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Research Article

Some New Classes of Generalized Apostol-Euler and Apostol-Genocchi Polynomials

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The main object of this paper is to introduce and investigate two new classes of generalized Apostol-Euler and Apostol-Genocchi polynomials. In particular, we obtain a new addition formula for the new class of the generalized Apostol-Euler polynomials. We also give an extension and some analogues of the Srivastava-Pintér addition theorem obtained in the works by Srivastava and Pintér (2004) and R. Tremblay, S. Gaboury, B.-J. Fugère, and Tremblay et al. (2011). for both classes.

1. Introduction

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree n as well as in α, are defined, respectively, by the following generating functions (see, [1, volume 3, page 253 et seq.], [2, Section 2.8], and [3]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x)\frac{t^k}{k!} \quad (|t|<2\pi;\, 1^{\alpha}:=1),$$
$$\left(\frac{2}{e^t+1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x)\frac{t^k}{k!} \quad (|t|<\pi;\, 1^{\alpha}:=1),$$
$$\left(\frac{2t}{e^t+1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x)\frac{t^k}{k!} \quad (|t|<\pi;\, 1^{\alpha}:=1). \tag{1.1}$$

The literature contains a large number of interesting properties and relationships involving these polynomials [1, 4–7]. These appear in many applications in combinatorics, number theory, and numerical analysis.

Many interesting extensions to these polynomials have been given. In particular, Luo and Srivastava [8, 9] introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$; Luo [10] invented the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ in [3]. These polynomials are defined, respectively, as follows.

Definition 1.1. The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!}$$
$$\left(|t| < 2\pi, \text{if } \lambda = 1; |t| < \left|\log\lambda\right|, \text{ if } \lambda \neq 1;\ 1^{\alpha} := 1\right) \tag{1.2}$$

with

$$B_n^{(\alpha)}(x) = \mathfrak{B}_n^{(\alpha)}(x;1). \tag{1.3}$$

Definition 1.2. The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \quad \left(|t| < \left|\log(-\lambda)\right|; 1^{\alpha} := 1\right) \tag{1.4}$$

with

$$E_n^{(\alpha)}(x) = \mathfrak{E}_n^{(\alpha)}(x;1). \tag{1.5}$$

Definition 1.3. The generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_k^{(\alpha)}(x;\lambda)\frac{t^k}{k!} \quad \left(|t| < \left|\log(-\lambda)\right|; 1^{\alpha} := 1\right) \tag{1.6}$$

with

$$G_n^{(\alpha)}(x) = \mathfrak{G}_n^{(\alpha)}(x;1). \tag{1.7}$$

Many authors have investigated these polynomials and numerous very interesting papers can be found in the literature. The reader should read [11–20].

Recently, the authors [21] studied a new family of generalized Apostol-Bernoulli polynomials of order α in the following form.

Definition 1.4. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^+$, the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x, b, c; \lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < 2\pi$ if $\lambda = 1$ or with $|t \log b| < |\log \lambda|$ if $\lambda \neq 1$, by means of the generating function:

$$\left(\frac{t^m}{\lambda b^t - \sum_{l=0}^{m-1} \left((t \log b)^l / l! \right)} \right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x, b, c; \lambda) \frac{t^k}{k!}. \tag{1.8}$$

It easy to see that if we set $m = 1$, $b = c = e$ in (1.8), we arrive at the following:

$$\left(\frac{t}{\lambda e^t - 1} \right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_n^{[0,\alpha]}(x, e, e; \lambda) \frac{t^k}{k!}. \tag{1.9}$$

This is the generating function for the generalized Apostol-Bernoulli polynomials of order α. Thus, we have

$$\mathfrak{B}_n^{[0,\alpha]}(x, e, e; \lambda) = \mathfrak{B}_n^{(\alpha)}(x; \lambda). \tag{1.10}$$

Obviously, when we set $\lambda = 1$ and $\alpha = 1$ in (1.10) we obtain

$$\mathfrak{B}_n^{[0,1]}(x, e, e; 1) = B_n(x), \tag{1.11}$$

where $B_n(x)$ are the classical Bernoulli polynomials.

