Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 920737, 9 pages
doi:10.1155/2011/920737

Research Article

Topological Aspects of the Product of Lattices

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Received 8 June 2011; Accepted 23 July 2011

Academic Editor: Alexander Rosa

Let $\mathbf{X}$ be an arbitrary nonempty set and $\mathbf{L}$ a lattice of subsets of $\mathbf{X}$ such that $\emptyset, \mathbf{X} \in \mathbf{L}$. $\mathbf{A}(\mathbf{L})$ denotes the algebra generated by $\mathbf{L}$, and $\mathbf{M}(\mathbf{L})$ denotes those nonnegative, finite, finitely additive measures on $\mathbf{A}(\mathbf{L})$. In addition, $\mathbf{I}(\mathbf{L})$ denotes the subset of $\mathbf{M}(\mathbf{L})$ which consists of the nontrivial zero-one valued measures. The paper gives detailed analysis of products of lattices, their associated Wallman spaces, and products of a variety of measures.

1. Introduction

It is well known that given two measurable spaces and measures on them, we can obtain the product measurable space and the product measure on that space. The purpose of this paper is to give detailed analysis of product lattices and their associated Wallman spaces and to investigate how certain lattice properties carry over to the product lattices. In addition, we proceed from a measure theoretic point of view. We note that some of the material presented here has been developed from a filter approach by Kost, but the measure approach lends to a generalization of measures and to an easier treatment of topological style lattice properties.

2. Background and Notations

In this section we introduce the notation and terminology that will be used throughout the paper. All is fairly standard, and we include it for the reader's convenience.

Let $\mathbf{X}$ be an arbitrary nonempty set and $\mathbf{L}$ a lattice of subsets of $\mathbf{X}$ such that $\emptyset, \mathbf{X} \in \mathbf{L}$. A lattice $\mathbf{L}$ is a partially ordered set any two elements (x, y) of which have both $\sup(x, y)$ and $\inf(x, y)$.

$\mathbf{A}(\mathbf{L})$ denotes the algebra generated by $\mathbf{L}$; $\sigma(\mathbf{L})$ is the σ algebra generated by $\mathbf{L}$; $\delta(\mathbf{L})$ is the lattice of all countable intersections of sets from $\mathbf{L}$; $\tau(\mathbf{L})$ is the lattice of arbitrary intersections of sets from $\mathbf{L}$; $\rho(\mathbf{L})$ is the smallest class closed under countable intersections and unions which contains $\mathbf{L}$.

2.1. Lattice Terminology

The lattice $\mathbf{L}$ is called:

> *δ-lattice* if $\mathbf{L}$ is closed under countable intersections; *complement generated* if $L \in \mathbf{L}$ implies $L = \cap L'_n$, $n = 1, \ldots, \infty$, $L_n \in \mathbf{L}$ (where prime denotes the complement); *disjunctive* if for $x \in \mathbf{X}$ and $L_1 \in \mathbf{L}$ such that $x \notin L_1$ there exists $L_2 \in \mathbf{L}$ with $x \in L_2$ and $L_1 \cap L_2 = \emptyset$; *separating* (or $\mathbf{T_1}$) if $x, y \in \mathbf{X}$ and $x \neq y$ implies there exists $L \in \mathbf{L}$ such that $x \in L$, $y \notin L$; $\mathbf{T_2}$ if for $x, y \in \mathbf{X}$ and $x \neq y$ there exist $L_1, L_2 \in \mathbf{L}$ such that $x \in L'_1$, $y \in L'_2$, and $L'_1 \cap L'_2 = \emptyset$; *normal* if for any $L_1, L_2 \in \mathbf{L}$ with $L_1 \cap L_2 = \emptyset$ there exist $L_3, L_4 \in \mathbf{L}$ with $L_1 \subset L'_3$, $L_2 \subset L'_4$, and $L'_3 \cap L'_4 = \emptyset$; *compact* if for any collection $\{L_\alpha\}$ of sets of $\mathbf{L}$ with $\cap_\alpha L_\alpha = \emptyset$, there exists a finite subcollection with empty intersection; *countably compact* if for any countable collection $\{L_\alpha\}$ of sets of $\mathbf{L}$ with $\cap_\alpha L_\alpha = \emptyset$, there exists a finite subcollection with empty intersection.

2.2. Measure Terminology

$\mathbf{M}(\mathbf{L})$ denotes those nonnegative, finite, finitely additive measures on $\mathbf{A}(\mathbf{L})$.

A measure $\mu \in \mathbf{M}(\mathbf{L})$ is called:

> *σ-smooth* on $\mathbf{L}$ if for all sequences $\{L_n\}$ of sets of $\mathbf{L}$ with $L_n \downarrow \emptyset$, $\mu(L_n) \to 0$;
>
> *σ-smooth* on $\mathbf{A}(\mathbf{L})$ if for all sequences $\{A_n\}$ of sets of $\mathbf{A}(\mathbf{L})$ with $A_n \downarrow \emptyset$, $\mu(A_n) \to 0$, that is, countably additive.

$\mathbf{L}$-*regular* if for any $A \in \mathbf{A}(\mathbf{L})$,

$$\mu(A) = \sup\{\mu(L) \mid L \subset A, L \in \mathbf{L}\}. \tag{2.1}$$

We denote by $\mathbf{M}_R(\mathbf{L})$ the set of $\mathbf{L}$-regular measures of $\mathbf{M}(\mathbf{L})$; $\mathbf{M}_\sigma(\mathbf{L})$ the set of σ-smooth measures on $\mathbf{L}$, of $\mathbf{M}(\mathbf{L})$; $\mathbf{M}^\sigma(\mathbf{L})$ the set of σ-smooth measures on $\mathbf{A}(\mathbf{L})$ of $\mathbf{M}(\mathbf{L})$; $\mathbf{M}_R^\sigma(\mathbf{L})$ the set of $\mathbf{L}$-regular measures of $\mathbf{M}^\sigma(\mathbf{L})$.

