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Research Article

Henstock-Kurzweil Integral Transforms

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We show conditions for the existence, continuity, and differentiability of functions defined by $\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t,s)dt$, where f is a function of bounded variation on $\mathbb{R}$ with $\lim_{|t|\to\infty} f(t) = 0$.

1. Introduction

Let g be a complex function defined on a certain subset of $\mathbb{R}^2$. Many functions on functional analysis are integrals of the following form:

$$\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t,s)dt. \tag{1.1}$$

We discuss the above function Γ, where the integral that we use is that of Henstock-Kurzweil. This integral introduced independently by Kurzweil and Henstock in 1957-58 encompasses the Riemann and Lebesgue integrals, as well as the Riemann and Lebesgue improper integrals.

In Lebesgue theory, there are well-known results about the existence, continuity, and differentiability of Γ. For Henstock-Kurzweil integrals also there are results about this, for example, Theorems 12.12 and 12.13 of [1]. However, they all need the stronger condition: $f(t)g(t,s)$ is bounded by a Henstock-Kurzweil integrable function $r(t)$. We provide other conditions for the existence, continuity, and differentiability of Γ.

2. Preliminaries

Let us begin by recalling the definition of Henstock-Kurzweil integral. For finite intervals in $\mathbb{R}$ it is defined in the following way.

Definition 2.1. Let $f : [a,b] \to \mathbb{R}$ be a function. One can say that f is Henstock-Kurzweil (shortly, HK-) integrable, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a function $\gamma_\epsilon : [a,b] \to (0,\infty)$ (named a gauge) with the property that for any δ_ϵ-fine partition $P = \{([x_{i-1},x_i],t_i)\}_{i=1}^n$ of $[a,b]$ (i.e., for each i, $[x_{i-1},x_i] \subset [t_i - \gamma_\epsilon(t_i), t_i + \gamma_\epsilon(t_i)]$), one has

$$\left|\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) - A\right| < \epsilon. \tag{2.1}$$

The number A is the integral of f over $[a,b]$ and it is denoted as $A = \int_a^b f$.

In the unbounded case, the Henstock-Kurzweil integral is defined as follows.

Definition 2.2. Given a gauge function $\gamma : [a,\infty] \to (0,\infty)$, one can say that a tagged partition $P = \{([x_{i-1},x_i],t_i)\}_{i=1}^{n+1}$ of $[a,\infty]$ is γ-fine, if

(a) $a = x_0$, $x_{n+1} = t_{n+1} = \infty$,

(b) $[x_{i-1},x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$ for all $i = 1,2,\dots,n$,

(c) $[x_n,\infty] \subseteq [1/\gamma(t_{n+1}),\infty]$.

Definition 2.3. A function $f : [a,\infty] \to \mathbb{R}$ is Henstock-Kurzweil integrable on $[a,\infty]$, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a gauge $\gamma_\epsilon : [a,\infty] \to (0,\infty)$ for which (2.1) is satisfied for all tagged partition P which is δ_ϵ-fine according to Definition 2.2.

Let f be a function defined on an infinite interval $[a,\infty)$, One can suppose that f is defined on $[a,\infty]$ assuming that $f(\infty) = 0$. Thus, f is Henstock-Kurzweil integrable on $[a,\infty)$ if f extended on $[a,\infty]$ is HK-integrable. For functions defined over intervals $(-\infty,a]$ and $(-\infty,\infty)$ One can makes similar considerations.

Let I be a finite or infinite interval. The space of all Henstock-Kurzweil integrable functions over I is denoted by $\mathcal{HK}(I)$. This space will be considered with the Alexiewicz seminorm, which it is defined as follows:

$$\|f\|_I = \sup_{J\subseteq I}\left|\int_J f\right|, \tag{2.2}$$

where the supremum is being taken over all intervals J contained in I.