Moreover, Srivastava et al. [22] introduced two new families of generalized Euler and Genocchi polynomials. They investigated the following forms.

Definition 1.5. Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\begin{gathered} \left(\frac{2}{\lambda b^t + a^t} \right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!} \\ \left(\left| t \log\left(\frac{b}{a} \right) \right| < \left| \log(-\lambda) \right|;\ 1^{\alpha} := 1;\ x \in \mathbb{R} \right). \end{gathered} \tag{1.12}$$

Definition 1.6. Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}$$
$$\left(\left|t\log\left(\frac{b}{a}\right)\right| < \left|\log(-\lambda)\right|;\ 1^{\alpha} := 1;\ x \in \mathbb{R}\right). \tag{1.13}$$

It is easy to see that setting $a = 1$ and $b = c = e$ in (1.12) and (1.13) yields the classical results for the Apostol-Euler and Apostol-Genocchi polynomials.

Lately, Kurt [23] presented a new interesting class of generalized Euler polynomials. Explicitly, he introduced the next definition.

Definition 1.7. For arbitrary real or complex parameter α, the generalized Euler polynomials $E_n^{[m-1,\alpha]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighborhood of $t = 0$ by means of the generating function:

$$\left(\frac{2^m}{e^t + \sum_{l=0}^{m-1}(t^l/l!)}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} E_n^{[m-1,\alpha]}(x)\frac{t^k}{k!}. \tag{1.14}$$

It is easy to see that if we set $m = 1$ in (1.14), we arrive at the following:

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x)\frac{t^k}{k!}, \tag{1.15}$$

which is the generating function for the generalized Euler polynomials of order α. Thus, we have

$$E_n^{[0,\alpha]}(x) = E_n^{(\alpha)}(x). \tag{1.16}$$

In this paper, we propose a further generalization of Apostol-Euler polynomials and the Apostol-Genocchi polynomials and we give some properties involving them. For the new class of Apostol-Euler polynomials, we establish a new addition theorem with the help of a result given by Srivastava et al. [24]. We also give an extension of the Srivastava-Pintér theorem [25, 26]. Finally, we exhibit some relationships between the generalized Apostol-Euler polynomials and other polynomials or special functions with the help of the new addition formula.

2. New Classes of Generalized Apostol-Euler and Apostol-Genocchi Polynomials

The following definitions provide a natural generalization of the Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$, $m \in \mathbb{N}$, of order $\alpha \in \mathbb{C}$ and the Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x;\lambda)$, $m \in \mathbb{N}$, of order $\alpha \in \mathbb{C}$.

Definition 2.1. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^{+}$, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < |\log(-\lambda)|$ by means of the generating function:

$$\left(\frac{2^m}{\lambda b^t + \sum_{l=0}^{m-1}\left((t\log b)^l / l!\right)}\right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)\frac{t^k}{k!}. \tag{2.1}$$

It is easy to see that if we set $m = 1$, $b = c = e$ in (2.1), we arrive at the following:

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_n^{[0,\alpha]}(x,e,e;\lambda)\frac{t^k}{k!}. \tag{2.2}$$

This is the generating function for the generalized Apostol-Euler polynomials of order α. Thus, we have

$$\mathfrak{E}_n^{[0,\alpha]}(x,e,e;\lambda) = \mathfrak{E}_n^{(\alpha)}(x;\lambda). \tag{2.3}$$

Definition 2.2. For arbitrary real or complex parameter α and for $b, c \in \mathbb{R}^{+}$, the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t = 0$, with $|t \log b| < |\log(-\lambda)|$ by means of the generating function:

$$\left(\frac{2^m t^m}{\lambda b^t + \sum_{l=0}^{m-1}\left((t\log b)^l / l!\right)}\right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)\frac{t^k}{k!}. \tag{2.4}$$

Obviously, if we set $m = 1$, $b = c = e$ in (2.4), we obtain

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_n^{[0,\alpha]}(x,e,e;\lambda)\frac{t^k}{k!}. \tag{2.5}$$

This is the generating function for the generalized Apostol-Genocchi polynomials of order α. Thus, we have

$$\mathfrak{G}_n^{[0,\alpha]}(x,e,e;\lambda) = \mathfrak{G}_n^{(\alpha)}(x;\lambda). \tag{2.6}$$

The generalized Apostol-Euler polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ defined by (2.1) possess the following interesting properties. These are stated as Theorems 2.3, 2.4, and 2.5 below.

