposition 3.1, the weak solution $y(t) = e^{tA} f$, $0 \leq t < \infty$, of (1.1) on $[0, \infty)$ is not once strongly differentiable on $(0, \infty)$.

Now, assume that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is unbounded. Therefore, there is a subsequence $\{\operatorname{Re}\lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow \infty \quad \text{or} \quad \operatorname{Re}\lambda_{n(k)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \tag{4.58}$$

Suppose that $\operatorname{Re}\lambda_{n(k)} \to \infty$ as $k \to \infty$.

Without restricting generality, we can regard that

$$\operatorname{Re}\lambda_{n(k)} \geq k, \quad k = 1, 2, \ldots. \tag{4.59}$$

Considering this case just like the analogous one in the proof of the "only if" part of Theorem 4.1, one can show that the vector

$$f := \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re}\lambda_{n(k)}} e_{n(k)} \tag{4.60}$$

belongs to $\bigcap_{0 \leq t < \infty} D(e^{tA})$ and, hence, (see Section 1) $y(t) := e^{tA} f$, $0 \leq t < \infty$, is a weak solution of (1.1) on $[0, \infty)$.

For any $\lambda \in \Delta_{n(k)}$, $k = 1, 2, \ldots$, by (4.46), (4.47), and (4.59),

$$\begin{aligned}
\operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} - \left(\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right) \\
&\geq \operatorname{Re}\lambda_{n(k)} - \left|\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda\right| \\
&\geq \operatorname{Re}\lambda_{n(k)} - \varepsilon_{n(k)} \geq \operatorname{Re}\lambda_{n(k)} - \frac{1}{n(k)} \geq k - 1 \geq 0, \\
\operatorname{Re}\lambda &< \max\Big(0, (2n(k))^{-1}\ln|\operatorname{Im}\lambda|\Big),
\end{aligned} \tag{4.61}$$

Hence, for $\lambda \in \Delta_{n(k)}$, $k = 1, 2, \ldots$,

$$|\lambda| \geq |\operatorname{Im}\lambda| \geq e^{2n(k)\operatorname{Re}\lambda} \geq e^{2n(k)(\operatorname{Re}\lambda_{n(k)} - 1/n(k))}. \tag{4.62}$$

Using this estimate, we have

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda| e^{\operatorname{Re}\lambda} dv(f, h^*, \lambda) \quad \text{considering (4.49), by (2.9);} \\
&\quad = \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} |\lambda| e^{\operatorname{Re}\lambda} dv\left(e_{n(k)}, h^*, \lambda\right) \\
&\quad \geq \sum_{k=1}^{\infty} e^{-n(k)\operatorname{Re}\lambda_{n(k)}} e^{2n(k)(\operatorname{Re}\lambda_{n(k)} - 1/n(k))} v\left(e_{n(k)}, h^*, \Delta_{n(k)}\right) \\
&\quad = e^{-2} \sum_{k=1}^{\infty} e^{n(k)\operatorname{Re}\lambda_{n(k)}} \left|\left\langle E_A\left(\Delta_{n(k)}\right) e_{n(k)}, h^*\right\rangle\right| \quad \text{by (4.49), (4.56), and (4.59);} \\
&\quad \geq e^{-2}\varepsilon \sum_{k=1}^{\infty} \frac{e^{n(k)}}{n(k)^2} = \infty.
\end{aligned} \tag{4.63}$$

Whence, by the properties of the *o.c.* and [2], Proposition 3.1, $y(1) = e^{A} f \notin D(A)$.

Therefore, by Proposition 3.1, the weak solution $y(t) = e^{tA} f$, $0 \leq t < \infty$, of (1.1) on $[0, \infty)$ is not once strongly differentiable on $(0, \infty)$.

Now, suppose that $\operatorname{Re}\lambda_{n(k)} \to -\infty$ as $k \to \infty$.

