Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 102980, 11 pages
doi:10.1155/2012/102980

Research Article

Common Fixed Points, Invariant Approximation and Generalized Weak Contractions

Sumit Chandok

Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar-143001, India

Correspondence should be addressed to Sumit Chandok, chansok.s@gmail.com

Received 23 March 2012; Revised 23 August 2012; Accepted 31 August 2012

Academic Editor: Harvinder S. Sidhu

Sufficient conditions for the existence of a common fixed point of generalized f-weakly contractive noncommuting mappings are derived. As applications, some results on the set of best approximation for this class of mappings are obtained. The proved results generalize and extend various known results in the literature.

1. Introduction and Preliminaries

It is well known that Banach's fixed point theorem for contraction mappings is one of the pivotal result of analysis. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *contraction* if there exists $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

If the metric space (X, d) is complete, then the mapping satisfying (1.1) has a unique fixed point.

A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity. Kannan [1, 2] proved the following result, giving an affirmative answer to the above question.

Theorem 1.1. *If $T : X \to X$, where (X, d) is a complete metric space, satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \tag{1.2}$$

where $0 \leq k < 1/2$ and $x, y \in X$, then T has a unique fixed point.

The mappings satisfying (1.2) are called Kannan type mappings. A similar type of contractive condition has been studied by Chatterjea [3] and he proved the following result.

Theorem 1.2. *If $T : X \to X$, where (X, d) is a complete metric space, satisfies*

$$d(Tx, Ty) \le k[d(x, Ty) + d(y, Tx)], \tag{1.3}$$

where $0 \le k < 1/2$ and $x, y \in X$, then T has a unique fixed point.

In Theorems 1.1 and 1.2 there is no requirement of continuity of T.

A map $T : X \to X$ is called a *weakly contractive* (see [4–6]) if for each $x, y \in X$,

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)), \tag{1.4}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

If we take $\psi(x) = (1 - k)x$, $0 \le k < 1$, then a weakly contractive mapping is called contraction.

A map $T : X \to X$ is called *f-weakly contractive* (see [7]) if for each $x, y \in X$,

$$d(Tx, Ty) \le d(fx, fy) - \psi(d(fx, fy)), \tag{1.5}$$

where $f : X \to X$ is a self-mapping, $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

If we take $\psi(x) = (1 - k)x$, $0 \le k < 1$, then a f-weakly contractive mapping is called f-contraction. Further, if $f =$ identity mapping and $\psi(x) = (1-k)x$, $0 \le k < 1$, then a f-weakly contractive mapping is called contraction.

A map $T : X \to X$ is called a *generalized weakly contractive* (see [5]) if for each $x, y \in X$,

$$d(Tx, Ty) \le \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)), \tag{1.6}$$

where $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If we take $\psi(x, y) = (1-(k/2))(x+y)$, $0 \le k < 1/2$, then inequality (1.6) reduces to (1.3). Choudhury [5] shows that generalized weakly contractive mappings are generalizations of contractive mappings given by Chatterjea (1.3), and it constitutes a strictly larger class of mappings than given by Chatterjea.

A map $T : X \to X$ is called a *generalized f-weakly contractive* [8] if for each $x, y \in X$,

$$d(Tx, Ty) \le \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \psi(d(fx, Ty), d(fy, Tx)), \tag{1.7}$$

where $f : X \to X$ is a self-mapping, $\psi : [0, \infty)^2 \to [0, \infty)$ is continuous such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If $f =$ identity mapping, then generalized f-weakly contractive mapping is generalized weakly contractive.

For a nonempty subset M of a metric space (X, d) and $x \in X$, an element $y \in M$ is said to be a *best approximant* to x or a *best M-approximant* to x if $d(x, y) = d(x, M) \equiv \inf\{d(x, k) : k \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

A subset M of a normed linear space X is said to be a *convex set* if $\lambda x + (1 - \lambda)y \in M$ for all $x, y \in M$ and $\lambda \in [0, 1]$. A set M is said to be *p-starshaped,* where $p \in M$, provided $\lambda x + (1 - \lambda)p \in M$ for all $x \in M$ and $\lambda \in [0, 1]$, that is, if the segment $[p, x] = \{\lambda x + (1 - \lambda)p : 0 \leq \lambda \leq 1\}$ joining p to x is contained in M for all $x \in M$. M is said to be *starshaped* if it is p-starshaped for some $p \in M$.