Theorem 2.3. *The generalized Apostol-Euler polynomials* $\mathfrak{E}_n^{[m-1,l]}(x,b,c;\lambda)$ *and the generalized Apostol-Bernoulli polynomials* $\mathfrak{B}_n^{[m-1,l]}(x,b,c;\lambda)$, $l \in \mathbb{N}_0$, *are related by*

$$\mathfrak{B}_n^{[m-1,l]}(x,b,c;\lambda) = \frac{(-1)^l n!}{2^{ml}(n-ml)!}\mathfrak{E}_{n-ml}^{[m-1,l]}(x,b,c;-\lambda) \quad (n,l,m \in \mathbb{N}_0, n \geq ml) \tag{2.7}$$

or, equivalently, by

$$\mathfrak{E}_n^{[m-1,l]}(x,b,c;\lambda) = \frac{(-2^m)^l n!}{(n-ml)!}\mathfrak{B}_{n+ml}^{[m-1,l]}(x,b,c;-\lambda) \quad (n,l,m \in \mathbb{N}_0). \tag{2.8}$$

Proof. Considering the generating function (2.1), the relations (2.7) and (2.8) follow easily. □

Theorem 2.4. *Let $b,c \in \mathbb{R}^+$, α an arbitrary complex number, and $m \in \mathbb{N}$. Then, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ satisfy the following relations:*

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x,b,c;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y,b,c;\lambda), \tag{2.9}$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x,b,c;\lambda)(y\log c)^{n-k}. \tag{2.10}$$

Proof. Considering the generating function (2.1) and comparing the coefficients of $t^n/n!$ in the both sides of the above equation, we arrive at (2.9). Proof of (2.10) is similar. □

Theorem 2.5. *The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ satisfy the following recurrence relation:*

$$\begin{aligned}
&\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1,b,c;\lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda) \\
&\quad = 2\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x,b,c;\lambda)\mathfrak{E}_{n-k}^{(-1)}(0;\lambda;1,c,a),
\end{aligned} \tag{2.11}$$

where $\mathfrak{E}_{n-1-k}^{(-1)}(0;\lambda;1,c,a)$ are the generalized Apostol-Euler polynomials defined by (1.12).

Proof. Considering the expression $\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1,b,c;\lambda) + \mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ and using the generating functions (2.1) and (1.12), (2.11) follows easily. □

Remark 2.6. Setting $m=1$ and $b=c=e$ in (2.11) and with the help of (2.3), we find

$$\lambda\mathfrak{E}_n^{(\alpha)}(x+1;\lambda) + \mathfrak{E}_n^{(\alpha)}(x;\lambda) = 2\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{(\alpha)}(x;\lambda)\mathfrak{E}_{n-k}^{(-1)}(0;\lambda). \tag{2.12}$$

Using the well-known result (see [8])

$$\mathfrak{E}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{(\alpha)}(x;\lambda)\mathfrak{E}_{n-k}^{(\beta)}(y;\lambda), \tag{2.13}$$

(2.12) becomes the familiar relation for the generalized Apostol-Euler polynomials (see [8]):

$$\lambda\mathfrak{E}_n^{(\alpha)}(x+1;\lambda) + \mathfrak{E}_n^{(\alpha)}(x;\lambda) = 2\mathfrak{E}_n^{(\alpha-1)}(x;\lambda). \tag{2.14}$$

Now, let us shift our focus on some interesting properties for the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ defined by (2.4). These are stated as Theorems 2.7, 2.8, and 2.9 below.