Without restricting generality, we can regard that

$$\operatorname{Re}\lambda_{n(k)} \leq -k, \quad k = 1, 2, \ldots. \tag{4.64}$$

Let

$$f := \sum_{k=1}^{\infty} \frac{1}{k^2} e_{n(k)}. \tag{4.65}$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) \quad \text{considering (4.49), by (2.9);} \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\Delta_{n(k)}} e^{t\operatorname{Re}\lambda} dv(e_{n(k)}, g^*, \lambda) \\
&\quad\quad \text{for } \lambda \in \Delta_{n(k)}, \text{by (4.42) and (4.64), } \operatorname{Re}\lambda = \operatorname{Re}\lambda_{n(k)} + \left(\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}\right) \\
&\quad\quad\quad \leq \operatorname{Re}\lambda_{n(k)} + \left|\lambda - \lambda_{n(k)}\right| \leq -k+1 \leq 0; \\
&\quad \leq \sum_{k=1}^{\infty} \frac{1}{k^2} v(e_{n(k)}, g^*, \Delta_{n(k)}) \quad \text{by (2.8);} \\
&\quad \leq \sum_{k=1}^{\infty} \frac{1}{k^2} 4M \|e_{n(k)}\| \|g^*\| = 4M \|g^*\| \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\end{aligned} \tag{4.66}$$

Analogously, for an arbitrary $t \geq 0$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\}} e^{t\operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\quad \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\Delta_{n(k)} \cap \{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\}} 1\, dv(e_{n(k)}, g^*, \lambda) \\
&\quad = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\}} 1\, dv(f, g^*, \lambda) \quad \text{by (2.7);} \\
&\quad \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\right\}\right) f \right\| \|g^*\| \\
&\quad = 4M \left\| E_A\left(\left\{\lambda \in \sigma(A) \mid e^{t\operatorname{Re}\lambda} > n\right\}\right) f \right\| \quad \text{by the strong continuity of the } s.m.; \\
&\quad \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.67}$$

From (4.66) and (4.67), by [2], Proposition 3.1, we infer that $f \in \bigcap_{0 \leq t < \infty} D(e^{tA})$.

Therefore (see Section 1), $y(t) := e^{tA} f$, $0 \leq t < \infty$, is a weak solution of (1.1) on $[0, \infty)$.

Let

$$h^* := \sum_{k=1}^{\infty} \frac{1}{k^2} e^*_{n(k)} \in X^* \tag{4.68}$$

(cf. (4.51)).

Taking into account (4.50) and (4.51), we have

$$\langle e_{n(k)}, h^* \rangle \geq \frac{\varepsilon}{k^2}, \quad k = 1, 2, \ldots. \tag{4.69}$$

For any $\lambda \in \Delta_{n(k)}$, $k = 1, 2, \ldots$, by (4.47) and (4.64),

$$\begin{aligned}
\operatorname{Re}\lambda &= \operatorname{Re}\lambda_{n(k)} + \left(\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}\right) \\
&\leq \operatorname{Re}\lambda_{n(k)} + \left|\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}\right| \\
&\leq \operatorname{Re}\lambda_{n(k)} + \varepsilon_{n(k)} \leq \operatorname{Re}\lambda_{n(k)} + 1 \leq -k + 1 \leq 0, \\
\operatorname{Re}\lambda &> \min(0, -b_{-}\ln|\operatorname{Im}\lambda|),
\end{aligned} \tag{4.70}$$

Hence, for $\lambda \in \Delta_{n(k)}$, $k = 1, 2, \ldots$,

$$|\lambda| \geq |\operatorname{Im}\lambda| \geq e^{b_{-}^{-1}[-\operatorname{Re}\lambda]}. \tag{4.71}$$