Clearly, each convex set M is starshaped but converse is not true.

Suppose that M is a subset of normed linear space X. A mapping T from M to X is said to be *demiclosed* if for every sequence $\{x_n\} \subseteq M$ such that x_n converges weakly to $x \in M$ and $\{T(x_n)\}$ converges strongly to $y \in X$ imply $y = Tx$. T is said to be *demiclosed at 0*, if for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converging strongly to 0, then $Tx = 0$.

For a convex subset M of a normed linear space X, a mapping $T : M \to M$ is said to be *affine* if for all $x, y \in M$, $T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$ for all $\lambda \in [0, 1]$.

The ordered pair (T, I) of two self-maps of a metric space (X, d) is called a *Banach operator pair* [9], if the set $F(I)$ of fixed points of I is T-invariant, that is, $T(F(I)) \subseteq F(I)$. A point $x \in X$ is a coincidence point (common fixed point) of I and T if $Ix = Tx (x = Ix = Tx)$. The set of fixed points (resp., coincidence points) of I and T is denoted by $F(I, T)$ (resp., $C(I, T)$). The pair (I, T) is called *commuting* if $TIx = ITx$ for all $x \in X$. Obviously, commuting pair, (T, I) is a Banach operator pair but not conversely (see [9]). If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (see [9]). If the self-maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and for some $k \geq 0$, $ITx = TIx$ whenever $x \in F(I)$, that is, $Tx \in F(I)$, then (T, I) is a Banach operator pair. This class of noncommuting mappings is different from the known classes of noncommuting mappings namely, R-weakly commuting, R-subweakly commuting, compatible, weakly compatible, C_q-commuting, and so forth, existing in the literature.

Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. For example, in theoretical economics, such as general equilibrium theory, a situation arises where one needs to know whether the solution to a system of equations necessarily exists, or, more specifically, under what conditions will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorems. Hence, finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect. Alber and Guerre-Delabriere [4] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, Rhoades [6] proved a fixed point theorem which is one of the generalizations of Banach's Fixed Point Theorem in 1922, because the weakly contractions contain many contractions as a special case, and he also showed that some results of [4] are true for any Banach spaces. In fact, weakly contractive mappings are closely related to the mappings introduced by Boyd and Wong [10] and Reich [11]. Many other nonlinear contractive type mappings like Chatterjea, Ciric, Kannan, Reich type, and so forth and their generalizations have been investigated by many authors. Fixed point and common fixed point theorems for different types of nonlinear contractive mappings have been investigated extensively by various researchers (see [1–21] and references cited therein). In this paper, sufficient conditions for the existence of a common fixed point of generalized f-weakly contractive noncommuting mappings are obtained. As applications, we also establish some results on the set of best approximation for this class of mappings. The proved results generalize and extend the corresponding results of [5, 8, 9, 12–14, 18–20] and of few others.

2. Main Results

The following result is a consequence of the main theorem of Choudhury [5].

Lemma 2.1. *Let M be a subset of a metric space (X,d) and T is a self-mapping of M such that $\operatorname{cl} T(M) \subseteq M$. If $\operatorname{cl} T(M)$ is complete and T satisfies*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)), \tag{2.1}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$, for all $x, y \in X$, then T has a unique fixed point in M.

Corollary 2.2 (see [5]). *Let T be a self-mapping of X, where (X,d) is a complete metric space. If T satisfies*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)), \tag{2.2}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$, for all $x, y \in X$, then T has a unique fixed point.

If $\psi(x,y) = ((1/2)-k)(x+y)$, $0 \leq k < (1/2)$, we have Theorem 1.2.

Theorem 2.3. *Let M be a subset of a metric space (X,d), and f and T are self-mappings of M such that $\operatorname{cl} T(F(f)) \subseteq F(f)$. If $\operatorname{cl} T(M)$ is complete, $F(f)$ is nonempty, and f and T satisfy*

$$d(Tx,Ty) \leq \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \psi(d(fx,Ty),d(fy,Tx)), \tag{2.3}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y) = 0$ if and only if $x = y = 0$, for all $x, y \in X$, then $M \cap F(T) \cap F(f)$ is a singleton.