Definition 2.4. Let $\varphi : I \to \mathbb{R}$ be a function, where $I \subseteq \mathbb{R}$ is a finite interval. The variation of φ over the interval I is defined as follows:

$$V_I\varphi = \sup\left\{\sum_{i=1}^n \left|\varphi(x_i) - \varphi(x_{i-1})\right| : P \text{ is partition of } I\right\}. \tag{2.3}$$

We say that the function φ is of bounded variation on I if $V_I\varphi < \infty$. Now if φ is a function defined on an infinite interval I, then φ is of bounded variation on I, if φ is of bounded variation on each finite subinterval of I and there is $M > 0$ such that $V_{[a,b]}\varphi \leq M$ for all $[a,b] \subseteq I$. The variation of φ on I is $V_I\varphi = \sup\{V_{[a,b]}\varphi | \ [a,b] \subseteq I\}$.

Given an interval I, the space of all bounded variation functions on I is denoted by $\mathcal{BV}(I)$. We set $\mathcal{BV}_0(\mathbb{R}) = \{f \in \mathcal{BV}(\mathbb{R}) \mid \lim_{|t|\to\infty} f(t) = 0\}$. The following are some classical theorems that are used throughout this paper. The first is given in [2, Lemma 24] and is an immediate consequence of [1, Theorem 10.12, and Corollary H.4].

Theorem 2.5. *If g is a HK-integrable function on $[a,b] \subseteq \mathbb{R}$ and f is a function of bounded variation on $[a,b]$, then fg is HK-integrable on $[a,b]$ and*

$$\left|\int_a^b fg\right| \leq \inf_{t\in[a,b]} |f(t)| \left|\int_a^b g(t)dt\right| + \|g\|_{[a,b]} V_{[a,b]} f. \tag{2.4}$$

Theorem 2.6 ([1] Chartier-Dirichlet's test). *Let f and g be functions defined on $[a,\infty)$. Suppose that*

(i) *$g \in \mathcal{HK}([a,c])$ for every $c \geq a$, and G defined by $G(x) = \int_a^x g$ is bounded on $[a,\infty)$;*

(ii) *f is of bounded variation on $[a,\infty)$ and $\lim_{x\to\infty} f(x) = 0$.*

Then $fg \in \mathcal{HK}([a,\infty))$.

Definition 2.7 (see [3]). Let $E \subseteq [a,b]$. A function $f : [a,b] \to \mathbb{R}$ is AC_δ on E, if for every $\epsilon > 0$, there exist $\eta_\epsilon > 0$ and a gauge δ_ϵ on E such that

$$\sum_{i=1}^{s} |f(v_i) - f(u_i)| < \epsilon, \tag{2.5}$$

whenever $P = \{([u_i, v_i], t_i)\}_{i=1}^s$ is a (δ_ϵ, E)-fine subpartition of $[a,b]$ (i.e., P is δ_ϵ-fine and the tags t_i belong to E) and $\sum_{i=1}^s |v_i - u_i| < \eta_\epsilon$.

We say that f is ACG_δ on $[a,b]$, if $[a,b]$ can be written as a countable union of sets on each of which the function f is AC_δ.

If $h(t,s)$ is a function on $\mathbb{R} \times \mathbb{R}$, then we use the notation D_2h for the partial derivative of h with respect to the second component s.

Theorem 2.8 ([4, Theorem 4]). *Let $a, b \in \mathbb{R}$. If $h : \mathbb{R} \times [a,b] \to \mathbb{C}$ is such that*

(i) *$h(t,\cdot)$ is ACG_δ on $[a,b]$ for almost all $t \in \mathbb{R}$;*

(ii) *$h(\cdot,s)$ is HK-integrable on $\mathbb{R}$ for all $s \in [a,b]$.*

Then $H := \int_{-\infty}^{\infty} h(t,\cdot)dt$ is ACG_δ on $[a,b]$ and $H'(s) = \int_{-\infty}^{\infty} D_2h(t,s)dt$ for almost all $s \in (a,b)$, if and only if,

$$\int_s^t \int_{-\infty}^{\infty} D_2h(t,s)dt\,ds = \int_{-\infty}^{\infty} \int_s^t D_2h(t,s)ds\,dt, \tag{2.6}$$

for all $[s,t] \subseteq [a,b]$. In particular,

$$H'(s_0) = \int_{-\infty}^{\infty} D_2h(t,s_0)dt, \tag{2.7}$$

when $H_2 := \int_{-\infty}^{\infty} D_2h(t,\cdot)dt$ is continuous at s_0.

