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Research Article

Some Properties of Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

J. Y. Kang

Department of Mathematics, Hannam University, Daejeon 306-791, Republic of Korea

Correspondence should be addressed to J. Y. Kang, rkdwjddnr2002@yahoo.co.kr

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The present paper deals with the various q-Genocchi numbers and polynomials. We define a new type of multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β by applying the method of p-adic q-integral. We will find a link between their numbers and polynomials with weight α and weak weight β. Also we will obtain the interesting properties of their numbers and polynomials with weight α and weak weight β. Moreover, we construct a Hurwitz-type zeta function which interpolates multiple generalized q-Genocchi polynomials with weight α and weak weight β and find some combinatorial relations.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ (see [1–21]). When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

Throughout this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Hence $\lim_{q\to 1}[x]_q = x$ for all $x \in \mathbb{Z}_p$ (see [1–14, 16, 18, 20, 21]).

We say that $g : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and we write $g \in \mathrm{UD}(\mathbb{Z}_p)$ if the difference quotients $\Phi_g : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_g(x,y) = \frac{g(x)-g(y)}{x-y} \tag{1.2}$$

have a limit $g'(a)$ as $(x,y) \to (a,a)$.

Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N, we set

$$\begin{gathered} \mathrm{X} = \mathrm{X}_d = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad \mathrm{X}_1 = \mathbb{Z}_p, \\ \mathrm{X}^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \left\{x \in \mathrm{X} \mid x \equiv a \left(\operatorname{mod} dp^N\right)\right\}, \end{gathered} \tag{1.3}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q} \tag{1.4}$$

is known to be a distribution on X.

For $g \in \mathrm{UD}(\mathbb{Z}_p)$, Kim defined the q-deformed fermionic p-adic integral on $\mathbb{Z}_p$:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{1.5}$$

(see [1–13]), and note that

$$\int_{\mathbb{Z}_p} g(x)\,d\mu_{-q}(x) = \int_{\mathrm{X}} g(x)\,d\mu_{-q}(x). \tag{1.6}$$

We consider the case $q \in (-1,0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given there in [9, 13, 14].

In [9, 12, 14, 19], we introduced multiple generalized Genocchi number and polynomials. Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. We assume that f

is odd. Then the multiple generalized Genocchi numbers, $G_{n,\chi}^{(r)}$, and the multiple generalized Genocchi polynomials, $G_{n,\chi}^{(r)}(x)$, associated with χ, are defined by

$$F_{\chi}^{(r)}(t)=\left(\frac{2t\sum_{a=0}^{f-1}\chi(a)(-1)^{a}e^{at}}{e^{ft}+1}\right)^{r}=\sum_{n=0}^{\infty}G_{n,\chi}^{(r)}\frac{t^{n}}{n!},$$
$$F_{\chi}^{(r)}(t,x)=\left(\frac{2t\sum_{a=0}^{f-1}\chi(a)(-1)^{a}e^{at}}{e^{ft}+1}e^{tx}\right)^{r}=\sum_{n=0}^{\infty}G_{n,\chi}^{(r)}(x)\frac{t^{n}}{n!}. \tag{1.7}$$

In the special case $x=0$, $G_{n,\chi}^{(r)}=G_{n,\chi}^{(r)}(0)$ are called the nth multiple generalized Genocchi numbers attached to χ.

Now, having discussed the multiple generalized Genocchi numbers and polynomials, we were ready to multiple-generalize them to their q-analogues. In generalizing the generating functions of the Genocchi numbers and polynomials to their respective q-analogues; it is more useful than defining the generating function for the Genocchi numbers and polynomials (see [12]).

Our aim in this paper is to define multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β. We investigate some properties which are related to multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β. We also derive the existence of a specific interpolation function which interpolate multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β at negative integers.

2. The Generating Functions of Multiple Generalized q-Genocchi Numbers and Polynomials with Weight α and Weak Weight β

Many mathematicians constructed various kinds of generating functions of the q-Gnocchi numbers and polynomials by using p-adic q-Vokenborn integral. First we introduce multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β.