Proof. $\operatorname{cl} T(F(f))$ being a subset of $\operatorname{cl} T(M)$ is complete and and $\operatorname{cl} T(F(f)) \subseteq F(f)$. So for all $x, y \in F(f)$, we have

$$\begin{aligned} d(Tx,Ty) &\leq \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \psi(d(fx,Ty),d(fy,Tx)) \\ &= \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)). \end{aligned} \tag{2.4}$$

Thus, by Lemma 2.1, T has a unique fixed point z in $F(f)$ and consequently, $M \cap F(T) \cap F(f)$ is a singleton. □

Corollary 2.4. *Let M be a subset of a metric space (X,d), and f and T are self-mappings of M. If* $\operatorname{cl} T(M)$ *is complete, (T,f) is a Banach operator pair, $F(f)$ is nonempty and closed, and f and T satisfy*

$$d(Tx,Ty) \le \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \psi(d(fx,Ty),d(fy,Tx)), \tag{2.5}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$, for all $x,y \in X$, then $M \cap F(T) \cap F(f)$ is a singleton.

Example 2.5. Let $X = \{p,q,r\}$ and d is a metric defined on X. Let T and f be self-mappings of X such that $Tp = fq$, $Tq = fq$, $Tr = fp$, $d(fp,fq) = 1$, $d(fq,fr) = 2$, $d(fr,fp) = 1.5$ and $\psi(a,b) = (1/2)\min\{a,b\}$. Then T is a generalized f-weakly contraction, and q is the coincidence point of T and f.

If f =identity mapping, this example given in [5].

If $\psi(x,y) = ((1/2)-k)(x+y)$, $0 \le k < 1/2$, we have the following result.

Corollary 2.6. *Let M be a subset of a metric space (X,d), and f and T be self-mappings of M such that* $\operatorname{cl} T(F(f)) \subseteq F(f)$. *If* $\operatorname{cl} T(M)$ *is complete, $F(f)$ is nonempty, and f and T satisfy*

$$d(Tx,Ty) \le k[d(fx,Ty)+d(fy,Tx)], \tag{2.6}$$

where $0 \le k < 1/2$, for all $x,y \in M$, then $M \cap F(T) \cap F(f)$ is a singleton.

Corollary 2.7. *Let M be a subset of a metric space (X,d) and f and T be self-mappings of M. If* $\operatorname{cl} T(M)$ *is complete, (T,f) is a Banach operator pair, $F(f)$ is nonempty and closed, and f and T satisfy*

$$d(Tx,Ty) \le k[d(fx,Ty)+d(fy,Tx)], \tag{2.7}$$

where $0 \le k < 1/2$, for all $x,y \in M$, then $M \cap F(T) \cap F(f)$ is a singleton.

Theorem 2.8. *Let M be a nonempty subset of a normed (resp., Banach) space X, and Tf be self-mappings of M. Suppose that $F(f)$ is q-starshaped,* $\operatorname{cl} T(F(f)) \subseteq F(f)$ *(resp.,* $w \operatorname{cl} T(F(f)) \subseteq F(f)$*),* $\operatorname{cl} T(M)$ *is compact (resp.,* $w \operatorname{cl} T(M)$ *is weakly compact, and $f-T$ is demiclosed at* 0*), and T satisfies*

$$\begin{aligned} \|Tx-Ty\| \le{} & \frac{1}{2}[\operatorname{dist}(fx,[Ty,q])+\operatorname{dist}(fy,[Tx,q])] \\ & - \psi(\operatorname{dist}(fx,[Ty,q]),\operatorname{dist}(fy,[Tx,q])), \end{aligned} \tag{2.8}$$

where $\psi : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$, for all $x,y \in M$. Then $M \cap F(T) \cap F(f) \ne \emptyset$.

Proof. For each n, define $T_n : M \to M$ by $T_n x = (1-k_n)q + k_n Tx$, $x \in M$ where (k_n) is a sequence in $(0,1)$ such that $k_n \to 1$. Since $F(f)$ is q-starshaped and $\operatorname{cl} T(F(f)) \subseteq F(f)$ (resp., $w \operatorname{cl} T(F(f)) \subseteq F(f)$), we have

$$T_n(x) = (1-k_n)q + k_n Tx = (1-k_n)fq + k_n Tx \in F(f), \tag{2.9}$$

for all $x \in F(f)$ and so $\operatorname{cl} T_n(F(f)) \subseteq F(f)$ (resp., $w \operatorname{cl} T_n(F(f)) \subseteq F(f)$) for each n. Consider