Theorem 2.7. *The generalized Apostol-Genocchi polynomials* $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$*, the generalized Apostol-Bernoulli polynomials* $\mathfrak{B}_n^{[m-1,\alpha]}(x,b,c;\lambda)$*, and the generalized Apostol-Euler polynomials* $\mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ *are related by*

$$\begin{aligned}\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda) &= (-2^m)^{\alpha}\mathfrak{B}_n^{[m-1,\alpha]}(x,b,c;-\lambda) \quad (\alpha\in\mathbb{C}),\\ \mathfrak{G}_n^{[m-1,l]}(x,b,c;\lambda) &= \frac{n!}{(n-ml)!}\mathfrak{E}_{n-ml}^{[m-1,l]}(x,b,c;\lambda) \quad (n,l,m\in\mathbb{N}_0, n\geq ml).\end{aligned} \tag{2.15}$$

Proof. Considering the generating function (2.4), the relations (2.15) follow easily. □

Theorem 2.8. *Let* $b,c\in\mathbb{R}^+$*,* α *an arbitrary complex number, and* $m\in\mathbb{N}$*. Then, the generalized Apostol-Genocchi polynomials* $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ *satisfy the following relations:*

$$\mathfrak{G}_n^{[m-1,\alpha+\beta]}(x+y,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k^{[m-1,\alpha]}(x,b,c;\lambda)\mathfrak{G}_{n-k}^{[m-1,\beta]}(y,b,c;\lambda), \tag{2.16}$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k^{[m-1,\alpha]}(x,b,c;\lambda)(y\log c)^{n-k}. \tag{2.17}$$

Proof . Considering the generating function (2.4) and comparing the coefficients of $t^n/n!$ in the both sides of the above equation, we arrive at (2.17). Proof of (2.18) is similar. □

Theorem 2.9. *The generalized Apostol-Genocchi polynomials* $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ *satisfy the following recurrence relation:*

$$\begin{aligned}&\lambda\mathfrak{G}_n^{[m-1,\alpha]}(x+1,b,c;\lambda)+\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)\\ &\quad = 2n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{G}_k^{[m-1,\alpha]}(x,b,c;\lambda)\mathfrak{G}_{n-1-k}^{(-1)}(0;\lambda;1,c,a),\end{aligned} \tag{2.18}$$

where $\mathfrak{G}_{n-1-k}^{(-1)}(0;\lambda;1,c,a)$ *are the generalized Apostol-Genocchi polynomials defined by* (1.13).

Proof. Considering the expression $\lambda\mathfrak{G}_n^{[m-1,\alpha]}(x+1,b,c;\lambda)+\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$ and using the generating functions (2.4) and (1.13), (2.19) follows easily. □

Remark 2.10. Putting $m=1$ and $b=c=e$ in (2.19) and with the help of (2.6), we find

$$\lambda\mathfrak{G}_n^{(\alpha)}(x+1;\lambda)+\mathfrak{G}_n^{(\alpha)}(x;\lambda) = 2n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{G}_k^{(\alpha)}(x;\lambda)\mathfrak{G}_{n-1-k}^{(-1)}(0;\lambda). \tag{2.19}$$

Using the well-known result (see [11])

$$\mathfrak{G}_n^{(\alpha+\beta)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k^{(\alpha)}(x;\lambda)\mathfrak{G}_{n-k}^{(\beta)}(y;\lambda), \tag{2.20}$$

(2.20) becomes the familiar relation for the generalized Apostol-Genocchi polynomials (see [11]):

$$\lambda\mathfrak{G}_n^{(\alpha)}(x+1;\lambda)+\mathfrak{G}_n^{(\alpha)}(x;\lambda)=2n\mathfrak{G}_{n-1}^{(\alpha-1)}(x;\lambda). \tag{2.21}$$

3. An Addition Theorem for the New Class of Generalized Apostol-Euler Polynomials

In this section, we establish a new addition theorem for the generalized Apostol-Euler polynomials. This new formula is based on a result due to Srivastava et al. [24].

The next theorem has been invented by Srivastava et al. [24]. However, the theorem is given without proof (see [24, pages 438–440]).