In addition, $\mathbf{I}(\mathbf{L})$, $\mathbf{I}_R(\mathbf{L})$, $\mathbf{I}_\sigma(\mathbf{L})$, $\mathbf{I}^\sigma(\mathbf{L})$, $\mathbf{I}_R^\sigma(\mathbf{L})$ are the subsets of the corresponding $\mathbf{M}$'s which consist of the nontrivial zero-one valued measures.

Finally, let $\mathbf{X}, \mathbf{Y}$ be abstract sets and $\mathbf{L_1}$ a lattice of subsets of $\mathbf{X}$ and $\mathbf{L_2}$ a lattice of subsets of $\mathbf{Y}$. Let $\mu_1 \in \mathbf{M}(\mathbf{L_1})$ and $\mu_2 \in \mathbf{M}(\mathbf{L_2})$.

The *product measure* $\mu_1 \times \mu_2 \in \mathbf{M}(\mathbf{L_1} \times \mathbf{L_2})$ is defined by

$$(\mu_1 \times \mu_2)(L_1 \times L_2) = \mu_1(L_1)\mu_2(L_2) \quad \forall L_1 \in \mathbf{A}(\mathbf{L_1}),\ L_2 \in \mathbf{A}(\mathbf{L_2}). \tag{2.2}$$

2.3. Lattice-Measure Correspondence

The *support* of $\mu \in \mathbf{M}(\mathbf{L})$ is $S(\mu) = \cap\{L \in \mathbf{L} / \mu(L) = \mu(X)\}$.

In case $\mu \in \mathbf{I}(\mathbf{L})$ then the support is $S(\mu) = \cap\{L \in \mathbf{L} / \mu(L) = 1\}$.

With this notation and in light of the above correspondences, we now note:

> For any $\mu \in \mathbf{I}(\mathbf{L})$, there exists $\nu \in \mathbf{I}_R(\mathbf{L})$ such that $\mu \leq \nu$ on $\mathbf{L}$ (i.e., $\mu(L) \leq \nu(L)$ for all $L \in \mathbf{L}$). For any $\mu \in \mathbf{I}(\mathbf{L})$, there exists $\nu \in \mathbf{I}_R(\mathbf{L}')$ such that $\mu \leq \nu$ on $\mathbf{L}'$.

L is *compact* if and only if $S(\mu) \neq \emptyset$ for every $\mu \in \mathbf{I}_R(\mathbf{L})$. **L** is *countably compact* if and only if $\mathbf{I}_R(\mathbf{L}) = \mathbf{I}_R^\sigma(\mathbf{L})$. **L** is *normal* if and only if for each $\mu \in \mathbf{I}(\mathbf{L})$, there exists a unique $\nu \in \mathbf{I}_R(\mathbf{L})$ such that $\mu \le \nu$ on **L**. **L** is *regular* if and only if whenever $\mu_1, \mu_2 \in \mathbf{I}(\mathbf{L})$ and $\mu_1 \le \mu_2$ on **L**, then $S(\mu_1) = S(\mu_2)$. **L** is *replete* if and only if for any $\mu \in \mathbf{I}_R^\sigma(\mathbf{L})$, $S(\mu) \neq \emptyset$. **L** is *prime-complete* if and only if for any $\mu \in \mathbf{I}_{\boldsymbol{\sigma}}(\mathbf{L})$, $S(\mu) \neq \emptyset$.

Finally, if μ_x is the measure concentrated at $x \in \mathbf{X}$, then $\mu_x \in \mathbf{I}_R(\mathbf{L})$, for all $x \in \mathbf{X}$ if and only if **L** is disjunctive.

For further results and related matters see [1–3].

2.4. The General Wallman Space and Wallman Topology

The *Wallman topology* in $\mathbf{I}_\mathbf{R}^\sigma(\mathbf{L})$ is obtained by taking all

$$W_\sigma(L) = \{\mu \in \mathbf{I}_\mathbf{R}^\sigma(\mathbf{L}) / \mu(L) = 1\}, \quad L \in \mathbf{L} \tag{2.3}$$

as a base for the closed sets in $\mathbf{I}_\mathbf{R}^\sigma(\mathbf{L})$ and then $\mathbf{I}_\mathbf{R}^\sigma(\mathbf{L})$ is called the *general Wallman space* associated with **X** and **L**. Assuming **L** is disjunctive, $\mathbf{W}_\sigma(\mathbf{L}) = \{W_\sigma(L)/L \in \mathbf{L}\}$ is a lattice in $\mathbf{I}_R^\sigma(\mathbf{L})$, isomorphic to **L** under the map $L \to W_\sigma(L)$, $L \in \mathbf{L}$. $\mathbf{W}_\sigma(\mathbf{L})$ is replete and a base for the closed sets $\mathbf{tW}_{\boldsymbol{\sigma}}(\mathbf{L})$, all arbitrary intersections of sets of $\mathbf{W}_{\boldsymbol{\sigma}}(\mathbf{L})$.

If $A \in \mathbf{A}(\mathbf{L})$, then $W_\sigma(A) = \{\mu \in \mathbf{I}_\mathbf{R}^\sigma(\mathbf{L}) / \mu(A) = 1\}$ and the following statements are true:

$$\begin{gathered} W_\sigma(A \cup B) = W_\sigma(A) \cup W_\sigma(B), \\ W_\sigma(A \cap B) = W_\sigma(A) \cap W_\sigma(B), \\ W_\sigma(A') = W_\sigma(A)', \\ A \supset B \text{ iff } W_\sigma(A) \supset W_\sigma(B), \\ \mathbf{A}(\mathbf{W}_{\boldsymbol{\sigma}}(\mathbf{L})) = \mathbf{W}_{\boldsymbol{\sigma}}(\mathbf{A}(\mathbf{L})). \end{gathered} \tag{2.4}$$

The Induced Measure

Let $\mu \in \mathbf{I}_\mathbf{R}^\sigma(\mathbf{L})$ and consider the *induced measure* $\underline{\mu} \in \mathbf{I}_\mathbf{R}^\sigma(\mathbf{W}_{\boldsymbol{\sigma}}(\mathbf{L}))$, defined by

$$\underline{\mu}(W_\sigma(A)) = \mu(A), \quad A \in \mathbf{A}(\mathbf{L}\ ). \tag{2.5}$$

The map $\mu \to \underline{\mu}$ is a bijection between $\mathbf{I}_\mathbf{R}^\sigma(\mathbf{L})$ and $\mathbf{I}_\mathbf{R}^\sigma(\mathbf{W}_{\boldsymbol{\sigma}}(\mathbf{L}))$.