$$\begin{aligned} \|T_n x - T_n y\| &= k_n \|Tx - Ty\| \\ &\le k_n \left[\frac{1}{2}[\operatorname{dist}(fx, [Ty,q]) + \operatorname{dist}(fy, [Tx,q])] \right. \\ &\quad \left. - \psi(\operatorname{dist}(fx, [Ty,q]), \operatorname{dist}(fy, [Tx,q]))\right] \\ &\le k_n \left[\frac{1}{2}[\operatorname{dist}(fx, [Ty,q]) + \operatorname{dist}(fy, [Tx,q])]\right] \\ &\le \frac{k_n}{2}[\|fx - T_n y\| + \|fy - T_n x\|], \end{aligned} \tag{2.10}$$

for all $x, y \in F(f)$. As $\operatorname{cl} T(M)$ is compact, $\operatorname{cl} T_n(M)$ is compact for each n and hence complete. Now by Corollary 2.6, there exists $x_n \in M$ such that x_n is a common fixed point of f and T_n for each n. The compactness of $\operatorname{cl} T(M)$ implies that there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to z \in \operatorname{cl} T(M)$. Since $\{Tx_n\}$ is a sequence in $T(F(f))$, $z \in \operatorname{cl} T(F(f)) \subseteq F(f)$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i} x_{n_i} = (1-k_{n_i})q + k_{n_i} T x_{n_i} \longrightarrow z \tag{2.11}$$

and $\|fx_{n_i} - Tx_{n_i}\| = \|x_{n_i} - Tx_{n_i}\| \to 0$. Further, we have

$$\begin{aligned} \|Tx_{n_i} - Tz\| &\le \frac{1}{2}[\operatorname{dist}(fx_{n_i}, [Tz,q]) + \operatorname{dist}(fz, [Tx_{n_i},q])] \\ &\quad - \psi(\operatorname{dist}(fx_{n_i}, [Tz,q]), \operatorname{dist}(fz, [Tx_{n_i},q])) \\ &= \frac{1}{2}[\operatorname{dist}(x_{n_i}, [Tz,q]) + \operatorname{dist}(fz, [Tx_{n_i},q])] \\ &\quad - \psi(\operatorname{dist}(x_{n_i}, [Tz,q]), \operatorname{dist}(fz, [Tx_{n_i},q])) \\ &\le \frac{1}{2}[\|x_{n_i} - Tz\| + \|fz - Tx_{n_i}\|], \end{aligned} \tag{2.12}$$

on taking limit, we get $z = Tz$ and so $M \cap F(T) \cap F(f) \neq \emptyset$.

Next, the weak compactness of $w \operatorname{cl} T(M)$ implies that $w \operatorname{cl} T_n(M)$ is weakly compact and hence complete. Hence, by Corollary 2.6, for each $n \in \mathbb{N}$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The weak compactness of $w \operatorname{cl} T(M)$ implies that there is subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \rightharpoonup y \in w \operatorname{cl} T(M)$. Since $\{Tx_n\}$ is a sequence in $T(F(f))$, $y \in w \operatorname{cl} T(F(f)) \subseteq F(f)$. Also, we have $fx_{n_i} - Tx_{n_i} = x_{n_i} - Tx_{n_i} \to 0$.

If $f - T$ is demiclosed at 0, then $fy = Ty = y$ and so $M \cap F(T) \cap F(f) \neq \emptyset$. □

Let M be a nonempty subset of a metric space (X, d). Suppose that $C = P_M(u) \cap C_M^f(u)$, where $C_M^f(u) = \{x \in M : fx \in P_M(u)\}$.

Corollary 2.9. *Let X be a normed (resp., Banach) space, and T, f are self-mappings of X. If $u \in X$, $D \subseteq C$, $G = D \cap F(f)$ is q-starshaped, $\operatorname{cl} T(G) \subseteq G$ (resp., $w \operatorname{cl} T(G) \subseteq G$), $\operatorname{cl} T(D)$ is compact (resp., $w \operatorname{cl} T(D)$ is weakly compact and $f - T$ is demiclosed at 0), and T satisfies the inequality (2.8) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.*

Corollary 2.10. *Let X be a normed (resp., Banach) space and T, f are self-mappings of X. If $u \in X$, $D \subseteq P_M(u)$, $G = D \cap F(f)$ is q-starshaped, $\operatorname{cl} T(G) \subseteq G$ (resp., $w \operatorname{cl} T(G) \subseteq G$), $\operatorname{cl} T(D)$ is compact (resp., $w \operatorname{cl} T(D)$ is weakly compact and $f - T$ is demiclosed at 0), and T satisfies the inequality (2.8) for all $x, y \in D$, then $P_M(u) \cap F(f, T)$ is nonempty.*

Remark 2.11. Theorem 2.8 extends and generalizes the corresponding results of [8, 9, 13, 14, 18, 19].