Theorem 3.1. *Let $B(z)$ and $z^{-1}C(z)$ be arbitrary functions which are analytic in the neighborhood of the origin, and assume (for sake of simplicity) that*

$$B(0)=C'(0)=1. \tag{3.1}$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ by means of

$$\sum_{n=0}^{\infty}f_n^{(\alpha)}(x)\frac{z^n}{n!}=[B(z)]^{\alpha}\exp(xC(z)), \tag{3.2}$$

where α and x are arbitrary complex numbers independent of z. Then, for arbitrary parameters σ and y,

$$f_n^{(\alpha+\sigma\gamma)}(x+\gamma y)=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}f_k^{(\alpha-\sigma k)}(x-ky)f_{n-k}^{(\sigma k+\sigma\gamma)}(ky+\gamma y), \tag{3.3}$$

provided that $\mathrm{Re}(\gamma)>0$.

Remark 3.2. The choice of 1 in the conditions of (3.3) is merely a convenient one. In fact, any nonzero constant values may be assumed for $B(0)$ and $C'(0)$.

Now, applying the last theorem with special choices of functions and parameters furnishes the next very interesting addition formula. This formula is contained in the following corollary.

Corollary 3.3. *Let $b, c \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then, for arbitrary complex parameters α, σ, x and y, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x, b, c; \lambda)$ satisfy the addition formula:*

$$\mathfrak{E}_n^{[m-1,\alpha+\sigma\gamma]}(x + \gamma y, b, c; \lambda) = \sum_{k=0}^{n} \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha-\sigma k]}(x - ky, b, c; \lambda) \mathfrak{E}_{n-k}^{[m-1,\sigma k+\sigma\gamma]}(ky + \gamma y, b, c; \lambda) \tag{3.4}$$

provided that $\mathrm{Re}(\gamma) > 0$.

Proof. Setting $B(z) = 2^m / (b^t + \sum_{l=0}^{m-1}((t \log b)^l / l!))$ and $C(z) = t \log c$ in Theorem 3.1, the result follows. □

Moreover, if we set $\sigma = 0$ in (3.4), we obtain

$$\begin{aligned}\mathfrak{E}_n^{[m-1,\alpha]}(x + \gamma y, b, c; \lambda) &= \sum_{k=0}^{n} \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x - ky, b, c; \lambda) \mathfrak{E}_{n-k}^{[m-1,0]}(ky + \gamma y, b, c; \lambda) \\ &= \sum_{k=0}^{n} \frac{\gamma + n}{\gamma + k} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x - ky, b, c; \lambda)(\gamma + k)^{n-k}(y \log c)^{n-k}.\end{aligned} \tag{3.5}$$

This result (3.5) will be very useful in the next section.

4. Some Analogues of the Srivastava-Pintér Addition Theorem

In this section, we give a generalization of the Srivastava-Pintér addition theorem and an analogue. We end this section by giving two interesting relationships involving the new addition formula (3.5).

Theorem 4.1. *The following relationship,*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x + y, b, c; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \left[2 \sum_{j=0}^{k} \binom{k}{j} \mathfrak{E}_j^{[m-1,\alpha]}(y, b, c; \lambda) \mathfrak{E}_{k-j}^{(-1)}(0; \lambda; 1, c, a) \right] \mathfrak{E}_{n-k}(x; \lambda)(\log c)^{n-k} \tag{4.1}$$

$$(\alpha, \lambda \in \mathbb{C}; n \in \mathbb{N}_0),$$

holds between the new class of generalized Apostol-Euler polynomials, the classical Apostol-Euler polynomials defined by (1.4), *and the generalized Apostol-Euler polynomials defined by* (1.12).

Table 1: x^n expressed in terms of sums of special polynomials or numbers.