3. The Case of Finite Product of Lattices

3.1. Notations

Let **X**, **Y** be abstract sets and $\mathbf{L_1}$ a lattice of subsets of **X** and $\mathbf{L_2}$ a lattice of subsets of **Y**. We denote:

(1) $\mathbf{L}^* = \mathbf{L_1} \times \mathbf{L_2} = \{L_1 \times L_2 / L_1 \in \mathbf{L_1}, L_2 \in \mathbf{L_2}\}$,

(2) $\mathbf{L} = \mathbf{L}(\mathbf{L}^*)$, the lattice generated by $\mathbf{L}^*$. ¡list-item¿¡label/¿

We have the following:

(3) $\mathbf{A}(\mathbf{L_1}) \times \mathbf{A}(\mathbf{L_2}) = \mathbf{A}(\mathbf{L_1} \times \mathbf{L_2})$,

(4) $\mathbf{A}(\mathbf{L}^*) = \mathbf{A}(\mathbf{L})$,

(5) $\mathbf{S_L}(\mu) = \mathbf{S_{L^*}}(\mu)$,

(6) $\mathbf{I}_\sigma(\mathbf{L}^*) = \mathbf{I}_\sigma(\mathbf{L})$,

(7) $\mathbf{I_R}(\mathbf{L}^*) = \mathbf{I_R}(\mathbf{L})$.

3.2. *Results*

Theorem 3.1 (the finite product of lattices/regular measures). *Let* $\mathbf{X}$, $\mathbf{Y}$ *be abstract sets and let* $\mathbf{L_1}$, $\mathbf{L_2}$ *be lattices of subsets of* $\mathbf{X}$ *and* $\mathbf{Y}$, *respectively. Then* $\mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2}) = \mathbf{I_R}(\mathbf{L})$.

Proof. For $A \in \mathbf{A}(\mathbf{L_1}) \times \mathbf{A}(\mathbf{L_2}) = \mathbf{A}(\mathbf{L_1} \times \mathbf{L_2})$, we have $A = \bigcup_{i=1}^{n} A_1^i \times A_2^i$, disjoint union and $A_1^i \in \mathbf{A}(\mathbf{L_1})$, $A_2^i \in \mathbf{A}(\mathbf{L_2})$.

Let $\mu \in \mathbf{I_R}(\mathbf{L_1})$ and $\nu \in \mathbf{I_R}(\mathbf{L_2})$ and consider $\mu \times \nu$ defined on $\mathbf{A}(\mathbf{L_1}) \times \mathbf{A}(\mathbf{L_2})$.

If $\mu \times \nu(A) = 1$, then $\mu \times \nu(A_1^i \times A_2^i) = 1$ for some i.

Then $\mu\ (A_1^i)\nu(A_2^i) = 1$, and since μ and ν are zero-one valued measures, $\mu(A_1^i)= 1$ and $\nu(A_2^i) = 1$. By the regularity of μ and ν there exist $L_1 \subset A_1^i$, $L_1 \in \mathbf{L_1}$ with $\mu(L_1) = 1$ and $L_2 \subset A_2^i$, $L_2 \in \mathbf{L_2}$ with $\nu(L_2) = 1$.

Therefore $\mu \times \nu(L_1 \times L_2) = \mu(L_1)\nu(L_2) = 1$ and $L_1 \times L_2 \in \mathbf{L}^*$.

If we let $M = L_1 \times L_2 \subset A_1^i \times A_2^i \subset A$, then

$$\mu \times \nu(A) = \sup\{\mu \times \nu(M)/M \subset A, M \in \mathbf{L}^*\} \Longrightarrow \mu \times \nu \in \mathbf{I_R}(\mathbf{L}^*). \tag{3.1}$$

□

Conversely, let $\mu \in \mathbf{I_R}(\mathbf{L}^*) = \mathbf{I_R}(\mathbf{L})$ and define μ_1 on $\mathbf{A}(\mathbf{L_1})$ by $\mu_1(A) = \mu(A \times \mathbf{Y})$, $A \in \mathbf{A}(\mathbf{L_1})$. Since μ is a zero-one measure on $\mathbf{A}(\mathbf{L_1} \times \mathbf{L_2})$, it follows that μ_1 is a zero-one measure on $\mathbf{A}(\mathbf{L_1})$, that is, $\mu_1 \in \mathbf{I_R}(\mathbf{L_1})$.

Suppose $\mu_1(A) = \mu(A \times \mathbf{Y}) = 1$; there exists $A \times \mathbf{Y} \supset L_1 \times L_2 \in \mathbf{L}^*$ such that $\mu(L_1 \times \mathbf{Y}) = 1$ and $\mu(L_1 \times L_2) = 1$. Then $\mu_1(L_1) = \mu(L_1 \times \mathbf{Y}) = 1$ and $L_1 \subset A$ which shows that $\mu_1 \in \mathbf{I_R}(\mathbf{L_1})$. Similarly take μ_2 on $\mathbf{A}(\mathbf{L_2})$ defined by $\mu_2(B) = \mu(\mathbf{X} \times B)$, $B \in \mathbf{A}(\mathbf{L_2})$.

Then, as before μ_2 is regular on $\mathbf{L_2}$.

Finally for any $A \in \mathbf{A}(\mathbf{L_1})$ and any $B \in \mathbf{A}(\mathbf{L_2})$ we have $\mu_1 \times \mu_2(A \times B) = \mu_1(A)\mu_2(B) = \mu(A \times \mathbf{Y})\mu(\mathbf{X} \times B) = \mu[(A \times \mathbf{Y}) \cap (\mathbf{X} \times B)] = \mu[(A \cap \mathbf{X}) \times (\mathbf{Y} \cap B)] = \mu(A \times B)$ which shows that $\mu = \mu_1 \times \mu_2$, and therefore $\mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2}) = \mathbf{I_R}(\mathbf{L})$.