Let G_0 denote the class of closed convex subsets of a normed space X containing 0. For $M \in G_0$ and $p \in X$, let $M_p = \{x \in M : \|x\| \le 2\|p\|\}$. Then $P_M(p) \subset M_p \in G_0$ (see [12, 20]).

Theorem 2.12. *Let X be a normed (resp., Banach) space, and T, g are self-mappings of X. If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\operatorname{cl} T(M_p)$ is compact (resp., $w \operatorname{cl} T(M_p)$ is weakly compact), and $\|Tx - p\| \le \|x - p\|$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed, and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped, $\operatorname{cl} T(G) \subseteq G$ (resp., $w \operatorname{cl} T(G) \subseteq G$ and $g - T$ is demiclosed at 0), and T satisfies inequality (2.8) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.*

Proof. If $p \in M$, then the results are obvious. So assume that $p \notin M$. If $x \in M \setminus M_p$, then $\|x - x_0\| > 2\|p - x_0\|$ and so $\|p - x\| \ge \|x\| - \|p\| > \|p\| \ge \operatorname{dist}(p, M)$. Thus $\alpha = \operatorname{dist}(p, M) \le \|p\|$. Since $\operatorname{cl}(T(M_p))$ is compact, and by the continuity of norm, there exists $z \in \operatorname{cl}(T(M_p))$ such that $\beta = \operatorname{dist}(p, \operatorname{cl}(T(M_p)) = \|z - p\|$.

On the other hand, if $w \operatorname{cl}(T(M_p))$ is weakly compact, then using Lemma 5.5 of Singh et al. [22, page 192], we can show that there exists $z \in w \operatorname{cl}(T(M_p))$ such that $\beta = \operatorname{dist}(p, w \operatorname{cl}(T(M_p))) = \|z - p\|$.

Hence, in both cases, we have

$$\begin{aligned} \alpha = \operatorname{dist}(p, M) &\le \operatorname{dist}(p, \operatorname{cl}(T(M_p))) \\ &= \beta \\ &= \operatorname{dist}(p, T(M_p)) \\ &\le \|Tx - p\| \\ &\le \|x - p\|, \end{aligned} \tag{2.13}$$

for all $x \in M_p$. Therefore, $\alpha = \beta = \operatorname{dist}(p, M)$, that is, $\operatorname{dist}(p, M) = \operatorname{dist}(p, \operatorname{cl}(T(M_p)) = d(p, z)$, that is, $z \in P_M(p)$ and so $P_M(p)$ is nonempty. The closedness and convexity of $P_M(p)$ follow from that of M. Now to prove $T(P_M(p) \subseteq P_M(p)$, let $y \in T(P_M(p))$. Then $y = Tx$ for $x \in P_M(p)$. Consider

$$d(p, y) = d(p, Tx) \le d(p, x) = \operatorname{dist}(p, M), \tag{2.14}$$

and so $y \in P_M(p)$ as $P_M(p) \subset M_p \Rightarrow T(P_M(p)) \subset M$, that is, $y \in M$.

The compactness of $\mathrm{cl}\, T(M_p)$ (resp., weakly compactness of $w\,\mathrm{cl}\, T(M_p)$) implies that $\mathrm{cl}\, T(D)$ is compact (resp., $w\,\mathrm{cl}\, T(D)$ is weakly compact). Hence, the result follows from Corollary 2.10. □

Corollary 2.13. *Let X be a normed (resp., Banach) space, and T, g are self-mappings of X. If $p \in X$ and $M \in G_0$ such that $T(M_p) \subseteq M$, $\mathrm{cl}\, T(M_p)$ is compact (resp., $w\,\mathrm{cl}\, T(M_p)$ is weakly compact), and $\|Tx - p\| \leq \|x - p\|$ for all $x \in M_p$, then $P_M(p)$ is nonempty, closed, and convex with $T(P_M(p)) \subseteq P_M(p)$. If, in addition, D is a subset of $P_M(p)$, $G = D \cap F(g)$ is q-starshaped and closed (resp., weakly closed and $g - T$ is demiclosed at 0), (T, g) is a Banach operator pair on D, and T satisfies inequality (2.8) for all $x, y \in D$, then $P_M(p) \cap F(g, T)$ is nonempty.*

Remark 2.14. Theorem 2.12 extends and generalizes the corresponding results of Al-Thagafi [12], Al-Thagafi and Shahzad [13], Habiniak [18], Narang and Chandok [20], and Shahzad [21].

