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Research Article

On Simultaneous Farthest Points in $L^{\infty}(I, X)$

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Let X be a Banach space and let G be a closed bounded subset of X. For $(x_1, x_2, \ldots, x_m) \in X^m$, we set $\rho(x_1, x_2, \ldots, x_m, G) = \sup\{\max_{1\le i\le m}\|x_i - y\| : y \in G\}$. The set G is called simultaneously remotal if, for any $(x_1, x_2, \ldots, x_m) \in X^m$, there exists $g \in G$ such that $\rho(x_1, x_2, \ldots, x_m, G) = \max_{1\le i\le m}\|x_i - g\|$. In this paper, we show that if G is separable simultaneously remotal in X, then the set of ∞-Bochner integrable functions, $L^{\infty}(I, G)$, is simultaneously remotal in $L^{\infty}(I, X)$. Some other results are presented.

1. Introduction

Let X be a Banach space and G a bounded subset of X. For $x \in X$, set $\rho(x, G) = \sup\{\|x - y\| : y \in G\}$. A point $g_0 \in G$ is called a farthest point of G if there exists $x \in X$ such that $\|x - g_0\| = \rho(x, G)$. For $x \in X$, the farthest point map $F_G(x) = \{g \in G : \|x - g\| = \rho(x, G)\}$, that is, the set of points of G farthest from x. Note that this set may be empty. Let $R(G, X) = \{x \in X : F_G(x) \neq \phi\}$. We call a closed bounded set G remotal if $R(G, X) = X$ and densely remotal if $R(G, X)$ is a norm dense in X. The concept of remotal sets in Banach spaces goes back to the sixties. However, almost all the results on remotal sets are concerned with the topological properties of such sets, see [1–4]. Remotal sets in vector valued continuous functions was considered in [5]. Related results on Bochner integrable function spaces, $L^p(I, X)$, $1 \le p \le \infty$, are given in [6–8].

The problem of approximating a set of points $x_1, x_2, \ldots, x_m$ simultaneously by a point g (farthest point) in a subset G of X can be done in several ways, see [9]. Here, we will use the following definition.

Definition 1.1. Let G be a closed bounded subset of X. A point $g \in G$ is called a simultaneous farthest point of $(x_1, x_2, \ldots, x_m) \in X^m$ if

$$\rho(x_1, x_2, \ldots, x_m, G) = \sup_{h \in G} \max_{1 \le i \le m} (\|x_i - h\|) = \max_{1 \le i \le m} (\|x_i - g\|). \tag{1.1}$$

We call a closed bounded set G of a Banach space X simultaneously remotal if each m-tuple $(x_1, x_2, \ldots, x_m) \in X^m$ admits a farthest point in G and simultaneously densely remotal if the set of points $R(G, X^m) = \{(x_1, x_2, \ldots, x_m) \in X^m : F_G(x_1, x_2, \ldots, x_m) \neq \phi\}$, where

$$F_G(x_1, x_2, \ldots, x_m) = \left\{ g \in G : \rho(x_1, x_2, \ldots, x_m, G) = \max_{1 \le i \le m} (\|x_i - g\|) \right\} \tag{1.2}$$

is norm dense in X^m.

Clearly, if $m = 1$, then simultaneously remotal is precisely remotal.

In this paper we consider the problem of simultaneous farthest point for bounded sets of the form $L^\infty(I, G)$ in the Banach space $L^\infty(I, X)$, where X is a Banach space.

Throughout this paper, X is a Banach space, G is a closed bounded subset of X and $L^\infty(I, X)$, the space of all X-valued essentially bounded functions on the unit interval I. For $f \in L^\infty(I, X)$, we set $\|f\|_\infty = \text{ess sup}\{\|f(s)\| : s \in I\}$. For $G \subset X$, we set $L^\infty(I, G) = \{f \in L^\infty(I, X) : f(s) \in G, \text{almost all } s \in I\}$.