Theorem 3.2 (the product of lattices/σ-smooth regular measures). *Let* $\mathbf{X}$, $\mathbf{Y}$ *be abstract sets and let* $\mathbf{L_1}$, $\mathbf{L_2}$ *be lattices of subsets of* $\mathbf{X}$ *and* $\mathbf{Y}$, *respectively. Then* $\mathbf{I_R^\sigma}(\mathbf{L_1}) \times \mathbf{I_R^\sigma}(\mathbf{L_2}) = \mathbf{I_R^\sigma}(\mathbf{L_1} \times \mathbf{L_2})$.

Proof. Let $\mu \in \mathbf{I_R^\sigma}(\mathbf{L_1})$ and $\nu \in \mathbf{I_R^\sigma}(\mathbf{L_2})$. Hence for $A_{1n} \in \mathbf{A}(\mathbf{L_1})$ with $A_{1n} \downarrow \emptyset$ we have $\mu(A_{1n}) \to 0$ and for $A_{2n} \in \mathbf{A}(\mathbf{L_2})$ with $A_{2n} \downarrow \emptyset$ we have $\nu(A_{2n}) \to 0$, $n = 1, 2, \ldots$.

Consider the sequence $\{B_n\}$ of sets from $\mathbf{A}(\mathbf{L_1}) \times \mathbf{A}(\mathbf{L_2})$. As in Theorem 3.1 $B_n = \bigcup_{i=1}^{k} A_{1n}^i \times A_{2n}^i$, disjoint union and $A_{1n}^i \in \mathbf{A}(\mathbf{L_1})$, $A_{2n}^i \in \mathbf{A}(\mathbf{L_2})$.

Suppose that $B_n \downarrow \emptyset$, that is, $A_{1n}^i \times A_{2n}^i \downarrow \emptyset$ for all i. Therefore $A_{1n} \downarrow \emptyset$ or $A_{2n} \downarrow \emptyset$ or both:

$$\begin{aligned} \mu \times \nu(B_n) &= \mu \times \nu\left[\bigcup_{i=1}^{k}\left(A_{1n}^i \times A_{2n}^i\right)\right] = \sum_{i=1}^{k} \mu \times \nu\left(A_{1n}^i \times A_{2n}^i\right) \\ &= \sum_{i=1}^{k} \mu\left(A_{1n}^i\right)\nu\left(A_{2n}^i\right) \longrightarrow 0, \text{ therefore } \mu \times \nu \in \mathbf{I_R^\sigma}(\mathbf{L_1} \times \mathbf{L_2}). \end{aligned} \tag{3.2}$$

□

Conversely, let $\mu \in \mathbf{I}_{\mathbf{R}}^{\sigma}(\mathbf{L_1} \times \mathbf{L_2})$ and define μ_1 on $\mathbf{A}(\mathbf{L_1})$ by

$$\mu_1(A) = \mu(A \times \mathbf{Y}), \quad A \in \mathbf{A}(\mathbf{L_1}). \tag{3.3}$$

If $\{A_n\}$ is a sequence of sets with $A_n \in \mathbf{A}(\mathbf{L_1})$ and $A_n \downarrow \emptyset$, then $A_n \times \mathbf{Y} \downarrow \emptyset$, and since $\mu \in \mathbf{I}^{\sigma}(\mathbf{L_1} \times \mathbf{L_2})$ it follows that $\mu(A_n \times \mathbf{Y}) \to 0$.

Therefore $\mu_1 \in \mathbf{I}^{\sigma}(\mathbf{L_1})$.

Similarly, defining μ_2 on $\mathbf{A}(\mathbf{L_2})$ by $\mu_2(B) = \mu(\mathbf{X} \times B)$, $B \in \mathbf{A}(\mathbf{L_2})$ we get $\mu_2 \in \mathbf{I}^{\sigma}(\mathbf{L_2})$.

Hence $\mu = \mu_1 \times \mu_2 \in \mathbf{I}^{\sigma}(\mathbf{L_1}) \times \mathbf{I}^{\sigma}(\mathbf{L_2}) = \mathbf{I}^{\sigma}(\mathbf{L_1} \times \mathbf{L_2})$.

Theorem 3.3 (product of supports of measures). *Let* $\mathbf{X}, \mathbf{Y}$ *be abstract sets and let* $\mathbf{L_1}, \mathbf{L_2}$ *be lattices of subsets of* $\mathbf{X}$ *and* $\mathbf{Y}$, *respectively. The following statements are true:*

(a) *if* $\mu = \mu_1 \times \mu_2 \in \mathbf{I}(\mathbf{L_1}) \times \mathbf{I}(\mathbf{L_2}) = \mathbf{I}(\mathbf{L_1} \times \mathbf{L_2})$ *then* $\mathbf{S}(\mu) = \mathbf{S}(\mu_1) \times \mathbf{S}(\mu_2)$;

(b) *if* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are* compact *lattices then* $\mathbf{L}$ *is* compact.

Proof. We have

(a) $\mathbf{S}(\mu) = \mathbf{S}(\mu_1 \times \mu_2) = \cap\{L_1 \times L_2 \in \mathbf{L_1} \times \mathbf{L_2} / \mu(L_1 \times L_2) = \mu(\mathbf{X} \times \mathbf{Y})\}$,

$$\mathbf{S}(\mu_1) \times \mathbf{S}(\mu_2) = \cap\left\{L_1 \times L_2 / L_1 \in \mathbf{L_1}, L_2 \in \mathbf{L_2}, \mu_1(L_1) = \mu_1(\mathbf{X}),\ \mu_2(L_2) = \mu_2(\mathbf{Y})\right\}, \tag{3.4}$$

But $\mu(L_1 \times L_2) = \mu_1 \times \mu_2(L_1 \times L_2) = \mu_1(L_1)\mu_2(L_2)$ and $\mu(\mathbf{X} \times \mathbf{Y}) = \mu_1 \times \mu_2(\mathbf{X} \times \mathbf{Y}) = \mu_1(\mathbf{X})\mu_2(\mathbf{Y})$,

(b) $\mathbf{S}(\mu) = \mathbf{S}(\mu_1) \times \mathbf{S}(\mu_2) \neq \emptyset$, since $\mathbf{S}(\mu_i) \neq \emptyset$, $\mathbf{L}_i$ being compact.