The following result will be used in the sequel.

Lemma 2.15. *Let C be a nonempty subset of a metric space (X, d), f, g self-maps of C, $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$. Suppose that $\mathrm{cl}(T(C))$ is complete, and T, f, g satisfy for all $x, y \in C$ and $0 \leq k < 1$,*

$$d(Tx, Ty) \leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)\}. \tag{2.15}$$

If $F(f) \cap F(g)$ is nonempty and $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, then there is a common fixed point of T, f and g.

Proof. $\mathrm{cl}\, T(F(f) \cap F(g))$, being a closed subset of the complete set $\mathrm{cl}\, T(C)$, is complete. Further for all $x, y \in F(f) \cap F(g)$, we have

$$\begin{aligned} d(Tx, Ty) &\leq k \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)\} \\ &= k \max\{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(Ty, x)\}. \end{aligned} \tag{2.16}$$

Hence, T is a generalized contraction on $F(f) \cap F(g)$ and $\mathrm{cl}\, T(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$. So by Lemma 3.1 of [13], T has a unique fixed point y in $F(f) \cap F(g)$ and consequently $F(T) \cap F(f) \cap F(g)$ is a singleton. □

Remark 2.16. If $f = g$, then Theorem 3.2 of Al-Thagafi and Shahzad [13] is a particular case of Lemma 2.15.

The following result extends and improves the corresponding results of [9, 12–14, 18, 20].

Theorem 2.17. *Let T, g, h be self-mappings of a Banach space X. If $p \in F(T) \cap F(g) \cap F(h)$ and $M \in G_\circ$ such that $T(M_p) \subseteq g(M) \subseteq M \subseteq h(M)$, $\mathrm{cl}\, g(M_p)$ is compact, and $\|Tx - p\| \leq \|gx - hp\|$, $\|gx - u\| \leq \|x - u\|$, $\|hx - u\| = \|x - u\|$ for all $x \in M_p$, then*

(i) *$P_M(p)$ is nonempty, closed, and convex,*

(ii) *$T(P_M(p)) \subseteq g(P_M(p)) \subseteq P_M(p) = h(P_M(p))$,*

(iii) $P_M(p) \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$, *provided* T *is continuous,* $F(g)$ *is* q*-starshaped,* $\operatorname{cl} g(F(h)) \subseteq F(h)$, *and the pair* (g,h) *satisfies the inequality* (2.8) *for all* $x,y \in P_M(p)$, $F(g)$ *is* q*-starshaped with* $q \in P_M(p) \cap F(g) \cap F(h) = G$, $\operatorname{cl} T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$ *and* T, g, h *satisfy for all* $q \in F(g) \cap F(h)$ *and* $x,y \in P_M(p)$

$$\begin{aligned} d(Tx,Ty) \le \max\{&d(hx,gy), \operatorname{dist}(hx,[q,Tx]), \operatorname{dist}(gy,[q,Ty]), \operatorname{dist}(hx,[q,Ty]),\\ &\operatorname{dist}(gy,[q,Tx])\}, \end{aligned} \tag{2.17}$$

then there is a common fixed point of $P_M(p)$, T, g, *and* h.

Proof. Proceeding as in Theorem 2.12, we can prove (i) and (ii).

By (ii), the compactness of $\operatorname{cl} g(M_p)$ implies that $\operatorname{cl} g(P_M(p))$ and $\operatorname{cl} T(P_M(p))$ are compact. Hence, Theorem 2.8 implies that $F(g) \cap F(h) \cap P_M(p) \neq \emptyset$.