No.	Special polynomials or numbers	Series representation for x^n
(1)	Hermite polynomials [27, page 194, (4)]	$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]}\frac{H_{n-2k}(x)}{k!(n-2k)!}$
(2)	Legendre polynomials [27, page 181, Theorem 65]	$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]}\frac{(2n-4k+1)P_{n-2k}(x)}{k!(3/2)_{n-k}}$
(3)	Generalized Laguerre polynomials [27, page 207, (2)]	$x^n = n!(1+\alpha)_n\sum_{k=0}^{n}\frac{(-1)^k L_k^{(\alpha)}(x)}{(1+\alpha)_k(n-k)!}$
(4)	Gegenbauer polynomials [27, page 283, (36)]	$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]}\frac{(\nu+n-2k)C_{n-2k}^{\nu}(x)}{k!(\nu)_{n+1-k}}$
(5)	Stirling numbers of the second kind [5, page 207, Theorem B]	$x^n = \sum_{k=0}^{n}\binom{x}{k}k!S(n,k)$
(6)	Bernoulli polynomials [7, page 26]	$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)$
(7)	Euler polynomials [7, page 30]	$x^n = \frac{1}{2}\left[E_n(x)+\sum_{k=0}^{n}\binom{n}{k}E_k(x)\right]$
(8)	Apostol-Bernoulli polynomials [8, page 634, (29)]	$x^n = \frac{1}{n+1}\left[\lambda\sum_{k=0}^{n+1}\binom{n+1}{k}\mathfrak{B}_k(x;\lambda)-\mathfrak{B}_{n+1}(x;\lambda)\right]$
(9)	Apostol-Euler polynomials [8, page 635, (32)]	$x^n = \frac{1}{2}\left[\lambda\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(x;\lambda)+\mathfrak{E}_n(x;\lambda)\right]$
(10)	Generalized Apostol-Euler polynomials [28, page 1325, (2.4)]	$x^n = \frac{1}{2^\beta}\sum_{k=0}^{\infty}\binom{\beta}{k}\lambda^k\mathfrak{E}_n^{(\beta)}(x+k;\lambda), \beta\in\mathbb{C}$
(11)	Generalized Bernoulli polynomials and Stirling numbers [28, page 1329, (2.16)]	$x^n = \sum_{l=0}^{n}\binom{n}{l}\binom{l+j}{j}^{-1}S(l+j,j)B_{n-l}^{(j)}(x), j\in\mathbb{N}_0$
(12)	Generalized Apostol-Bernoulli polynomials and generalized Stirling numbers [28, page 1329, (2.15)]	$x^n = n!\sum_{l=-j}^{n}\frac{j!}{(l+j)!(n-l)!}S(l+j,j;\lambda)\mathfrak{B}_{n-l}^{(j)}(x;\lambda), j\in\mathbb{N}_0$
(13)	Generalized Bernoulli polynomials [29, page 158, (2.6)]	$x^n = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(k+m)!}B_{n-k}^{[m-1]}(x), m\in\mathbb{N}$

Proof. First of all, if we substitute the entry (9) for x^n from Table 1 into the right-hand side of (2.10), we get

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y,b,c;\lambda)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)(\log c)^{n-k}\left[\mathfrak{E}_{n-k}(x;\lambda)+\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}\mathfrak{E}_j(x;\lambda)\right]\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)(\log c)^{n-k}\mathfrak{E}_{n-k}(x;\lambda)\\
&\qquad+\frac{\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)(\log c)^{n-k}\sum_{j=0}^{n-k}\binom{n-k}{j}\mathfrak{E}_j(x;\lambda),
\end{aligned}\tag{4.2}$$

which, upon inverting the order of summation and using the following elementary combinatorial identity:

$$\binom{m}{l}\binom{l}{n} = \binom{m}{n}\binom{m-n}{m-l} \quad (m \geq l \geq n;\ l, m, n \in \mathbb{N}_0), \tag{4.3}$$

yields

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) \\
&\quad= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)\mathfrak{E}_{n-k}(x;\lambda)(\log c)^{n-k} \\
&\qquad+ \frac{\lambda}{2}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{E}_j(x;\lambda)(\log c)^j\sum_{k=0}^{n-j}\binom{n-j}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)(\log c)^{n-j-k}.
\end{aligned} \tag{4.4}$$

The innermost sum in (4.4) can be calculated with the help of (2.10) with, of course,

$$x = 1 \quad n \longmapsto n-j \quad (0 \leq j \leq n;\ n, j \in \mathbb{N}_0). \tag{4.5}$$