2. Distance Formula

The farthest distance formula is important in the study of farthest point. In this section, we compute the ∞-farthest distance from an element $f \in L^\infty(I, X)$ to a bounded set $L^\infty(I, G)$. We begin with the following proposition.

Proposition 2.1. *Let $f_1, f_2, \ldots, f_m$, then* $\max_{1 \le i \le m} \|f_i\|_\infty = \text{ess sup } \max_{1 \le i \le m} (\|f_i(t)\|)$.

Proof. For $1 \le i \le m$,

$$\|f_i\|_\infty = \text{ess sup} \|f_i(t)\| \le \text{ess sup} \max_{1 \le i \le m} (\|f_i(t)\|) \le \max_{1 \le i \le m} (\|f_i\|_\infty). \tag{2.1}$$

Hence, $\max_{1 \le i \le m} \|f_i\|_\infty = \text{ess sup} \max_{1 \le i \le m} (\|f_i(t)\|)$ □

Theorem 2.2. *Let X be a Banach space and let G be a closed bounded subset of X. If a function $\Phi : I \to \mathbb{R}$ defined by $\Phi(t) = \rho(f_1(t), f_2(t), \ldots, f_m(t), G)$, where $f_1, f_2, \ldots, f_m \in L^\infty(I, X)$, then $\Phi \in L^\infty(I)$ and*

$$\rho(f_1, f_2, \ldots, f_m, L^\infty(I, G)) = \|\Phi\|_\infty = \sup_{g \in L^\infty(I,G)} \max_{1 \le i \le m} (\|f_i - g\|_\infty). \tag{2.2}$$

Proof. Let $f_1, f_2, \ldots, f_m \in L^\infty(I, X)$. Being strongly measurable, there exist m sequences of simple functions (f_{in}), $1 \le i \le m$ such that $\|f_{in}(t) - f_i(t)\| \to 0$ as $n \to \infty$ for almost all

$t \in I$. We may write $f_{jn} = \sum_{i=1}^{m(n)} \chi_{A_{(i,n)}} x_{in}^{(j)}$. Since $\rho(x_1, x_2, \ldots, x_m, G)$ is a continuous function of $(x_1, x_2, \ldots, x_m) \in X^m$, the inequality

$$\left|\rho\left(f_{1n}(t), f_{2n}(t), \ldots, f_{mn}(t), G\right) - \rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right)\right| \leq \max_{1\leq i\leq m}\left\|f_{in}(t) - f_i(t)\right\| \longrightarrow 0 \tag{2.3}$$

implies that

$$\left|\rho\left(f_{1n}(t), f_{2n}(t), \ldots, f_{mn}(t), G\right) - \rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right)\right| \longrightarrow 0. \tag{2.4}$$

Set $\Phi_n(t) = \rho(f_{1n}(t), f_{2n}(t), \ldots, f_{mn}(t), G)$. Then,

$$\begin{aligned}
\Phi_n(t) &= \sup_{g\in G}\max_{1\leq k\leq m}\left(\left\|f_{kn}(t) - g\right\|\right) \\
&= \sup_{g\in G}\max_{1\leq k\leq m}\left\|\sum_{i=1}^{m(n)} \chi_{A_{in}}(t)\left(x_{in}^{(k)} - g\right)\right\| \\
&= \sum_{i=1}^{m(n)} \chi_{A_{in}}(t) \sup_{g\in G}\max_{1\leq k\leq m}\left\|x_{in}^{(k)} - g\right\|.
\end{aligned} \tag{2.5}$$

So Φ_n is a simple function for each n and $\mathrm{Lim}_{n\to\infty}\|\Phi_n(t) - \Phi(t)\| = 0$ for almost all $t \in I$. Hence Φ is measurable. Furthermore, for each $w \in L^\infty(I, G)$,

$$\begin{aligned}
\max_{1\leq i\leq m}\left\|f_i - w\right\|_\infty &= \max_{1\leq i\leq m} \text{ess sup}\left(\left\|f_i(t) - w(t)\right\|\right) \\
&= \text{ess sup}\max_{1\leq i\leq m}\left(\left\|f_i(t) - w(t)\right\|\right), \quad \text{Proposition 2.1} \\
&\leq \text{ess sup}\sup_{g\in G}\max_{1\leq i\leq m}\left(\left\|f_i(t) - g\right\|\right) \\
&= \left\|\rho\left(f_1(t), f_2(t), f_3(t), \ldots, f_m(t), G\right)\right\|_\infty \\
&= \|\Phi\|_\infty.
\end{aligned} \tag{2.6}$$