□

Theorem 3.4 (product of Wallman spaces/Wallman topologies). *Consider the spaces* $\mathbf{I_R}(\mathbf{L}_i)$ *with the* Wallman topologies $\mathbf{tW}_i(\mathbf{L}_i)$, $i = 1, 2$.

It is known that the topological spaces $(\mathbf{I_R}(\mathbf{L}_i), \mathbf{tW}_i(\mathbf{L}_i))$ *are compact and* $\mathbf{T_1}$. *Then the topological space* $(\mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2}),\ \mathbf{tW_1}(\mathbf{L_1}) \times \mathbf{tW_2}(\mathbf{L_2}))$ *is also* compact *and* $\mathbf{T_1}$.

Proof. Since $\mathbf{I_R}(\mathbf{L}_i)$ are compact topological spaces,

$$\begin{gathered}\mathbf{S}_{\mathbf{L_1}}\left(\underline{\mu}\right) = \cap\left\{W_1(L_1) \in \mathbf{W_1}(\mathbf{L_1}) / \underline{\mu}(W_1(L_1)) = 1\right\} \neq \emptyset, \\ \mathbf{S}_{\mathbf{L_2}}(\underline{\nu}) = \cap\left\{W_2(L_2) \in \mathbf{W_2}(\mathbf{L_2}) / \underline{\nu}(W_2(L_2)) = 1\right\} \neq \emptyset.\end{gathered} \tag{3.5}$$

We have

$$\begin{gathered}\underline{\mu}(W_1(A)) = \mu(A), \qquad \mu \in \mathbf{I_R}(\mathbf{L_1}), \qquad A \in \mathbf{A}(\mathbf{L_1}), \qquad \underline{\mu} \in \mathbf{I_R}(\mathbf{W_1}(\mathbf{L_1})), \\ \underline{\nu}(W_2(B)) = \nu(B), \qquad \nu \in \mathbf{I_R}(\mathbf{L_2}), \qquad B \in \mathbf{A}(\mathbf{L_2}), \qquad \underline{\nu} \in \mathbf{I_R}(\mathbf{W_2}(\mathbf{L_2})).\end{gathered} \tag{3.6}$$

Therefore

$$\begin{gathered}\underline{\mu \times \nu}(W_1(A) \times W_2(B)) = \mu \times \nu(A \times B) = \mu(A)\nu(B), \\ \underline{\mu} \times \underline{\nu}(W_1(A) \times W_2(B)) = \underline{\mu}(W_1(A))\ \underline{\nu}(W_2(B)) = \mu(A)\nu(B),\end{gathered} \tag{3.7}$$

so that $\underline{\mu \times \nu} = \underline{\mu} \times \underline{\nu} \in \mathbf{I_R}(\mathbf{W_1}(\mathbf{L_1})) \times \mathbf{I_R}(\mathbf{W_2}(\mathbf{L_2}))$, and then $S_{L_1 \times L_2}(\underline{\mu \times \nu}) = S_{L_1 \times L_2}(\underline{\mu} \times \underline{\nu}) = S_{L_1}(\underline{\mu}) \times S_{L_2}(\underline{\nu}) \neq \emptyset \Rightarrow \mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2})$ is *compact*. □

To show that $\mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2})$ is a $\mathbf{T_1}$-*space*, let $\mu, \nu \in \mathbf{I_R}(\mathbf{L})$ and suppose $\mu \neq \nu$. Since $\mu = \mu_1 \times \mu_2$ with $\mu_1, \mu_2 \in \mathbf{I_R}(\mathbf{L_1})$ and $\nu = \nu_1 \times \nu_2$ with $\nu_1, \nu_2 \in \mathbf{I_R}(\mathbf{L_2})$ we get $\mu_1 \neq \nu_1$ and $\mu_2 \neq \nu_2$. There exist $L_1, \widetilde{L}_1 \in \mathbf{L_1}$ and $L_2, \widetilde{L}_2 \in \mathbf{L_2}$ with

$$\begin{aligned} &\mu_1 \in W_1(L_1), \quad \nu_1 \in W_1(L_1)'; \quad \nu_1 \in W_1\left(\widetilde{L}_1\right), \quad \mu_1 \in W_1\left(\widetilde{L}_1\right)', \\ &\mu_2 \in W_2(L_2), \quad \nu_2 \in W_2(L_2)'; \quad \nu_2 \in W_2\left(\widetilde{L}_2\right), \quad \mu_2 \in W_2\left(\widetilde{L}_2\right)'. \end{aligned} \tag{3.8}$$

Therefore $\mu_1(L_1) = \mu_2(L_2) = 1$, $\nu_1(L_1) = \nu_2(L_2) = 0$, $\mu_1(\widetilde{L}_1) = \mu_2(\widetilde{L}_2) = 0$, $\nu_1(\widetilde{L}_1) = \nu_2(\widetilde{L}_2) = 1$ which implies $\mu \in W(L_1 \times L_2)$, $\nu \in W(L_1 \times L_2)'$; $\nu \in W(\widetilde{L}_1 \times \widetilde{L}_2)$, $\mu \in W(\widetilde{L}_1 \times \widetilde{L}_2)'$.

Theorem 3.5 (product of normal lattices). *Let* $\mathbf{X}$, $\mathbf{Y}$ *be abstract sets and let* $\mathbf{L_1}$, $\mathbf{L_2}$ *be* normal *lattices of subsets of* $\mathbf{X}$ *and* $\mathbf{Y}$, *respectively. Then* $\mathbf{L}$ *is a* normal *lattice of subsets of* $\mathbf{X} \times \mathbf{Y}$.

Proof. Let $\mu \in \mathbf{I}(\mathbf{L})$ and $\nu, \rho \in \mathbf{I_R}(\mathbf{L})$ such that $\mu \leq \nu, \rho$ on $\mathbf{L}$.