Let us define the generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β, respectively,

$$F_{\chi,q}^{(\alpha,\beta)}(t)=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^{n}}{n!}=\int_{X}t\chi(x)e^{[x]_{q^{\alpha}}t}d\mu_{-q^{\beta}}(x),$$
$$F_{\chi,q}^{(\alpha,\beta)}(t,x)=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}(x)\frac{t^{n}}{n!}=\int_{X}t\chi(y)e^{[x+y]_{q^{\alpha}}t}d\mu_{-q^{\beta}}(y). \tag{2.1}$$

By using the Taylor expansion of $e^{[x]_{q^{\alpha}}t}$, we have

$$\sum_{n=0}^{\infty}\int_{X}\chi(x)[x]_{q^{\alpha}}^{n}d\mu_{-q^{\beta}}(x)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^{n-1}}{n!}=G_{0,\chi,q}^{(\alpha,\beta)}+\sum_{n=0}^{\infty}\frac{G_{n+1,\chi,q}^{(\alpha,\beta)}}{n+1}\frac{t^{n}}{n!}. \tag{2.2}$$

By comparing the coefficient of both sides of $t^n/n!$ in (2.2), we get

$$\frac{G_{n+1,\chi,q}^{(\alpha,\beta)}}{n+1} = \frac{[2]_{q^\beta}}{(1-q^\alpha)^n}\sum_{a=0}^{f-1}(-1)^a q^{\beta a}\chi(a)\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha a l}\frac{1}{1+q^{f(\alpha l+\beta)}}. \tag{2.3}$$

From (2.2) and (2.3), we can easily obtain that

$$\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(t\int_X \chi(x)[x]_{q^\alpha}^n d\mu_{-q^\beta}(x)\right)\frac{t^n}{n!} = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[l]_{q^\alpha}t}. \tag{2.4}$$

Therefore, we obtain

$$F_{\chi,q}^{(\alpha,\beta)}(t) = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[l]_{q^\alpha}t} = \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^n}{n!}. \tag{2.5}$$

Similarly, we find the generating function of generalized q-Genocchi polynomials with weight α and weak weight β:

$$G_{0,\chi,q}^{(\alpha,\beta)}(x) = 0, \qquad \frac{G_{n+1,\chi,q}^{(\alpha,\beta)}(x)}{n+1} = \int_X \chi(y)[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) = [2]_{q^\beta}\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)[x+l]_{q^\alpha}^n. \tag{2.6}$$

From (2.6), we have

$$F_{\chi,q}^{(\alpha,\beta)}(t,x) = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[x+l]_{q^\alpha}t} = \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!}. \tag{2.7}$$

Observe that $F_{\chi,q}^{(\alpha,\beta)}(t) = F_{\chi,q}^{(\alpha,\beta)}(t,0)$. Hence we have $G_{n,\chi,q}^{(\alpha,\beta)} = G_{n,\chi,q}^{(\alpha,\beta)}(0)$. If $q \to 1$ into (2.7), then we easily obtain $F_\chi(t,x)$.

First, we define the multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ with weight α and weak weight β:

$$\begin{aligned}
F_{\chi,q}^{(\alpha,\beta,r)}(t) &= [2]_{q^\beta}^r t^r \sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r\chi(k_i)\right)e^{[\sum_{i=1}^r k_i]_{q^\alpha}t} \\
&= t^r\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(x_1)\cdots\chi(x_r)e^{[x_1+\cdots+x_r]_{q^\alpha}t}d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r) \\
&= \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(x_1)\cdots\chi(x_r)[x_1+\cdots+x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^{n-r}}{n!}=\sum_{n=0}^{r-1}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^{n-r}}{n!}+\sum_{n=0}^{\infty}\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r}r!}\frac{t^n}{n!},
\end{aligned}\tag{2.9}$$

where $\binom{n+r}{r}=(n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.9), we obtain the following theorem.