Thus,

$$\|\Phi\|_\infty \geq \rho\left(f_1, f_2, \ldots, f_m, L^\infty(I, G)\right). \tag{2.7}$$

To prove the reverse inequality. Let $\epsilon > 0$ be given, since countably valued functions are dense in $L^\infty(I, X)$, there exist countably valued functions $\gamma_1, \gamma_2, \ldots, \gamma_m$ in $L^\infty(I, X)$ such that $\|f_i - \gamma_i\| < \epsilon$, $1 \leq i \leq m$. We may write $\gamma_j = \sum_{i=1}^\infty \chi_{A_i} x_i^{(j)}$, as in [10]. We may assume $\sum_{i=1}^\infty \chi_{A_i} = 1$, $\mu(A_i) > 0$ for all i. For each i, Choose $h_i \in G$ such that

$$\max_{1\leq k\leq m}\left\|x_i^{(k)} - h_i\right\| > \rho\left(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}, G\right) - \epsilon. \tag{2.8}$$

Now, set $g \in L^{\infty}(I,G)$ as $g(s) = \sum_{i=1}^{\infty} \chi_{A_i}(s) h_i$. The inequality

$$\|\gamma_j - g\|_{\infty} \leq \|\gamma_j - f_j\|_{\infty} + \|f_j - g\|_{\infty} \tag{2.9}$$

implies

$$\max_{1\leq j\leq m} \|\gamma_j - g\|_{\infty} \leq \epsilon + \max_{1\leq i\leq m} \|f_j - g\|_{\infty}. \tag{2.10}$$

Further,

$$\begin{aligned}
\max_{1\leq j\leq m} \|f_j - g\|_{\infty} &\geq \max_{1\leq j\leq m} \|\gamma_j - g\|_{\infty} - \epsilon \\
&= \max_{1\leq j\leq m} \left(\operatorname{ess\,sup}\left(\|\gamma_j(t) - g(t)\|\right)\right) - \epsilon \\
&= \operatorname{ess\,sup} \max_{1\leq j\leq m} \left\| \sum_{i=1}^{\infty} \chi_{A_i}(t) \left(x_i^{(j)} - h_i\right) \right\| - \epsilon, \quad \text{Proposition 2.1} \\
&= \operatorname{ess\,sup} \max_{1\leq j\leq m} \sum_{i=1}^{\infty} \chi_{A_i}(t) \left\| x_i^{(j)} - h_i \right\| - \epsilon \\
&= \operatorname{ess\,sup} \sum_{i=1}^{\infty} \chi_{A_i}(t) \max_{1\leq j\leq m} \left\| x_i^{(j)} - h_i \right\| - \epsilon \\
&\geq \operatorname{ess\,sup} \sum_{i=1}^{\infty} \chi_{A_i}(t)\, \rho\left(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}, G\right) - 2\epsilon \\
&= \operatorname{ess\,sup} \rho\left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t), G\right) - 2\epsilon.
\end{aligned} \tag{2.11}$$

For $1 \leq j \leq m$ and $a \in G$, the inequality

$$\|f_j(t) - a\| \leq \|f_j(t) - \gamma_j(t)\| + \|\gamma_j(t) - a\| \tag{2.12}$$

implies

$$\rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right) \leq \max_{1\leq j\leq m} \|f_j(t) - \gamma_j(t)\| + \rho\left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t), G\right). \tag{2.13}$$