Then, since $\mu = \mu_1 \times \mu_2 \in \mathbf{I}(\mathbf{L_1}) \times \mathbf{I}(\mathbf{L_2})$, $\nu = \nu_1 \times \nu_2 \in \mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2})$ and $\rho = \rho_1 \times \rho_2 \in \mathbf{I_R}(\mathbf{L_1}) \times \mathbf{I_R}(\mathbf{L_2})$, we obtain $\mu_i \leq \nu_i, \rho_i$ on $\mathbf{L}_i$, $i = 1, 2$.

$\mathbf{L}_i$ normal lattices $\Rightarrow \nu_i = \rho_i$; therefore $\nu_1 \times \nu_2 = \rho_1 \times \rho_2$, that is, $\nu = \rho$. □

3.3. Examples

(1) Let $\mathbf{X}$, $\mathbf{Y}$ be topological spaces and let $\mathbf{O}_1$, $\mathbf{O}_2$ be the lattices of *open* sets of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Consider the product space $\mathbf{X} \times \mathbf{Y}$ with a base of *open* sets given by

$$\{O_1 \times O_2 / O_1 \in \mathbf{O}_1, O_2 \in \mathbf{O}_2\}. \tag{3.9}$$

We have

$$\begin{aligned} (O_1 \times O_2)' &= \{(x, y) \in \mathbf{X} \times \mathbf{Y} / (x, y) \notin (O_1 \times O_2)\} \\ &= \{(x, y) / (x, y) \in (\mathbf{X} \times O_2') \text{ or } (x, y) \in (O_1' \times \mathbf{Y})\} \\ &= (\mathbf{X} \times O_2') \cup (O_1' \times \mathbf{Y}) = (\mathbf{X} \times F_2) \cup (F_1 \times \mathbf{Y}). \end{aligned} \tag{3.10}$$

Hence $\mathbf{F} = \mathbf{t}(\mathbf{L}(\mathbf{F}_1 \times \mathbf{F}_2))$ where $\mathbf{F}_1$, $\mathbf{F}_2$ are the lattices of *closed* sets of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

(2) Let $\mathbf{X}$, $\mathbf{Y}$ be topological $\mathbf{T_{3.5}}$-spaces and let $\mathbf{Z}_1$, $\mathbf{Z}_2$ be the lattices of *zero* sets of continuous functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Then for the product space $\mathbf{X} \times \mathbf{Y}$ we consider a base of open sets given by

$$\{Z_1' \times Z_2' / Z_1 \in \mathbf{Z}_1, Z_2 \in \mathbf{Z}_2\} \tag{3.11}$$

such that any open set from $\mathbf{X} \times \mathbf{Y}$ is of the form $O = \bigcup_\alpha Z_{1\alpha}' \times Z_{2\alpha}'$ and any closed set is

$$F = O' = \bigcap_\alpha (Z_{1\varepsilon} \times Y) \cup (X \times Z_{2\alpha}) \in \mathbf{t}(\mathbf{L}(\mathbf{Z}_1 \times \mathbf{Z}_2)) \tag{3.12}$$

and then $\mathbf{F} = \mathbf{t}(\mathbf{L}(\mathbf{Z}_1 \times \mathbf{Z}_2))$.

4. The General Case of Product of Lattices

Let $\{\mathbf{X}_\alpha\}_{\alpha\in\Lambda}$ be a collection of abstract sets (Λ an arbitrary index set) and let $\mathbf{L}_\alpha$ be the lattice of subsets of $\mathbf{X}_\alpha$ for all α.

We denote

$$\mathbf{L}^* = \prod_{\alpha\in\Lambda}\mathbf{L}_\alpha = \left\{\prod_{\alpha\in\Lambda}L_\alpha / L_\alpha \in \mathbf{L}_\alpha, L_\alpha = \mathbf{X}_\alpha \text{ for almost all } \alpha\right\}. \tag{4.1}$$

4.1. Results

Theorem 4.1 (the product of lattices/regular measures). *One has*

$$\prod_{\alpha\in\Lambda}\mathbf{I}_\mathbf{R}(\mathbf{L}_\alpha) = \mathbf{I}_\mathbf{R}(\mathbf{L}) = \mathbf{I}_\mathbf{R}\left(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha\right). \tag{4.2}$$

Proof. We note that $\prod_{\alpha\in\Lambda}\mathbf{A}(\mathbf{L}_\alpha) = \mathbf{A}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha) = \mathbf{A}(\mathbf{L})$ and that $\prod_{\alpha\in\Lambda}\mathbf{A}(\mathbf{L}_\alpha)$ is the collection of all finite cylinder sets which means that if $B \in \prod_{\alpha\in\Lambda}\mathbf{A}(\mathbf{L}_\alpha)$ then B is a cylinder set for which there exists a nonempty finite subset $F = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of Λ and a subset $E_F \in \prod_{\alpha\in F}\mathbf{A}(\mathbf{L}_\alpha)$ such that $B = \mathrm{P}_F^{-1}(E_F)$ with

$$P_F : \prod_{\alpha\in\Lambda}\mathbf{X}_\alpha \longrightarrow \prod_{\alpha\in\Lambda}\mathbf{X}_\alpha = \mathbf{X}_{\alpha_1}\times\mathbf{X}_{\alpha_2}\times\cdots\times\mathbf{X}_{\alpha_n}, \qquad P_\alpha : \prod_{\alpha\in\Lambda}\mathbf{X}_\alpha \longrightarrow \mathbf{X}_\alpha. \tag{4.3}$$

Let $\mu_\alpha \in \mathbf{I}_\mathbf{R}(\mathbf{L}_\alpha)$ for all $\alpha\in\Lambda$ with $\mu_\alpha : \mathbf{A}(\mathbf{L}_\alpha)$ and define

$$\mu = \prod_{\alpha\in\Lambda}\mu_\alpha \in \prod_{\alpha\in\Lambda}\mathbf{I}_\mathbf{R}(\mathbf{L}_\alpha), \qquad \mu : \prod_{\alpha\in\Lambda}\mathbf{A}(\mathbf{L}_\alpha). \tag{4.4}$$