3. Main Results

All results in this paper are based on functions in the vector space $\mathcal{BV}_0(\mathbb{R})$. Note that $\mathcal{BV}_0(\mathbb{R}) \not\subseteq L(\mathbb{R})$, where $L(\mathbb{R})$ is the space of Lebesgue integrable functions. Indeed, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 1), \\ \dfrac{1}{x} & \text{if } x \in [1, \infty), \end{cases} \tag{3.1}$$

is in $\mathcal{BV}_0(\mathbb{R}) \setminus L(\mathbb{R})$. However, for bounded intervals I, functions in $\mathcal{BV}(I)$ are Lebesgue integrables on I.

To facilitate the statement of these results, it seems appropriate to introduce some additional terminology. If $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a function and $s_0 \in \mathbb{R}$, we say that s_0 satisfies Hypothesis (**H**) relative to g if

(**H**) there exist $\delta = \delta(s_0) > 0$ and $M = M(s_0) > 0$, such that, if $|s - s_0| < \delta$ then

$$\left| \int_u^v g(t,s)\,dt \right| \le M, \tag{3.2}$$

for all $[u, v] \subseteq \mathbb{R}$.

This type of condition plays a major role in the results of the present work.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions. If $f \in \mathcal{BV}_0(\mathbb{R})$, and $s_0 \in \mathbb{R}$ satisfies Hypothesis* (**H**) *relative to g, then*

$$\Gamma(s) = \int_{-\infty}^{\infty} f(t) g(t,s)\,dt \tag{3.3}$$

exists for all s in a neighborhood of s_0.

Proof. It follows by Theorem 2.6. □

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions such that*

(i) *$f \in \mathcal{BV}_0(\mathbb{R})$, g is bounded, and*

(ii) *$g(t, \cdot)$ is continuous for all $t \in \mathbb{R}$.*

If $s_0 \in \mathbb{R}$ satisfies Hypothesis (**H**) *relative to g, then the function Γ defined in Theorem 3.1 is continuous at s_0.*

Proof. There exist $\delta_1 > 0$ and $M > 0$, such that, if $|s - s_0| < \delta_1$ then

$$\left| \int_u^v g(t,s)\,dt \right| \le M, \tag{3.4}$$

for all $[u, v] \subseteq \mathbb{R}$. From Theorem 3.1, $\Gamma(s)$ exists for all $s \in B_{\delta_1}(s_0)$.

Let $\epsilon > 0$ be given. By Hake's Theorem, there exists $K_1 > 0$ such that

$$\left|\int_{|t|\geq u} f(t)g(t,s_0)dt\right| < \frac{\epsilon}{3}, \tag{3.5}$$

for all $u \geq K_1$. On the other hand, as

$$\lim_{t\to-\infty} V_{(-\infty,t]}f = 0, \qquad \lim_{t\to\infty} V_{[t,\infty)}f = 0, \tag{3.6}$$

there is $K_2 > 0$ such that for each $t > K_2$,

$$V_{(-\infty,-t]}f + V_{[t,\infty)}f < \frac{\epsilon}{3M}. \tag{3.7}$$

Let $K = \max\{K_1, K_2\}$. From Theorem 2.5, it follows that for every $v \geq K$ and every $s \in B_{\delta_1}(s_0)$,

$$\begin{aligned}\left|\int_K^v f(t)g(t,s)dt\right| &\leq \|g(\cdot,s)\|_{[K,v]}\left[\inf_{t\in[K,v]}|f(t)| + V_{[K,v]}f\right] \\ &\leq M\left[|f(v)| + V_{[K,\infty)}f\right],\end{aligned} \tag{3.8}$$

where the second inequality is true due to (3.4). This implies, since $\lim_{t\to\infty}|f(t)| = 0$, that

$$\left|\int_K^\infty f(t)g(t,s)dt\right| \leq M\cdot V_{[K,\infty)}f. \tag{3.9}$$