Therefore,

$$\begin{aligned}
\max_{1\leq j\leq m} \|f_j - g\|_{\infty} &\geq \operatorname{ess\,sup} \begin{pmatrix} \rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right) \\ -\max_{1\leq j\leq m} \|\gamma_j(t) - f_j(t)\| \end{pmatrix} - 2\epsilon \\
&\geq \|\Phi\|_{\infty} - \operatorname{ess\,sup} \max_{1\leq j\leq m} \|\gamma_j(t) - f_j(t)\| - 2\epsilon \\
&\geq \|\Phi\|_{\infty} - 3\epsilon.
\end{aligned} \tag{2.14}$$

Hence, $\rho(f_1, f_2, \ldots, f_m, L^\infty(I,G)) + 3\epsilon \geq \|\Phi\|_\infty$. Since ϵ was arbitrary, we have $\|\Phi\|_\infty = \rho(f_1, f_2, \ldots, f_m, L^\infty(I,G))$. □

Corollary 2.3. *Let g be a strongly measurable function from I to a closed bounded subset G of a Banach space X, and $f_1, f_2, \ldots, f_m \in L^\infty(I,X)$. If $g(t)$ is a simultaneous farthest point of $f_1(t), f_2(t), \ldots, f_m(t)$ in G, then g is a simultaneous farthest point of $f_1, f_2, \ldots, f_m$ in $L^\infty(I,G)$.*

Proof. By assumption, $\max_{1\leq i\leq m}(\|f_i(t) - g(t)\|) = \rho(f_1(t), f_2(t), \ldots, f_m(t), G)$ for almost $t \in I$. Since G is bounded, it follows that $g \in L^\infty(I,G)$ and

$$\text{ess sup} \max_{1\leq i\leq m}\left(\|f_i(t) - g(t)\|\right) = \text{ess sup}\, \rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right). \tag{2.15}$$

Theorem 2.2 and Proposition 2.1 implies that

$$\max_{1\leq i\leq m}\left(\|f_i - g\|_\infty\right) = \text{ess sup} \max_{1\leq i\leq m}\left(\|f_i(t) - g(t)\|\right) = \rho\left(f_1, f_2, \ldots, f_m, L^\infty(I,G)\right) \tag{2.16}$$

and g is a simultaneous farthest point of $f_1, f_2, \ldots, f_m$ in $L^\infty(I,G)$. □

3. Remotal Sets in $L^\infty(I,X)$

In this section, we raise the question: if G is a simultaneously remotal set in X, is $L^\infty(I,G)$ simultaneously remotal in $L^\infty(I,X)$? We get a positive answer to this question in the case that G is a separable simultaneously remotal subset of X or in the case that $\operatorname{Span} G$ is finite dimensional subspace of X. We begin with the following theorem.

Theorem 3.1. *If G is simultaneously densely remotal in X, then $L^\infty(I,G)$ is simultaneously densely remotal in $L^\infty(I,X)$.*

Proof. Let $f_1, f_2, \ldots, f_m \in L^\infty(I,X)$. Then, there exist $\gamma_1, \gamma_2, \ldots, \gamma_m$ simple functions such that $\|f_i - \gamma_i\|_\infty < \epsilon/2$, $1 \leq i \leq m$. Now, we can write without lost of generality, $\gamma_i = \sum_{k=1}^n \chi_{A_k} x_k^{(i)}$. Since G is simultaneously densely remotal, then there exist an m-tuple $(y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(m)})$ and $g_k \in G$ such that $\max_{1\leq i\leq m}\|x_k^{(i)} - y_k^{(i)}\| < \epsilon/2$ and

$$\max_{1\leq i\leq m}\left\|y_k^{(i)} - g_k\right\| = \rho\left(y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(m)}, G\right) = \sup_{g\in G}\max_{1\leq i\leq m}\left\|y_k^{(i)} - g\right\|. \tag{3.1}$$