Theorem 2.1. *Let $q\in\mathbb{C}_p$ with $|1-q|_p<1$ and $n\in\mathbb{Z}_+$. Then one has*

$$\begin{aligned}
G_{0,\chi,q}^{(\alpha,\beta,r)}&=G_{1,\chi,q}^{(\alpha,\beta,r)}=\cdots=G_{r-1,\chi,q}^{(\alpha,\beta,r)}=0,\\
\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r}r!}&=\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(x_1)\cdots\chi(x_r)[x_1+\cdots+x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r)\\
&=\frac{[2]_{q^\beta}^r}{(1-q^\alpha)^n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}\left(\prod_{i=1}^{r}\chi(a_i)\right)\frac{(-1)^{l+\sum_{i=1}^r a_i}q^{(\alpha l+\beta)\sum_{i=1}^r a_i}}{\left(1+q^{f(\alpha l+\beta)}\right)^r}\\
&=[2]_{q^\beta}^r\sum_{m=0}^{\infty}\sum_{a_1,\ldots,a_r=0}^{f-1}\binom{m+r-1}{m}(-1)^{\sum_{i=1}^r a_i+m}q^{\beta(\sum_{i=1}^r a_i+fm)}\times\left(\prod_{i=1}^{r}\chi(a_i)\right)\left[\sum_{i=1}^{r}a_i+fm\right]_{q^\alpha}^n.
\end{aligned}\tag{2.10}$$

From now on, we define the multiple generalized q-Genocchi polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β.

$$\begin{aligned}
F_{\chi,q}^{(\alpha,\beta,r)}(t,x)&=[2]_{q^\beta}^r t^r\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^{r}\chi(k_i)\right)e^{[\sum_{i=1}^r k_i+x]_{q^\alpha}t}\\
&=t^r\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)e^{[x+y_1+\cdots+y_r]_{q^\alpha}t}d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r)\\
&=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}.
\end{aligned}\tag{2.11}$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)\,[x+y_1+\cdots+y_r]_{q^\alpha}^n d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^{n-r}}{n!}=\sum_{n=0}^{r-1}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^{n-r}}{n!}+\sum_{n=0}^{\infty}\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!}\frac{t^n}{n!},
\end{aligned}\tag{2.12}$$

where $\binom{n+r}{r}=(n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.12), we have the following theorem.

Theorem 2.2. *Let $q\in\mathbb{C}_p$ with $|1-q|_p<1$ and $n\in\mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&G_{0,\chi,q}^{(\alpha,\beta,r)}(x)=G_{1,\chi,q}^{(\alpha,\beta,r)}(x)=\cdots=G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x)=0,\\
&\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!}=\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)\,[x+y_1+\cdots+y_r]_{q^\alpha}^n d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r)\\
&\quad=\frac{[2]_{q^\beta}^r}{(1-q^\alpha)^n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}\left(\prod_{i=1}^{r}\chi(a_i)\right)\frac{(-1)^{l+\sum_{i=1}^r a_i}q^{\alpha lx+(\alpha l+\beta)\sum_{i=1}^r a_i}}{\left(1+q^{f(\alpha l+\beta)}\right)^r}\\
&\quad=[2]_{q^\beta}^r\sum_{m=0}^{\infty}\sum_{a_1,\ldots,a_r=0}^{f-1}\binom{m+r-1}{m}(-1)^{\sum_{i=1}^r a_i+m}q^{\beta(\sum_{i=1}^r a_i+fm)}\\
&\qquad\times\left(\prod_{i=1}^{r}\chi(a_i)\right)\left[\sum_{i=1}^{r}a_i+fm+x\right]_{q^\alpha}^n.
\end{aligned}\tag{2.13}$$

In (2.11), we simply identify that

$$\begin{aligned}
\lim_{q\to1}F_{\chi,q}^{(\alpha,\beta,r)}(t,x)&=2^rt^r\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}\left(\prod_{i=1}^{r}\chi(k_i)\right)e^{(\sum_{i=1}^r k_i+x)t}\\
&=\left(\frac{2t\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}}{1+e^{ft}}\right)^r e^{tx}=F_\chi^{(r)}(t,x).
\end{aligned}\tag{2.14}$$

So far, we have studied the generating functions of the multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β.