Analogously we have that

$$\left|\int_{-\infty}^{-K} f(t)g(t,s)dt\right| \leq M\cdot V_{(-\infty,-K]}f. \tag{3.10}$$

Therefore, for each $s \in B_{\delta_1}(s_0)$,

$$\left|\int_{|t|\geq K} f(t)g(t,s)dt\right| \leq M\left[V_{(-\infty,-K]}f + V_{[K,\infty)}f\right] < M\frac{\epsilon}{3M} = \frac{\epsilon}{3}. \tag{3.11}$$

By hypothesis, f is Lebesgue integrable on $[-K,K]$, g is bounded, and $g(t,\cdot)$ is continuous for all $t \in \mathbb{R}$. From this it is easy to see, for example using [1, Theorem 12.12], that $\Gamma_K : \mathbb{R} \to \mathbb{R}$ defined as

$$\Gamma_K(s) = \int_{-K}^{K} f(t)g(t,s)dt, \quad s \in \mathbb{R}, \tag{3.12}$$

is continuous at s_0. This implies that there is $\delta_2 > 0$ such that for every $s \in B_{\delta_2}(s_0)$,

$$\left|\int_{-K}^{K} f(t)\left[g(t,s) - g(t,s_0)\right]dt\right| < \frac{\epsilon}{3}. \tag{3.13}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $s \in B_\delta(s_0)$,

$$\begin{aligned} |\Gamma(s) - \Gamma(s_0)| \le{}& \left|\int_{-K}^{K} f(t)\left[g(t,s) - g(t,s_0)\right]dt\right| \\ &+ \left|\int_{|t|\ge K} f(t)g(t,s)dt\right| + \left|\int_{|t|\ge K} f(t)g(t,s_0)dt\right|. \end{aligned} \tag{3.14}$$

Thus, from (3.5), (3.11), and (3.13), $|\Gamma(s) - \Gamma(s_0)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$, for all $s \in B_\delta(s_0)$. □

Theorem 3.3. *Let $a, b \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times [a,b] \to \mathbb{C}$ are functions such that*

(i) *$f \in \mathcal{BV}_0(\mathbb{R})$, g is measurable, bounded, and*

(ii) *for all $s \in [a,b]$, s satisfies hypothesis (**H**) relative to g,*

then

$$\int_a^b \int_{-\infty}^{\infty} f(t)g(t,s)dt\,ds = \int_{-\infty}^{\infty}\int_a^b f(t)g(t,s)ds\,dt. \tag{3.15}$$

Proof. From condition (ii) and by the compactness of $[a,b]$, we claim that there exists $M > 0$ such that, for each $s \in [a,b]$, $|\int_u^v g(t,s)dt| \le M$, for all $[u,v] \subseteq \mathbb{R}$.

For each $r > 0$ and $s \in [a,b]$, let $\Gamma_r(s) = \int_{-r}^{r} f(t)g(t,s)dt$. Observe, by Theorem 2.5,

$$\begin{aligned} |\Gamma_r(s)| &= \left|\int_{-r}^{r} f(t)g(t,s)dt\right| \\ &\le \left\|g(\cdot,s)\right\|_{[-r,r]}\left[\inf_{t\in[-r,r]}\left|f(t)\right| + V_{[-r,r]}f\right] \\ &\le M\left[\left|f(0)\right| + Vf\right], \end{aligned} \tag{3.16}$$

for all $s \in [a,b]$.

So, for each $r > 0$, Γ_r is HK-integrable on $[a,b]$ and is bounded for a fixed constant. Moreover, by Theorem 3.1 and Hake's Theorem,

$$\lim_{r\to\infty} \Gamma_r(s) = \Gamma(s), \tag{3.17}$$

for all $s \in [a,b]$.