Set $\phi = \sum_{k=1}^n \chi_{A_k} g_k$ and $h_i = \sum_{k=1}^n \chi_{A_k} y_k^{(i)}$. Then,

$$\begin{aligned}\max_{1\leq i\leq m}\|h_i - \phi\|_\infty &= \max_{1\leq i\leq m} \text{ess sup}\|h_i(t) - \phi(t)\| \\ &= \text{ess sup} \max_{1\leq i\leq m}\left\|\sum_{k=1}^n \chi_{A_k}(t)\left(y_k^{(i)} - g_k\right)\right\|, \quad \text{Proposition 2.1}\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{ess\,sup} \max_{1\le i\le m} \sum_{k=1}^{n} \chi_{A_k}(t) \left\| y_k^{(i)} - g_k \right\| \\
&\ge \operatorname{ess\,sup} \max_{1\le i\le m} \sum_{k=1}^{n} \chi_{A_k}(t) \left\| y_k^{(i)} - g \right\|
\end{aligned}
\tag{3.2}
$$

for every $g \in G$. In particular, for any $w \in L^{\infty}(I, G)$, using Proposition 2.1,

$$
\begin{aligned}
\max_{1\le i\le m} \left\| h_i - \phi \right\|_{\infty} &\ge \operatorname{ess\,sup} \max_{1\le i\le m} \| h_i(t) - w(t) \| \\
&\ge \max_{1\le i\le m} \operatorname{ess\,sup} \| h_i(t) - w(t) \| = \max_{1\le i\le m} \| h_i - w \|_{\infty},
\end{aligned}
\tag{3.3}
$$

Hence, ϕ is a farthest point from the m-tuple $(h_1, h_2, \ldots, h_m)$. But

$$
\begin{aligned}
\max_{1\le i\le m} \left\| h_i - \gamma_i \right\|_{\infty} &= \max_{1\le i\le m} \operatorname{ess\,sup} \left\| \sum_{k=1}^{n} \chi_{A_k} \left(y_k^{(i)} - x_k^{(i)} \right) \right\| \\
&= \max_{1\le i\le m} \operatorname{ess\,sup} \sum_{k=1}^{n} \chi_{A_k} \left\| y_k^{(i)} - x_k^{(i)} \right\| \\
&= \operatorname{ess\,sup} \max_{1\le i\le m} \sum_{k=1}^{n} \chi_{A_k} \left\| y_k^{(i)} - x_k^{(i)} \right\|, \quad \text{Proposition 2.1} \\
&= \operatorname{ess\,sup} \sum_{k=1}^{n} \chi_{A_k} \max_{1\le i\le m} \left\| y_k^{(i)} - x_k^{(i)} \right\| \le \frac{\epsilon}{2}.
\end{aligned}
\tag{3.4}
$$

Hence,

$$
\max_{1\le i\le m} \left\| f_i - h_i \right\|_{\infty} \le \max_{1\le i\le m} \left\| f_i - \gamma_i \right\| + \max_{1\le i\le m} \left\| \gamma_i - h_i \right\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.5}
$$

This complete the proof of the theorem. □

For a remotal set $G \subseteq X$, the map $T : X^m \to 2^G$ defined by

$$
\begin{aligned}
T(x_1, x_2, \ldots, x_m) &= F(x_1, x_2, \ldots, x_m, G) \\
&= \left\{ g \in G : \rho(x_1, x_2, \ldots, x_m, G) = \max_{1\le i\le m} \left\| x_i - g \right\| \right\}
\end{aligned}
\tag{3.6}
$$

is a multivalued map in general. Hence for any $f \in L^{\infty}(I, X)$, the map $T \circ f$ is a multivalued map from I into G.

Before proceeding we remind the reader for some facts regarding mult-valued maps. For a Banach space X and a measurable space I a function $f : I \to X$ is said to be strongly measurable if it is the pointwise limit of a sequence of simple functions almost everywhere.