3. Modified Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

In this section, we will investigate about modified multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β. Also, we will find their relations in multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β.

Firstly, we modify generating functions of $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We access some relations connected to these numbers and polynomials with weight α and weak weight β. For this reason, we assign generating function of modified multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β which are implied by $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We give relations between these numbers and polynomials with weight α and weak weight β.

We modify (2.11) as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = F_{\chi,q}^{(\alpha,\beta,r)}\left(q^{-\alpha x}t,x\right), \tag{3.1}$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t,x)$ is defined in (2.11).

From the above we know that

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = \sum_{n=0}^{\infty} q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}. \tag{3.2}$$

After some elementary calculations, we attain

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = q^{-\alpha r x} e^{\left(q^{-\alpha x}[x]_{q^\alpha} t\right)} F_{\chi,q}^{(\alpha,\beta,r)}(t), \tag{3.3}$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t)$ is defined in (2.8).

From the above, we can assign the modified multiple generalized q-Genocchi polynomials $\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = \sum_{n=0}^{\infty} \varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}. \tag{3.4}$$

Then we have

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x). \tag{3.5}$$

Theorem 3.1. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{i=0}^{n} \binom{n}{i} q^{\alpha i x} [x]_{q^\alpha}^{n-i} G_{i,\chi,q}^{(\alpha,\beta,r)}. \tag{3.6}$$

Corollary 3.2. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by using* (3.7), *one easily obtains*

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j, l, n-j-l} \binom{n-j+m-1}{m} (-1)^l q^{\alpha\{(j+l)x+m\}} G_{j,\chi,q}^{(\alpha,\beta,r)}. \quad (3.7)$$

Secandly, by using generating function of the multiple generalized q-Genocchi polynomials with weight α and weak weight β, which is defined by (2.11), we obtain the following identities.

By using (2.13), we find that

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{m=0}^{\infty} \sum_{a_1,\ldots,a_r=0}^{f-1} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i + m} \\ &\quad \times q^{\beta(\sum_{i=1}^r a_i + fm)} \left(\prod_{i=1}^r \chi(a_i) \right) \left[\sum_{i=1}^r a_i + fm + x \right]_{q^\alpha}^n \\ &= [2]_{q^\beta}^r \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{l=0}^{n} \sum_{a=0}^{l} \binom{n}{a, l-a, n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{f\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.8)$$

Thus we have the following theorem.

Theorem 3.3. *Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $r \in \mathbb{N}$. Then one has*

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{l=0}^{n} \sum_{a=0}^{l} \binom{n}{a, l-a, n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{f\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.9)$$

By using (2.13), we have

$$\begin{aligned} F_{\chi,q}^{(\alpha,\beta,r)}(t,x) &= [2]_{q^\beta}^r t^r \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{\alpha l x}}{(1-q)^n} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} \\ &\quad \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)} \left(\prod_{i=1}^r \chi(a_i) \right) \sum_{m=0}^{\infty} \binom{m+r-1}{m} \left(-q^{f(\alpha l+\beta)} \right)^m \frac{t^n}{n!}. \end{aligned} \quad (3.10)$$

Thus we have

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[2]_{q^\beta}^r t^r \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}(1-q^\alpha)^{-n}\sum_{a_1,\ldots,a_r=0}^{f-1}(-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(1+q^{f(\alpha l+\beta)}\right)^{-r}\frac{t^n}{n!}. \tag{3.11}$$

By comparing the coefficients of both sides of $(n+r)!/t^{n+r}$ in the above, we arrive at the following theorem.

Theorem 3.4. *Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $r \in \mathbb{N}$. Then one has*

$$\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!} = [2]_{q^\beta}^r \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}(1-q^\alpha)^{-n}\sum_{a_1,\ldots,a_r=0}^{f-1}(-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(1+q^{f(\alpha l+\beta)}\right)^{-r}. \tag{3.12}$$

From (2.12), we easily know that

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[2]_{q^\beta}^r \binom{n+r}{r}r!\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i} q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r \chi(k_i)\right) \times \left[x+\sum_{i=1}^r k_i\right]_{q^\alpha}^n \frac{t^{n+r}}{(n+r)!}. \tag{3.13}$$

From the above, we get the following theorem.