Therefore, by dominated convergence theorem, Γ is HK-integrable on $[a,b]$ and

$$\int_a^b \Gamma(s)ds = \lim_{r\to\infty}\int_a^b \Gamma_r(s)ds. \tag{3.18}$$

Now, since f is Lebesgue integrable on $[-r,r]$, g is measurable and bounded; it follows by Fubini's Theorem that

$$\int_a^b\int_{-r}^r f(t)g(t,s)dt\,ds = \int_{-r}^r\int_a^b f(t)g(t,s)ds\,dt. \tag{3.19}$$

Consequently,

$$\lim_{r\to\infty}\int_{-r}^r\int_a^b f(t)g(t,s)ds\,dt = \lim_{r\to\infty}\int_a^b \Gamma_r(s)ds = \int_a^b \Gamma(s)ds. \tag{3.20}$$

So by Hake's Theorem,

$$\int_{-\infty}^{\infty}\int_a^b f(t)g(t,s)ds\,dt = \int_a^b \Gamma(s)ds = \int_a^b\int_{-\infty}^{\infty} f(t)g(t,s)dt\,ds. \tag{3.21}$$

□

Theorem 3.4. *Consider $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ functions, where $f\in\mathcal{BV}_0(\mathbb{R})$ and the partial derivative D_2g exists on $\mathbb{R}\times\mathbb{R}$ and is bounded and continuous. If $s_0\in\mathbb{R}$ such that*

(i) *there is $K>0$ for which $\|g(\cdot,s_0)\|_{[u,v]}\le K$ for all $[u,v]\subseteq\mathbb{R}$, and*

(ii) *s_0 satisfies Hypothesis (**H**) relative to D_2g;*

then Γ is differentiable at s_0, and

$$\Gamma'(s_0) = \int_{-\infty}^{\infty} f(t)D_2g(t,s_0)dt. \tag{3.22}$$

Proof. It is not difficult to prove, using conditions (i), (ii), and the Mean Value Theorem, that there exist $\delta>0$ and $M>0$ such that, for each $s\in(s_0-\delta,s_0+\delta)$,

$$\left|\int_u^v D_2g(t,s)dt\right| < M, \qquad \left|\int_u^v g(t,s)dt\right| < M, \tag{3.23}$$

for all $[u,v]\subseteq\mathbb{R}$.

Consider $a,b\in\mathbb{R}$ with $s_0-\delta<a<s_0<b<s_0+\delta$. In order to show (3.22), we use Theorem 2.8. The function $f(t)g(t,\cdot)$ is differentiable on $[a,b]$ for each $t\in\mathbb{R}$, so $f(t)g(t,\cdot)$ is ACG_δ on $[a,b]$ for all $t\in\mathbb{R}$. Also, by (3.23) and Theorem 2.6, $f(\cdot)g(\cdot,s)$ is HK-integrable on $\mathbb{R}$ for all $s\in[a,b]$. Then

$$\Gamma'(s_0) = \int_{-\infty}^{\infty} f(t)D_2g(t,s_0)dt, \tag{3.24}$$

if

$$\Gamma_2 := \int_{-\infty}^{\infty} f(t) D_2 g(t,\cdot) dt \tag{3.25}$$

is continuous at s_0, and

$$\int_s^t \int_{-\infty}^{\infty} f(t) D_2 g(t,s) dt\, ds = \int_{-\infty}^{\infty} \int_s^t f(t) D_2 g(t,s) ds\, dt, \tag{3.26}$$

for all $[s,t] \subseteq [a,b]$. The first affirmation is true by (3.23) and Theorem 3.2. The second affirmation is true due to (3.23) and Theorem 3.3. □

Remark 3.5. In the previous theorems the kernel $g(t,s)$ satisfies $|\int_u^v g(t,s)dt| \leq M$, for all $[u,v] \subseteq \mathbb{R}$. Moreover, if g will satisfy

$$\left| \int_u^v g(t,s) dt \right| \leq \frac{M_0}{|s|}, \tag{3.27}$$

for all $[u,v] \subseteq \mathbb{R}$, then $\lim_{|s|\to\infty} \Gamma(s) = 0$, when $f \in \mathcal{BV}_0(\mathbb{R})$ (a version of Riemann-Lebesgue Lemma).