We thus find from (4.4) that

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) \\
&\quad= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)\mathfrak{E}_{n-k}(x;\lambda)(\log c)^{n-k} \\
&\qquad+ \frac{\lambda}{2}\sum_{j=0}^{n}\binom{n}{n-j}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y+1,b,c;\lambda)\mathfrak{E}_j(x;\lambda)(\log c)^j \\
&\quad= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda) + \lambda\mathfrak{E}_k^{[m-1,\alpha]}(y+1,b,c;\lambda)\right]\mathfrak{E}_{n-k}(x;\lambda)(\log c)^{n-k}
\end{aligned} \tag{4.6}$$

which, with the relation (2.11), leads us to the relationship (4.7) asserted by Theorem 4.1. □

Theorem 4.2. *The following relationship,*

$$\begin{aligned}
\mathfrak{G}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}(x;\lambda)(\log c)^{n-k} \\
&\quad\cdot\left[2k\sum_{j=0}^{k-1}\binom{k-1}{j}\mathfrak{G}_j^{[m-1,\alpha]}(y,b,c;\lambda)\mathfrak{G}_{k-1-j}^{(-1)}(0;\lambda;1,c,a)\right] \\
&\qquad\qquad (\alpha, \lambda \in \mathbb{C};\ n \in \mathbb{N}_0),
\end{aligned} \tag{4.7}$$

holds between the new class of generalized Apostol-Genocchi polynomials, the classical Apostol-Euler polynomials defined by (1.4)*, and the generalized Apostol-Genocchi polynomials defined by* (1.13)*.*

Making use of Table 1 that contains a list of series representation for x^n in terms of special polynomials or numbers, we can find some analogues of the Srivastava-Pintér addition theorem. Let us give an example of such formula.

Theorem 4.3. *The following relationship,*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y,b,c;\lambda)(\log c)^{n-k}\frac{(n-k)!}{2^{(n-k)}}\sum_{j=0}^{[(n-k)/2]}\frac{H_{n-k-2j}(x)}{j!(n-k-2j)!}, \tag{4.8}$$

holds between the new class of generalized Apostol-Euler polynomials and the Hermite polynomials defined by

$$e^{(2xt-t^2)} = \sum_{n=0}^{\infty} H_n(x)t^n. \tag{4.9}$$

Proof. We derived the Proof from the addition theorem (2.10) and entry 1. □

We end this paper by giving two special cases of the addition theorem (3.4) involving the new class of generalized Apostol-Euler polynomials. These are contained in the two next theorems.

Theorem 4.4. *The following relationship,*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+\gamma y,b,c;\lambda) = \sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x-ky,b,c;\lambda)(\gamma+k)^{n-k}(\log c)^{n-k}\sum_{j=0}^{n-k}\binom{y}{j}j!S(n-k,j), \tag{4.10}$$

holds between the new class of generalized Apostol-Euler polynomials and the Stirling numbers of the second kind that could be computed by the formula [30, page 58, (1.5)*]*

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \tag{4.11}$$

Proof. We derived the Proof from the addition theorem (3.5) and entry 5. □

Theorem 4.5. *The following relationship,*

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+\gamma y,b,c;\lambda)\\
&\quad=\sum_{k=0}^{n}\frac{\gamma+n}{\gamma+k}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x-ky,b,c;\lambda)(\gamma+k)^{n-k}(\log c)^{n-k}\\
&\qquad\cdot(n-k)!\sum_{l=-j}^{n-k}\frac{j!}{(l+j)!(n-k-l)!}S(l+j,j;\lambda)\mathfrak{B}_{n-k-l}^{(j)}(y;\lambda)\quad(j\in\mathbb{N}_0),
\end{aligned}\tag{4.12}$$

holds between the new class of generalized Apostol-Euler polynomials and the classical Apostol-Bernoulli polynomials and the generalized Stirling numbers.

Proof. We derived the Proof from the addition theorem (3.5) and entry 12. □

It could be interesting to apply the addition formula (3.3) to other family of polynomials in conjunction with series representation involving some special functions for x^n in order to derive some analogues of the Srivastava-Pintér addition theorem.

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