Theorem 3.5. *Let $r \in \mathbb{N}$, $k \in \mathbb{Z}_+$. Then one has*

$$G_{0,\chi,q}^{(\alpha,\beta,r)}(x) = G_{1,\chi,q}^{(\alpha,\beta,r)}(x) = \cdots = G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x) = 0,$$
$$G_{l+r,\chi,q}^{(\alpha,\beta,r)}(x) = [2]_{q^\beta}^r\binom{l+r}{r}r!\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i} q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r \chi(k_i)\right)\left[x+\sum_{i=1}^r k_i\right]_{q^\alpha}^l. \tag{3.14}$$

From (2.13), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,s)}(x)\frac{t^n}{n!} \\
&\quad = [2]_{q^\beta}^{r+s} t^{r+s} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^{r} a_i+m} q^{\beta(\sum_{i=1}^{r} a_i+fm)} \\
&\qquad \times \left(\prod_{i=1}^{r}\chi(a_i)\right) e^{[\sum_{i=1}^{r} a_i+fm+x]_{q^\alpha} t} \sum_{b_1,\ldots,b_s=0}^{f-1}\sum_{k=0}^{\infty}\binom{k+s-1}{k}(-1)^{\sum_{i=1}^{s} b_i+k} \\
&\qquad \times q^{\beta(\sum_{i=1}^{s} b_i+fk)}\left(\prod_{i=1}^{s}\chi(b_i)\right) e^{[\sum_{i=1}^{r} b_i+fk+x]_{q^\alpha} t}.
\end{aligned} \tag{3.15}$$

By using Cauchy product in (3.15), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{n-j,\chi,q}^{(\alpha,\beta,s)}(x)\frac{t^n}{n!} \\
&\quad = [2]_{q^\beta}^{r+s} t^{r+s} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{b_1,\ldots,b_s=0}^{f-1}\binom{j+r-1}{j}\binom{n-j+s-1}{n-j} \\
&\qquad \times (-1)^{\sum_{i=1}^{r} a_i+\sum_{i=1}^{s} b_i+n} q^{\beta(\sum_{i=1}^{r} a_i+\sum_{i=1}^{s} b_i+fn)}\left(\prod_{i=1}^{r}\chi(a_i)\right)\left(\prod_{i=1}^{s}\chi(b_i)\right) \\
&\qquad \times e^{[\sum_{i=1}^{r} a_i+fj+x]_{q^\alpha} t} e^{[\sum_{i=1}^{s} b_i+f(n-j)+x]_{q^\alpha} t}.
\end{aligned} \tag{3.16}$$

From (3.16), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{m-j,\chi,q}^{(\alpha,\beta,s)}(x)\right)\frac{t^m}{m!} \\
&\quad = \sum_{m=0}^{\infty}[2]_{q^\beta}^{r+s} t^{r+s} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{b_1,\ldots,b_s=0}^{f-1}\binom{j+r-1}{j}\binom{n-j+s-1}{n-j} \\
&\qquad \times (-1)^{\sum_{i=1}^{r} a_i+\sum_{i=1}^{s} b_i+n} q^{\beta(\sum_{i=1}^{r} a_i+\sum_{i=1}^{s} b_i+fn)}\left(\prod_{i=1}^{r}\chi(a_i)\right)\left(\prod_{i=1}^{s}\chi(b_i)\right) \\
&\qquad \times \left(\left[\sum_{i=1}^{r} a_i+fj+x\right]_{q^\alpha}+\left[\sum_{i=1}^{s} b_i+f(n-j)+x\right]_{q^\alpha}\right)^m \frac{t^m}{m!}.
\end{aligned} \tag{3.17}$$

By comparing the coefficients of both sides of $t^{m+r+s}/(m+r+s)!$ in (3.17), we have the following theorem.