Let $A \in \prod_{\alpha\in\Lambda}\mathbf{A}(\mathbf{L}_\alpha)$ with $\mu(A) = 1$. Then $(\prod_{\alpha\in\Lambda}\mu_\alpha)(A) = (\prod_{\alpha\in\Lambda}\mu_\alpha)(P_F^{-1}(E_F)) = 1$ that is

$$\prod_{\alpha\in F}A(L_\alpha) \xrightarrow{P_F^{-1}} \prod_{\alpha}A(L_\alpha) \xrightarrow{\prod_\alpha \mu_\alpha} \{0,1\} \tag{4.5}$$

and for $E_F \in \prod_{\alpha\in F}\mathbf{A}(\mathbf{L}_\alpha)$ we have

$$\begin{aligned}\left(\prod_{\alpha\in\Lambda}\mu_\alpha P_F^{-1}\right)(E_F) &= \left(\prod_{\alpha\in\Lambda}\mu_\alpha\right)\left(P_F^{-1}(E_F)\right) = \left(\prod_{\alpha\in F}\right)(E_F) \\ &= \left(\mu_{\alpha_1}\times\mu_{\alpha_2}\times\cdots\times\mu_{\alpha_n}\right)(E_F) = 1.\end{aligned} \tag{4.6}$$

As in the finite case, we get $E_F \supset L_{\alpha_1}\times L_{\alpha_2}\times\cdots L_{\alpha_n}$ where $L_{\alpha_i}\in\mathbf{L}_{\alpha_i}$ and $\mu_{\alpha_i}(L_{\alpha_i}) = 1$ for all $i = 1, 2, \dots, n$. Then $A = P_F^{-1}(E_F) \supset P_F^{-1}(L_{\alpha_1}\times L_{\alpha_2}\times\cdots L_{\alpha_n})$ and $P_F^{-1}(L_{\alpha_1}\times L_{\alpha_2}\times\cdots L_{\alpha_n}) = 1$, which shows that $\mu(A) = \sup\{\mu(P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n}))/P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n}) \subset A$ and $P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n}) \in \prod_{\alpha\in\Lambda}\mathbf{L}_\alpha = \mathbf{L}^*\}$; hence μ is $\mathbf{L}$-regular.

Conversely, let $\mu \in \mathbf{I_R}(\mathbf{L}) = \mathbf{I_R}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ and define μ_α on $\mathbf{A}(\mathbf{L}_\alpha)$ by

$$\mu_\alpha(A) = \mu\left(A \times \prod_{\beta\in\Lambda-\{\alpha\}} \mathbf{X}_\beta\right), \quad A \in \mathbf{A}(\mathbf{L}_\alpha), \text{ that is, } \mu_\alpha(A) = \mu\left(P_\alpha^{-1}(A)\right). \tag{4.7}$$

Since μ is a zero-one valued measure on $\mathbf{A}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ it follows from the above definition that $\mu_\alpha \in \mathbf{I}(\mathbf{L}_\alpha)$. If $\mu_\alpha(A) = 1$, then $\mu(P_\alpha^{-1}(A)) = 1$, and since μ is $\mathbf{L}$-regular, there exists $\prod_{\beta\in\Lambda}L_\beta$ such that $P_\alpha^{-1}(A) \supset \prod_{\beta\in\Lambda}L_\beta \in \mathbf{L}^*$ and $\mu(\prod_{\beta\in\Lambda}L_\beta) = 1$.

Then $P_\alpha^{-1}(L_\alpha) \subset P_\alpha^{-1}(A)$ and $\mu_\alpha(L_\alpha) = \mu_\alpha(P_\alpha^{-1}(L_\alpha)) = 1$.

Therefore $\mu_\alpha(A) = \sup\{\mu_\alpha(L_\alpha)/L_\alpha \subset A, L_\alpha \in \mathbf{L}_\alpha\}$, that is, $\mu_\alpha \in \mathbf{I_R}(\mathbf{L}_\alpha)$. Next, if $B \in \mathbf{L}^*$, we may consider $B = P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n})$ and then $\prod_{\alpha\in\Lambda}\mu_\alpha(B) = \prod_{\alpha\in\Lambda}\mu_\alpha(P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n})) = \prod_{\alpha\in F}\mu_\alpha(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n}) = (\mu_{\alpha_1} \times \mu_{\alpha_2} \times \cdots \times \mu_{\alpha_n})(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n}) = \mu_{\alpha_1}(L_{\alpha_1})\mu_{\alpha_2}(L_{\alpha_2})\cdots\mu_{\alpha_n}(L_{\alpha_n}) = \mu(P_{\alpha_1}^{-1}(L_{\alpha_1}))\cdots\mu(P_{\alpha_n}^{-1}(L_{\alpha_n}))$. If $\prod_{\alpha\in\Lambda}\mu_\alpha(B) = 1$, then $\mu(P_{\alpha_i}^{-1}(L_{\alpha_i}))$ for all i; hence $\mu(\bigcap_{i=1}^n P_{\alpha_i}^{-1}(L_{\alpha_i})) = 1$ and $\mu(P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \cdots L_{\alpha_n})) = \mu(\prod_\alpha L_\alpha) = 1$, that is, $\mu(B) = 1$. Thus $\mu = \prod_\alpha\mu_\alpha \in \mathbf{I_R}(\mathbf{L}^*)$, and then $\mu = \prod_\alpha\mu_\alpha$ on $\prod_\alpha A(L_\alpha)$. □

Theorem 4.2 (the product of normal lattices). *Let $\mathbf{L}_\alpha$ be a lattice of subsets of $\mathbf{X}_\alpha$. Then*

(a) *if $\mu = \prod_\alpha\mu_\alpha \in \mathbf{I}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha) = \prod_{\alpha\in\Lambda}\mathbf{I}(\mathbf{L}_\alpha)$ we have $\mathbf{S}(\mu) = \prod_{\alpha\in\Lambda}\mathbf{S}(\mu_\alpha)$;*