4. Applications

If $f : \mathbb{R} \to \mathbb{R}$, then its Fourier transform at $s \in \mathbb{R}$ is defined as follows:

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-its} dt. \tag{4.1}$$

Talvila in [2] has done an extensive work about the Fourier transform using the Henstock-Kurzweil integral: existence, continuity, inversion theorems and so forth. Nevertheless, there are some omissions in those results that use [2, Lemma 25(a)]. Also Mendoza Torres et al. in [5] have studied existence, continuity, and Riemann-Lebesgue Lemma about the Fourier transform of functions belonging to $\mathcal{HK}(R) \cap \mathcal{BV}(\mathbb{R})$. Following the line of [5], in Theorem 4.2, we include some results from them as consequences of theorems in the above section.

Let f and g be real-valued functions on $\mathbb{R}$. The convolution of f and g is the function $f * g$ defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy, \tag{4.2}$$

for all x such that the integral exists. Various conditions can be imposed on f and g to guarantee that $f * g$ is defined on $\mathbb{R}$, for example, if f is HK-integrable and g is of bounded variation.

Lemma 4.1. *If* $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$, *then* $\lim_{|x|\to\infty} f(x) = 0$.

Proof. Since f is of bounded variation on $\mathbb{R}$, then $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ exist. Suppose that $\lim_{x\to\infty} f(x) = \alpha \neq 0$. If $\alpha > 0$, there exists $A > 0$ such that $\alpha/2 < f(x)$, for all $x > A$. If $\alpha < 0$, there is $B > 0$ such that $-\alpha/2 < -f(x)$, for all $x > B$. This shows that $f \notin \mathcal{HK}([A,\infty))$ or $-f \notin \mathcal{HK}([B,\infty))$, which contradicts $f \in \mathcal{HK}(\mathbb{R})$, so $\lim_{x\to\infty} f(x) = 0$. Using a similar argument, we show that $\lim_{x\to-\infty} f(x) = 0$. □

As consequence of Lemma 4.1, the vector space $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$ is contained in $\mathcal{BV}_0(\mathbb{R})$. So the next theorem is an immediate consequence of the above section.

Theorem 4.2. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$, then*

(a) $\widehat{f}$ *exists on* $\mathbb{R}$.

(b) $\widehat{f}$ *is continuous on* $\mathbb{R} \setminus \{0\}$.

(c) $\lim_{|s|\to\infty} \widehat{f}(s) = 0$.

(d) *Define $g(t) = tf(t)$ and suppose that $g \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$, then $\widehat{f}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and*

$$\widehat{f}'(s) = -i\widehat{g}(s), \quad \forall s \in \mathbb{R} \setminus \{0\}. \tag{4.3}$$

(e) *If $h \in L(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$, then $\widehat{f * h}(s) = \widehat{f}(s)\widehat{h}(s)$ for all $s \in \mathbb{R}$.*

Proof. First observe that

$$\left| \int_u^v e^{-its} dt \right| \leq \frac{2}{|s|}, \tag{4.4}$$

for all $[u,v] \subseteq \mathbb{R}$. Then, each $s_0 \neq 0$ satisfies Hypothesis (**H**) relative to e^{-its}.

(a) Theorem 3.1 implies that $\widehat{f}(s_0)$ exists for all $s_0 \neq 0$ and, since $f \in \mathcal{HK}(\mathbb{R})$, $\widehat{f}(0)$ exists. Thus, $\widehat{f}$ exists on $\mathbb{R}$.

(b) From Theorem 3.2, $\widehat{f}$ is continuous at s_0, for all $s_0 \neq 0$.

(c) It follows by Remark 3.5 and (4.4).