On the other hand f is said to measurable in the classical sense if $f^{-1}(K)$ is measurable in I for any closed set K in X, see [10]. A multivalued function $T : I \to X$ is said to measurable in the classical sense if $T^{-1}(K)$ is measurable in I for any closed set K in X, here $T^{-1}(K) = \{t \in I : T(t) \cap K \neq \phi\}$. A measurable in the classical sense g may be extracted from a measurable multivalued function $T : I \to X$ where X is a separable Banach space provided that $T(t)$ is a closed subset of X for each $t \in I$ and such that $g(t) \in T(t)$ for each $t \in I$, see [11, page 289].

Theorem 3.2. *Let G be a closed bounded simultaneously remotal in X such that* span(G) *is finite dimensional subspace of X, then $L^\infty(I, G)$ is simultaneously remotal in $L^\infty(I, X)$.*

Proof. First, we will prove that T is a closed valued map. If $T(x_1, x_2, \ldots, x_m)$ is finite set, then it is closed. If $T(x_1, x_2, \ldots, x_n)$ is not finite, let $y \in \overline{T(x_1, x_2, \ldots, x_m)}$, then there exists $y_n \in T(x_1, x_2, \ldots, x_m)$ such that $y_n \to y$. This implies

$$\max_{1\le i\le m}\|x_i - g\| \le \max_{1\le i\le m}\|x_i - y_n\|, \tag{3.7}$$

for every $g \in G$. Taking the limit as $n \to \infty$, we get $\max\|x_i - g\| \le \max\|x_i - y\|$ for every $g \in G$, and this implies that $y \in T(x_1, x_2, \ldots, x_m)$ and T is a closed multivalued map.

To prove that T is measurable in the classical sense let B be any closed subset of G. If $(x_1, x_2, \ldots, x_m) \in \overline{T^{-1}(B)}$, then there exists $(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}) \in T^{-1}(B)$ that converges to $(x_1, x_2, \ldots, x_m)$. Since $(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}) \in T^{-1}(B)$, then $T(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}) \cap B \neq \phi$. Choose $y_n \in T(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}) \cap B$. Then, y_n has a convergent subsequence $y_{nk} \to y$ being a sequence in a bounded closed subset of a finite dimensional space. But

$$\max_{1\le i\le m}\left\|x_i^{(n_k)} - y_{n_k}\right\| \ge \max_{1\le i\le m}\left\|x_i^{(n_k)} - g\right\| \tag{3.8}$$

for every $g \in G$. Taking the limit as $n_k \to \infty$, we get

$$\max_{1\le i\le m}\|x_i - y\| \ge \max_{1\le i\le m}\|x_i - g\| \tag{3.9}$$

for every $g \in G$. Hence $y \in T(x_1, x_2, \ldots, x_m) \cap B \neq \emptyset$. Therefore $(x_1, x_2, \ldots, x_m) \in T^{-1}(B)$ and $T^{-1}(B)$ is closed. Hence, T is measurable and if γ is the vector valued map $\gamma : I \to X^m$, $\gamma(t) = (f_1(t), f_2(t), \ldots, f_m(t))$, then $T \circ \gamma$ is a measurable closed multivalued map. By Theorem 6.6.4 in [11, page 289], $T \circ \gamma$ has a measurable selection say g. Further,

$$\max_{1\le i\le m}\|f_i(t) - g(t)\| \ge \max_{1\le i\le m}\|f_i(t) - h(t)\|, \tag{3.10}$$

for every $h \in L^\infty(I, G)$ and so

$$\begin{aligned} \text{ess sup}\max_{1\le i\le m}\|f_i(t) - g(t)\| &= \max_{1\le i\le m}\text{ess sup}\|f_i(t) - g(t)\| \\ &\ge \max_{1\le i\le m}\text{ess sup}\|f_i(t) - h(t)\|. \end{aligned} \tag{3.11}$$

Hence,

$$\max_{1\le i\le m}\|f_i - g\|_\infty \ge \max_{1\le i\le m}\|f_i - h\|_\infty, \tag{3.12}$$

for every $h \in L^\infty(I,G)$ and $L^\infty(I,G)$ is remotal in $L^\infty(I,X)$. □

Finally, following the steps in the prove of Theorem 3.6 in [6], we proof the following main result in the paper.