For each $n \in \mathbb{N}$, define $T_n : X \to X$ by $T_n x = (1-k_n)q + k_n Tx$, for each $x \in X$, where $\{k_n\}$ is a sequence in $(0,1)$ such that $k_n \to 1$. Then each T_n is a self-mapping of C. Since $\operatorname{cl} T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$, $F(g) \cap F(h)$ is q-starshaped with $q \in G$, so $\operatorname{cl} T_n(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$ for each n. Consider

$$\begin{aligned} \|T_n x - T_n y\| &= k_n \|Tx - Ty\| \\ &\le k_n \max\{\|hx-gy\|, \operatorname{dist}(hx,[q,Tx]), \operatorname{dist}(gy,[q,Ty]), \operatorname{dist}(hx,[q,Ty]),\\ &\qquad \operatorname{dist}(gy,[q,Tx])\} \\ &\le k_n \max\{\|hx-gy\|, \|hx - T_n x\|, \|gy - T_n y\|, \|hx - T_n y\|, \|gy - T_n x\|\}, \end{aligned} \tag{2.18}$$

for all $x, y \in P_M(p)$. As $\operatorname{cl}(T(P_M(p)))$ is compact, $\operatorname{cl}(T_n(P_M(p)))$ is compact for each n and hence complete. Now by Lemma 2.15, there exists $x_n \in M$ such that x_n is a common fixed point of g, h and T_n for each n. The compactness of $\operatorname{cl}(T(P_M(p)))$ implies that there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to z \in \operatorname{cl} T(P_M(p))$. Since $\{Tx_n\}$ is a sequence in $T(F(g) \cap F(h))$, and $\operatorname{cl} T(F(g) \cap F(h)) \subseteq F(g) \cap F(h)$, then $z \in F(g) \cap F(h)$. Now, as $k_{n_i} \to 1$, we have

$$x_{n_i} = T_{n_i} x_{n_i} = (1-k_{n_i})q + k_{n_i} T x_{n_i} \longrightarrow z, \tag{2.19}$$

T is continuous, we have $Tz = z$ and hence $P_M(p) \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$. □

Remark 2.18. (i) Let $M = [0,1]$. Let d be defined by $d(x,y) = |x-y|$. We set $Tx = x/4$ and $fx = x/2$ for all $x \in M$. Define $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\psi(t,s) = \frac{t+s}{8}. \tag{2.20}$$

Then for $x, y \in M$, we have

$$d(Tx, Ty) = d\left(\frac{x}{4}, \frac{y}{4}\right) = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}|x - y|, \tag{2.21}$$

$$\begin{aligned} &\frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)) \\ &\quad = \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] - \varphi\left(\left|\frac{x}{2} - \frac{y}{4}\right|, \left|\frac{y}{2} - \frac{x}{4}\right|\right) \\ &\quad = \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] - \frac{1}{8}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] \\ &\quad = \frac{3}{8}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right]. \end{aligned} \tag{2.22}$$

From (2.21) and (2.22), without loss of generality assume that $x \geq y$. Hence, we have two cases:

Case 1. If $2y \geq x$, from (2.22), we have

$$\begin{aligned} &\frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)) \\ &\quad = \frac{3}{8}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] = \frac{3}{8}\left[\left(\frac{x}{2} - \frac{y}{4}\right) + \left(\frac{y}{2} - \frac{x}{4}\right)\right] \\ &\quad = \frac{3}{8}\left(\frac{x}{4} + \frac{y}{4}\right) = \frac{3}{32}(x + y) \geq \frac{1}{4}(x - y). \end{aligned} \tag{2.23}$$

Case 2. If $2y < x$, from (2.22), we have

$$\begin{aligned} &\frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)) \\ &\quad = \frac{3}{8}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{y}{2} - \frac{x}{4}\right|\right] = \frac{3}{8}\left[\left(\frac{x}{2} - \frac{y}{4}\right) + \left(-\frac{y}{2} + \frac{x}{4}\right)\right] \\ &\quad = \frac{3}{8}\left(\frac{3x}{4} + \frac{3y}{4}\right) = \frac{9}{32}(x - y) \geq \frac{1}{4}(x - y). \end{aligned} \tag{2.24}$$

Thus inequality (2.3) is satisfied and by Theorem 2.3; 0 is a common fixed point of T and f.

(ii) It may be noted that the assumption of linearity or affinity for I is necessary in almost all known results about common fixed points of maps T, I such that T is I-nonexpansive under the conditions of commuting, weakly commuting, R-subweakly commuting, or compatibility (see [9, 12, 14, 20, 21] and the literature cited therein), but our results in this paper are independent of the linearity or affinity.