(b) *if $\mathbf{L}_\alpha$ disjunctive for all $\alpha \in \Lambda$, then $\mathbf{L} = \mathbf{L}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ is a disjunctive lattice of subsets of $\prod_{\alpha\in\Lambda}\mathbf{X}_\alpha$;*

(c) *suppose that $\mathbf{L}_\alpha$ is a* normal *lattice of subsets of $\mathbf{X}_\alpha$ for all $\alpha \in \Lambda$; then $\mathbf{L} = \mathbf{L}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ is a* normal *lattice of subsets of $\prod_{\alpha\in\Lambda}\mathbf{X}_\alpha$.*

Proof. (a) We have $\mathbf{S}(\mu_\alpha) = \cap\{L_\alpha \in \mathbf{L}_\alpha/\mu_\alpha(L_\alpha) = \mu_\alpha(X_\alpha) = 1\}$ and $\mathbf{S}(\mu) = \mathbf{S}(\prod_{\alpha\in\Lambda}\mu_\alpha) = \cap\{\prod_{\alpha\in\Lambda}L_\alpha \in \prod_{\alpha\in\Lambda}\mathbf{L}_\alpha/\mu(\prod_\alpha L_\alpha) = \mu(\prod_\alpha X_\alpha) = 1\}$. But $\mu(\prod_\alpha L_\alpha) = 1$ implies $\prod_\alpha\mu_\alpha(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha) = 1$. Then $\mathbf{S}(\mu) = \prod_{\alpha\in\Lambda}\mathbf{S}(\mu_\alpha)$.

(b) Let $x = (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda}X_\alpha$ Since $\prod_{\alpha\in\Lambda}\mu_{x_\alpha}(\prod_{\alpha\in\Lambda}A_\alpha) = \mu_x(\prod_{\alpha\in\Lambda}A_\alpha)$ we get $\prod_{\alpha\in\Lambda}\mu_{x_\alpha} = \mu_x$.

$\mathbf{L}_\alpha$ disjunctive implies $\mu_{x_\alpha} \in \mathbf{I_R}(\mathbf{L}_\alpha)$ for all $\alpha \in \Lambda$ and then $\prod_{\alpha\in\Lambda}\mu_{x_\alpha} \in \prod_{\alpha\in\Lambda}\mathbf{I_R}(\mathbf{L}_\alpha) = \mathbf{I_R}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$; therefore $\mu_x \in \mathbf{I_R}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ which proves that $\mathbf{L} = \mathbf{L}(\prod_{\alpha\in\Lambda}\mathbf{L}_\alpha)$ is disjunctive.

(c) Let $\mu \in \mathbf{I}(\mathbf{L})$ and $\nu, \rho \in \mathbf{I}_R(\mathbf{L})$ such that $\mu \le \nu, \rho$ on $\mathbf{L}$.

But $\mu = \prod_\alpha\mu_\alpha \in \prod_{\alpha\in\Lambda}I(L_\alpha)$ and both $\nu = \prod_{\alpha\in}\nu_\alpha$, $\rho = \prod_{\alpha\in}\rho_\alpha \in \prod_{\alpha\in\Lambda}I_R(L_\alpha)$ and then $\prod_\alpha\mu_\alpha \le \prod_{\alpha\in}\nu_\alpha$ and $\prod_\alpha\mu_\alpha \le \prod_{\alpha\in}\rho_\alpha$ on $\mathbf{L}$ with $\mu_\alpha(A_\alpha) = \mu(P_\alpha^{-1}(A_\alpha))$, $\nu_\alpha(A_\alpha) = \nu(P_\alpha^{-1}(A_\alpha))$ for $A_\alpha \in A(L_\alpha)$. By the previous work we get $\mu_\alpha \le \nu_\alpha$ and $\mu_\alpha \le \rho_\alpha$ on $\mathbf{L}_\alpha$.

Since each $\mathbf{L}_\alpha$ is normal it follows that $\nu_\alpha = \rho_\alpha$ for all $\alpha \in \Lambda$, and therefore $\nu = (\nu_\alpha)_{\alpha\in\Lambda} = (\rho_\alpha)_{\alpha\in\Lambda} = \rho$ which proves that $\mathbf{L}$ is a normal lattice. □

4.2. Examples

(3) Let $\mathbf{X}_\alpha$ be a topological $\mathbf{T_{3.5}}$-spaces and let $\mathbf{L}_\alpha = \mathbf{Z}_\alpha$ be the replete lattices of *zero* sets of continuous functions of $\mathbf{X}_\alpha$ for all $\alpha \in A$.

Then each $\mathbf{X}_\alpha$ is said to be *realcompact*.

Consider a lattice $\mathbf{Z}$ of subsets of $\prod_{\alpha\in A}\mathbf{X}_\alpha$ such that

$$\prod_{\alpha\in A}\mathbf{Z}_\alpha \subset t\left(\prod_{\alpha\in A}\mathbf{Z}_\alpha\right). \tag{4.8}$$

Then $\mathbf{Z}$ is *replete*, and $\prod_{\alpha\in A}\mathbf{X}_\alpha$ is *realcompact*.

(4) Let $\mathbf{X}_\alpha$ be a T_2 and 0-dimensional space and let $\mathbf{L}_\alpha = \mathbf{C}_\alpha$ be the *replete* lattice of *clopen* sets for all $\alpha \in A$. Then each $\mathbf{X}_\alpha$ is said to be $\mathbf{N}$-*compact*. Consider any lattice $\mathbf{C}$ of subsets of $\prod_{\alpha\in A}\mathbf{X}_\alpha$ such that $\prod_{\alpha\in A}\mathbf{C}_\alpha \subset t(\prod_{\alpha\in A}\mathbf{C}_\alpha) \subset t(\prod_{\alpha\in A}\mathbf{Z}_\alpha) = \mathbf{F}$ and $\mathbf{C}$ is *replete* and $\prod_{\alpha\in A}\mathbf{X}_\alpha$ is $\mathbf{N}$-*compact*.

Acknowledgment

The author is grateful to the late Dr. George Bachman for suggesting this research.

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