Theorem 3.3. *Let G be a separable simultaneously remotal subset of X. Then $L^\infty(I,G)$ is simultaneously remotal in $L^\infty(I,X)$.*

Proof. Let $(f_1, f_2, \ldots, f_m) \in (L^\infty(I,X))^m$. Using Corollary 2.3, it is sufficient to show that there exists a measurable function g defined on I such that $g(t)$ is a simultaneous farthest point of $(f_1(t), f_2(t), \ldots, f_m(t))$ in G. Since $f_1, f_2, \ldots, f_m$ are strongly measurable. We may assume that $f_1(I), f_2(I), \ldots, f_m(I)$ are separable sets in X. So there exist $\{I_{n_i}\}_{n=1}^\infty$a countable partition of I such that $\operatorname{diam}(f_i(I_{n_i})) < 1/2$, for each i, $1 \le i \le m$, for all n; where

$$\operatorname{diam}(S) = \sup\{\|x-y\| : x, y \in S\}. \tag{3.13}$$

Consider the partition $I_{k_1k_2\cdots k_m} = \bigcap_{i=1}^m I_{k_i}$, where $(I_{k_1}, I_{k_2}, \ldots, I_{k_m}) \subseteq \prod_{i=1}^m \{I_{n_i}\}$, for $1 \le n < \infty$. Then $\operatorname{diam}(f_i(I_{k_1k_2\cdots k_m})) < 1/2$. For simplicity write $I_{k_1k_2\cdots k_m}$ as $\{I_n\}_{n=1}^\infty$. For each $t \in I$, let $g_0(t)$ be a simultaneous farthest point from $(f_1(t), f_2(t), \ldots, f_m(t))$ in G. Define the map g_0 from I into G by $g_0(t)$ is a simultaneous farthest point from $(f_1(t), f_2(t), \ldots, f_m(t))$. Apply Lemma 3.1 in [6] with $\epsilon = 1/2$ and $I = I_n = A$, we get countable partitions in each I_n and therefore countable partition in the whole of I in measurable sets $\{E_n\}_{n=1}^\infty$ and a sequence of subsets $\{A_n\}_{n=1}^\infty$ such that

$$\begin{gathered} A_n \subseteq E_n, \quad \mu^*(A_n) = \mu(E_n), \\ \operatorname{diam}(g_0(A_n)) < \frac{1}{2} \text{ and } \operatorname{diam}(f_i(E_n)) < \frac{1}{2}, \quad 1 \le i \le m. \end{gathered} \tag{3.14}$$

Repeat the same argument in each E_n with $\epsilon = 1/2^2$, $I = E_n$, and $A = A_n$. For each n, we get a countable partition $\{E_{(n,k)} : 1 \le k < \infty\}$ of E_n in measurable sets and a sequence $\{A_{(n,k)} : 1 \le k < \infty\}$ of subsets of I such that

$$\begin{gathered} A_{(n,k)} \subseteq E_{(n,k)} \cap A_n, \quad \mu^*(A_{(n,k)}) = \mu(E_{(n,k)}), \\ \operatorname{diam}(g_0(A_{(n,k)})) < \frac{1}{2^2} \text{ and } \operatorname{diam}(f_i(E_{(n,k)})) < \frac{1}{2^2}, \quad i = 1, 2, \ldots, m. \end{gathered} \tag{3.15}$$

Now, we will use mathematical induction for each n; let Δ_n be the set of n-tuples of natural numbers and $\Delta = \bigcup\{\Delta_n : 1 \le n < \infty\}$. On this Δ consider the partial order defined by $(m_1, m_2, \ldots, m_i) < (n_1, n_2, \ldots, n_j)$ if and only if $i \le j$ and $m_k = n_k$, $k = 1, 2, \ldots, i$. Then by induction for each n, we can find a countable partition $\{E_\alpha : \alpha \in \Delta_n\}$ of I of measurable sets

and a collection $\{A_\alpha : \alpha \in \Delta_n\}$ of subsets of I such that:

(1) $A_\alpha \subseteq E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$,

(2) $E_\beta \subseteq E_\alpha$ and $A_\beta \subseteq A_\alpha$ if $\alpha \leq \beta$,

(3) $\operatorname{diam}(f_i(E_\alpha)) < 1/2^n$ for $i = 1, 2, \ldots, m$ and $\operatorname{diam}(g_0(A_\alpha)) < 1/2^n$ for $\alpha \in \Delta_n$.