Theorem 3.6. *Let $r \in \mathbb{N}$and $s \in \mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&\frac{\sum_{j=0}^{l+r+s}\binom{l+r+s}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+s-j,\chi,q}^{(\alpha,\beta,s)}(x)}{\binom{l+r+s}{l}(r+s)!} \\
&\quad = [2]_{q^\beta}^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{b_1,\ldots,b_s=0}^{f-1} \binom{j+r-1}{j}\binom{n-j+s-1}{n-j} \\
&\qquad \times (-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)} \left(\prod_{i=1}^r \chi(a_i)\right)\left(\prod_{i=1}^s \chi(b_i)\right) \\
&\qquad \times \left(\left[\sum_{i=1}^r a_i + fj + x\right]_{q^\alpha} + \left[\sum_{i=1}^s b_i + f(n-j) + x\right]_{q^\alpha}\right)^l.
\end{aligned} \tag{3.18}$$

Corollary 3.7. *In* (3.18) *setting $s = 1$, one has*

$$\begin{aligned}
&\frac{\sum_{j=0}^{l+r+1}\binom{l+r+1}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+1-j,\chi,q}^{(\alpha,\beta,1)}(x)}{\binom{l+r+1}{l}(r+1)!} \\
&\quad = [2]_{q^\beta}^{r+1} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{b_1=0}^{f-1} \binom{j+r-1}{j} (-1)^{\sum_{i=1}^r a_i+b_1+n} q^{\beta(\sum_{i=1}^r a_i+b_1+fn)} \\
&\qquad \times \left(\chi(b_1)\prod_{i=1}^r \chi(a_i)\right)\left(\left[\sum_{i=1}^r a_i + fj + x\right]_{q^\alpha} + \left[b_1 + f(n-j) + x\right]_{q^\alpha}\right)^l.
\end{aligned} \tag{3.19}$$

By using (2.13) we have the following theorem.

Theorem 3.8. *Distribution theorem is as follows:*

$$\begin{aligned}
G_{n+r,\chi,q}^{(\alpha,\beta,r)} &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta \sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i)\right) G_{n+r,q^f}^{(\alpha,\beta,r)}\left(\frac{a_1+\cdots+a_r}{f}\right), \\
G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x) &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta \sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i)\right) G_{n+r,q^f}^{(\alpha,\beta,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).
\end{aligned} \tag{3.20}$$

4. Interpolation Function of Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

In this section, we see interpolation function of multiple generalized q-Genocchi polynomials with weak weight α and find some relations.

Let us define interpolation function of the $G_{k+r,q}^{(\alpha,\beta,r)}(x)$ as follows.

Definition 4.1. Let $q, s \in \mathbb{C}$ with $|q| < 1$ and $0 < x \leq 1$. Then one defines

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(s,x) = [2]_{q^\beta}^r \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right)}{\left[x + \sum_{i=1}^r k_i\right]_{q^\alpha}^s}. \tag{4.1}$$

We call $\zeta_q^{(\alpha,\beta,r)}(s,x)$ the multiple generalized Hurwitz type q-zeta funtion.

In (4.1), setting $r = 1$, we have

$$\zeta_{\chi,q}^{(\alpha,\beta,1)}(s,x) = [2]_{q^\beta} \sum_{l=0}^{\infty} \frac{(-1)^l q^{\beta l} \chi(l)}{[x+l]_{q^\alpha}^s} = \zeta_{\chi,q}^{(\alpha,\beta)}(s,x). \tag{4.2}$$

Remark 4.2. It holds that

$$\lim_{q \to 1} \zeta_{\chi,q}^{(\alpha,\beta,r)}(s,x) = 2^r \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right)}{\left(x + \sum_{i=1}^r k_i\right)^s}. \tag{4.3}$$

Substituting $s = -n, n \in \mathbb{Z}_+$ into (4.1), then we have,

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n,x) = [2]_{q^\beta}^r \sum_{k_1,\ldots,k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right) \left[x + \sum_{i=1}^r k_i\right]_{q^\alpha}^n. \tag{4.4}$$

Setting (3.14) into the above, we easily get the following theorem.

Theorem 4.3. *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$. Then one has*

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n,x) = \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!}. \tag{4.5}$$

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