(iii) Consider $M = \mathbb{R}^2$ with usual metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$. Define T and I on M as $T(x, y) = ((x - 2)/2, (x^2 + y - 4)/2)$ and $I(x, y) = ((x-2)/2, x^2+y-4)$. Obviously, T is I-nonexpansive, I-asymptotically nonexpansive, but I is not linear or affine. Moreover, $F(T) = (-2, 0)$, $F(I) = \{(-2, y) : y \in \mathbb{R}\}$ and $C(I, T) = \{(x, y) : y = 4 - x^2, x \in \mathbb{R}\}$. Thus, $\operatorname{cl} T(F(I)) \subset F(I)$, which is not a compatible pair (see [9]), $F(I)$ is convex, starshaped for any $z \in F(I)$, and $(-2, 0)$ is a common fixed point of I and T.

Acknowledgment

The author is thankful to the learned referees for the valuable suggestions.

References

[1] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.
[2] R. Kannan, "Some results on fixed points. II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.
[3] S. K. Chatterjea, "Fixed-point theorems," *Comptes Rendus de l'Académie Bulgare des Sciences*, vol. 25, pp. 727–730, 1972.
[4] Ya. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New Results in Operator Theory*, I. Gohberg and Yu. Lyubich, Eds., vol. 8 of *Operator Theory: Advances and Applications*, pp. 7–22, Birkhäuser, Basel, Switzerland, 1997.
[5] B. S. Choudhury, "Unique fixed point theorem for weakly C-contractive mappings," *Kathmandu University Journal of Science, Engineering and Technology*, vol. 5, pp. 6–13, 2009.
[6] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis*, vol. 47, pp. 2683–2693, 2001.
[7] L. Ćirić, N. Hussain, and N. Cakić, "Common fixed points for Ćirić type f-contraction with applications," *Publicationes Mathematicae Debrecen*, vol. 4317, pp. 1–19, 2009.
[8] S. Chandok, "Some common fixed point theorems for generalized f-weakly contractive mappings," *Journal of Applied Mathematics & Informatics*, vol. 29, no. 1-2, pp. 257–265, 2011.
[9] J. Chen and Z. Li, "Common fixed-points for Banach operator pairs in best approximation," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1466–1475, 2007.
[10] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.
[11] S. Reich, "Some fixed point problems," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 57, no. 3-4, pp. 194–198, 1974.
[12] M. A. Al-Thagafi, "Common fixed points and best approximation," *Journal of Approximation Theory*, vol. 85, no. 3, pp. 318–323, 1996.
[13] M. A. Al-Thagafi and N. Shahzad, "Banach operator pairs, common fixed-points, invariant approximations, and $*$-nonexpansive multimaps," *Nonlinear Analysis A*, vol. 69, no. 8, pp. 2733–2739, 2008.
[14] S. Chandok and T. D. Narang, "Some common fixed point theorems for Banach operator pairs with applications in best approximation," *Nonlinear Analysis A*, vol. 73, no. 1, pp. 105–109, 2010.
[15] S. Chandok and T. D. Narang, "Common fixed points and invariant approximation for Gregus type contraction mappings," *Rendiconti del Circolo Matematico di Palermo*, vol. 60, no. 1-2, pp. 203–214, 2011.
[16] S. Chandok and T. D. Narang, "Common fixed points and invariant approximation for banach operator Pairs with ćirić type nonexpansive mappings," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 6, pp. 871–883, 2011.
[17] S. Chandok, J. Liang, and D. O'Regan, "Common fixed points and invariant approximations for noncommuting contraction mappings in strongly convex metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 14, 2013.
[18] L. Habiniak, "Fixed point theorems and invariant approximations," *Journal of Approximation Theory*, vol. 56, no. 3, pp. 241–244, 1989.
[19] T. D. Narang and S. Chandok, "Fixed points and best approximation in metric spaces," *Indian Journal of Mathematics*, vol. 51, no. 2, pp. 293–303, 2009.
[20] T. D. Narang and S. Chandok, "Common fixed points and invariant approximation of R-subweakly commuting maps in convex metric spaces," *Ukrainian Mathematical Journal*, vol. 62, no. 10, pp. 1585–1596, 2011.
[21] N. Shahzad, "Invariant approximations and R-subweakly commuting maps," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 1, pp. 39–45, 2001.
[22] S. Singh, B. Watson, and P. Srivastava, *Fixed Point Theory and Best Approximation: The KKM-map Principle*, vol. 424, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.