We may assume $A_\alpha \neq \phi$, for all α. For each $\alpha \in \Delta$, let $t_\alpha \in A_\alpha$ and define g_n from I into G by $g_n(t) = \sum_{\alpha\in\Delta_n} \chi_{E_\alpha}(t) g_0(t_\alpha)$. Then, for each $t \in I$ and $n \leq m$, we have

$$\begin{aligned}
\|g_n(t) - g_m(t)\| &= \left\| \sum_{\alpha\in\Delta_n} \chi_{E_\alpha}(t) g_0(t_\alpha) - \sum_{\beta\in\Delta_m} \chi_{E_\beta}(t) g_0(t_\beta) \right\| \\
&\leq \left\| \sum_{\beta\in\Delta_m} \chi_{E_\beta}(t) \left(g_0(t_\alpha) - g_0(t_\beta)\right) \right\| \\
&\leq \sum_{\beta\in\Delta_m} \left\| \left(g_0(t_\alpha) - g_0(t_\beta)\right) \right\| \chi_{E_\beta} \\
&\leq \frac{1}{2^n}.
\end{aligned} \tag{3.16}$$

Hence, $(g_n(t))$ is a Cauchy sequence in G for all $t \in I$. Therefore $(g_n(t))$ is a convergent sequence. Let $g : I \to G$ be defined to be the pointwise limit of (g_n). Since g_n is strongly measurable for each n, we have g is strongly measurable. Further for $t \in I$, $1 \leq n < \infty$, and $t \in E_\alpha$ for some $\alpha \in \Delta_n$, we have

$$\begin{aligned}
\max_{1\leq i\leq m} \|f_i(t) - g_n(t)\| &= \max_{1\leq i\leq m} \|f_i(t) - g_0(t_\alpha)\| \\
&\geq \max_{1\leq i\leq m} \left| \|f_i(t_\alpha) - g_0(t_\alpha)\| - \|f_i(t) - f_i(t_\alpha)\| \right| \\
&\geq \max_{1\leq i\leq m} \left| \|f_i(t_\alpha) - g_0(t_\alpha)\| - \frac{1}{2^n} \right| \\
&\geq \rho\left(f_1(t_\alpha), f_2(t_\alpha), \ldots, f_m(t_\alpha), G\right) - \frac{1}{2^n}.
\end{aligned} \tag{3.17}$$

For $1 \leq i \leq m$, the inequality

$$\|f_i(t) - a\| \leq \|f_i(t) - f_i(t_\alpha)\| + \|f_i(t_\alpha) - a\| \leq \frac{1}{2^n} + \|f_i(t_\alpha) - a\| \tag{3.18}$$

implies that

$$\rho\left(f_1(t), f_2(t), \ldots, f_m(t), G\right) \leq \frac{1}{2^n} + \rho\left(f_1(t_\alpha), f_2(t_\alpha), \ldots, f_m(t_\alpha), G\right). \tag{3.19}$$

Therefore,

$$\max_{1\le i\le m}\left\|f_i(t)-g_n(t)\right\| \ge \rho\left(f_1(t),f_2(t),\ldots,f_m(t),G\right)-\frac{1}{2^{n-1}}. \tag{3.20}$$

Taking limit as $n\to\infty$, we get

$$\rho\left(f_1(t),f_2(t),\ldots,f_m(t),G\right)=\max_{1\le i\le m}\left\|f_i(t)-g(t)\right\|. \tag{3.21}$$

This completes the proof of the theorem. □

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