(d) It follows by Theorem 2.8 in a similar way to the proof of Theorem 3.4.

(e) Take $s \in \mathbb{R}$ and let $k(x,y) = f(y-x)e^{-iys}$. Then, for each $y \in \mathbb{R}$ and all $[u,v] \subseteq \mathbb{R}$,

$$\left| \int_u^v k(x,y)dx \right| = \left| \int_u^v f(y-x)dx \right| = \left| \int_{y-u}^{y-v} f(z)dz \right| \leq \|f\|. \tag{4.5}$$

Thus, for every $y \in \mathbb{R}$, y satisfies Hypothesis (**H**) relative to k. Now, observe that $h \in \mathcal{BV}_0(\mathbb{R})$ and k is measurable and bounded. So by Theorem 3.3,

$$\int_{-a}^{a} \int_{-\infty}^{\infty} h(x)k(x,y)dx\,dy = \int_{-\infty}^{\infty} \int_{-a}^{a} h(x)k(x,y)dy\,dx, \tag{4.6}$$

for all $a > 0$.

On the other hand,

$$\left|h(x)\int_{-a}^{a} f(y-x)e^{-iys}dy\right| \leq |h(x)|\left|\int_{-a-x}^{a-x} f(z)e^{-izs}dz\right| \\ \leq |h(x)|\left\|f(\cdot)e^{-i(\cdot)s}\right\|. \tag{4.7}$$

Thus, since $h \in L(\mathbb{R})$, dominated convergence theorem implies that

$$\begin{aligned}\widehat{f}(s)\widehat{h}(s) &= \int_{-\infty}^{\infty} h(x)\int_{-\infty}^{\infty} f(y-x)e^{-iys}dy\,dx \\ &= \lim_{a\to\infty}\int_{-\infty}^{\infty} h(x)\int_{-a}^{a} f(y-x)e^{-iys}dy\,dx,\end{aligned} \tag{4.8}$$

but from (4.6), we have that

$$\begin{aligned}\widehat{f}(s)\widehat{h}(s) &= \lim_{a\to\infty}\int_{-a}^{a}\int_{-\infty}^{\infty} h(x)f(y-x)e^{-iys}dx\,dy \\ &= \lim_{a\to\infty}\int_{-a}^{a}(f*h)(y)e^{-iys}dy.\end{aligned} \tag{4.9}$$

Therefore, by Hake's Theorem,

$$\widehat{f*h}(s) = \widehat{f}(s)\widehat{h}(s). \tag{4.10}$$

□

If $f : [0,\infty) \to \mathbb{R}$, then its Laplace transform at $z \in \mathbb{C}$ is defined as follows:

$$L(f)(z) = \int_{0}^{\infty} f(t)e^{-zt}dt. \tag{4.11}$$

Here, also the Laplace transform is considered as Henstock-Kurzweil integral.

Theorem 4.3. *If* $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$, *then*

(a) $L(f)(z)$ *exists for all* $z \in \mathbb{C}$ *with* $Re\ z \geq 0$.

(b) *If* $F(x,y) = L(f)(x+iy)$, *then* $F(\cdot,y)$ *is continuous on* $\mathbb{R}^+ \cup \{0\}$ *for all* $y \neq 0$, *and* $F(x,\cdot)$ *is continuous on* $\mathbb{R}$ *for all* $x > 0$.

Proof. It is an easy consequence from Theorems 3.1 and 3.2, since $|\int_u^v e^{-(x+iy)t}dt| \leq 2/|x+iy|$ for all $u, v, x \in \mathbb{R}^+ \cup \{0\}$, $y \in \mathbb{R}$ with $x+iy \neq 0$. □

Moreover, the Riemann-Lebesgue Lemma holds the following.

Theorem 4.4. *If* $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$ *and* $z = x+iy$, *with* $x \geq 0$, *then* $\lim_{y\to\infty} L(f)(z) = 0$.

Proof. It results by Remark 3.5 and (4.4), because $f(\cdot)e^{-x(\cdot)}$ is in $\mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$. □

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