Using these estimates, we have

$$\begin{aligned}
&\int_{\sigma(A)} |\lambda| e^{(b_{-}^{-1}/2)\operatorname{Re}\lambda} dv\left(f, h^{*}, \lambda\right) \quad \text{considering (4.49), by (2.9);} \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k^{2}} \int_{\Delta_{n(k)}} |\lambda| e^{(b_{-}^{-1}/2)\operatorname{Re}\lambda} dv\left(e_{n(k)}, h^{*}, \lambda\right) \\
&\quad \geq \sum_{k=1}^{\infty} \frac{1}{k^{2}} \int_{\Delta_{n(k)}} e^{(b_{-}^{-1}/2)[-\operatorname{Re}\lambda]} dv\left(e_{n(k)}, h^{*}, \lambda\right) \geq \sum_{k=1}^{\infty} \frac{e^{(b_{-}^{-1}/2)[k-1]}}{k^{2}} v\left(e_{n(k)}, h^{*}, \Delta_{n(k)}\right) \\
&\quad \geq e^{-b_{-}^{-1}/2} \sum_{k=1}^{\infty} \frac{e^{(b_{-}^{-1}/2)k}}{k^{2}} \left|\left\langle E_{A}\left(\Delta_{n(k)}\right) e_{n(k)}, h^{*}\right\rangle\right| \quad \text{by (4.49), (4.64), and (4.69);} \\
&\quad \geq e^{-b_{-}^{-1}/2} \varepsilon \sum_{k=1}^{\infty} \frac{e^{(b_{-}^{-1}/2)k}}{k^{4}} = \infty.
\end{aligned} \tag{4.72}$$

Whence, by the properties of the *o.c.* and [2], Proposition 3.1, $y(b_{-}/2) = e^{(b_{-}^{-1}/2)A} f \notin D(A)$.

Therefore, by Proposition 3.1, the weak solution $y(t) = e^{tA} f$, $0 \leq t < \infty$, of (1.1) on $[0, \infty)$ is not once strongly differentiable on $(0, \infty)$.

With every possibility concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ considered, we infer that the opposite to the assumption that there is a $b_{+} > 0$ such that, for any $b_{-} > 0$, the set $\sigma(A) \setminus \mathcal{L}_{b_{-}, b_{+}}$ is bounded, allows to single out a weak solution of (1.1) on $[0, \infty)$ that is not once strongly differentiable on $(0, \infty)$, much less strongly infinite differentiable on $(0, \infty)$.

Thus, the "only if" part has been proved by contrapositive. □

Theorem 5.2 of [3] and Theorem 4.1 of [4] have been generalized.

5. Certain Effects of Smoothness Improvement

As we observed in the proofs of the "only if" parts of Theorems 4.1 and 4.2, the opposits to the "if" parts' premises imply that there is a weak solution of (1.1) on $[0, \infty)$, which is not strongly differentiable at 0 or, respectively, once strongly differentiable on $(0, \infty)$.

Therefore, the case of finite strong differentiability of the weak solutions is superfluous and we obtain the following effects of smoothness improvement.

Proposition 5.1. *If every weak solution of* (1.1) *on* $[0, \infty)$ *is strongly differentiable at 0, then all of them are strongly infinite differentiable on* $[0, \infty)$.

Proposition 5.2. *If every weak solution of* (1.1) *on* $[0, \infty)$ *is once strongly differentiable on* $(0, \infty)$, *then all of them are strongly infinite differentiable on* $(0, \infty)$.

These statements generalize Propositions 6.1 and 6.2 of [3], respectively, the latter agreeing with the case when A is a linear operator (not necessarily spectral of scalar type) generating a C_0-semigroup (cf. [1, 13]).

6. Final Remarks

Due to the *scalar type spectrality* of the operator A, all the above criteria are formulated exclusively in terms of its *spectrum*, no restrictions on the operator's *resolvent* behavior necessary, which makes them inherently qualitative and more transparent than similar results for C_0-semigroups (cf. [14]).

If a scalar type spectral A generates a C_0-semigroup (cf. [15]), we immediately obtain the results of paper [4] regarding infinite differentiability.

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