Research Article

Refinements of Inequalities among Difference of Means

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In this paper, for the difference of famous means discussed by Taneja in 2005, we study the Schur-geometric convexity in $(0, \infty) \times (0, \infty)$ of the difference between them. Moreover some inequalities related to the difference of those means are obtained.

1. Introduction

In 2005, Taneja [1] proved the following chain of inequalities for the binary means for $(a, b) \in R_+^2 = (0, \infty) \times (0, \infty)$:

$$H(a, b) \leq G(a, b) \leq N_1(a, b) \leq N_3(a, b) \leq N_2(a, b) \leq A(a, b) \leq S(a, b), \quad (1.1)$$

where

$$\begin{aligned} A(a, b) &= \frac{a+b}{2}, \\ G(a, b) &= \sqrt{ab}, \\ H(a, b) &= \frac{2ab}{a+b}, \\ N_1(a, b) &= \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 = \frac{A(a, b) + G(a, b)}{2}, \end{aligned} \quad (1.2)$$

$$\begin{aligned}
N_3(a, b) &= \frac{a + \sqrt{ab} + b}{3} = \frac{2A(a, b) + G(a, b)}{3}, \\
N_2(a, b) &= \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right) \left(\sqrt{\frac{a+b}{2}} \right), \\
S(a, b) &= \sqrt{\frac{a^2 + b^2}{2}}.
\end{aligned} \tag{1.3}$$

The means A, G, H, S, N_1 and N_3 are called, respectively, the arithmetic mean, the geometric mean, the harmonic mean, the root-square mean, the square-root mean, and Heron's mean. The N_2 one can be found in Taneja [2, 3].

Furthermore Taneja considered the following difference of means:

$$\begin{aligned}
M_{SA}(a, b) &= S(a, b) - A(a, b), \\
M_{SN_2}(a, b) &= S(a, b) - N_2(a, b), \\
M_{SN_3}(a, b) &= S(a, b) - N_3(a, b), \\
M_{SN_1}(a, b) &= S(a, b) - N_1(a, b), \\
M_{SG}(a, b) &= S(a, b) - G(a, b), \\
M_{SH}(a, b) &= S(a, b) - H(a, b), \\
M_{AN_2}(a, b) &= A(a, b) - N_2(a, b), \\
M_{AG}(a, b) &= A(a, b) - G(a, b), \\
M_{AH}(a, b) &= A(a, b) - H(a, b), \\
M_{N_2N_1}(a, b) &= N_2(a, b) - N_1(a, b), \\
M_{N_2G}(a, b) &= N_2(a, b) - G(a, b)
\end{aligned} \tag{1.4}$$

and established the following.

Theorem A. *The difference of means given by (1.4) is nonnegative and convex in $R_+^2 = (0, \infty) \times (0, \infty)$.*

Further, using Theorem A, Taneja proved several chains of inequalities; they are refinements of inequalities in (1.1).

Theorem B. *The following inequalities among the mean differences hold:*

$$M_{SA}(a, b) \leq \frac{1}{3}M_{SH}(a, b) \leq \frac{1}{2}M_{AH}(a, b) \leq \frac{1}{2}M_{SG}(a, b) \leq M_{AG}(a, b), \quad (1.5)$$

$$\frac{1}{8}M_{AH}(a, b) \leq M_{N_2N_1}(a, b) \leq \frac{1}{3}M_{N_2G}(a, b) \leq \frac{1}{4}M_{AG}(a, b) \leq M_{AN_2}(a, b), \quad (1.6)$$

$$M_{SA}(a, b) \leq \frac{4}{5}M_{SN_2}(a, b) \leq 4M_{AN_2}(a, b), \quad (1.7)$$

$$M_{SH}(a, b) \leq 2M_{SN_1}(a, b) \leq \frac{3}{2}M_{SG}(a, b), \quad (1.8)$$

$$M_{SA}(a, b) \leq \frac{3}{4}M_{SN_3}(a, b) \leq \frac{2}{3}M_{SN_1}(a, b). \quad (1.9)$$

For the difference of means given by (1.4), we study the Schur-geometric convexity of difference between these differences in order to further improve the inequalities in (1.1). The main result of this paper reads as follows.

Theorem I. *The following differences are Schur-geometrically convex in $R_+^2 = (0, \infty) \times (0, \infty)$:*

$$\begin{aligned} D_{SH-SA}(a, b) &= \frac{1}{3}M_{SH}(a, b) - M_{SA}(a, b), \\ D_{AH-SH}(a, b) &= \frac{1}{2}M_{AH}(a, b) - \frac{1}{3}M_{SH}(a, b), \\ D_{SG-AH}(a, b) &= M_{SG}(a, b) - M_{AH}(a, b), \\ D_{AG-SG}(a, b) &= M_{AG}(a, b) - \frac{1}{2}M_{SG}(a, b), \\ D_{N_2N_1-AH}(a, b) &= M_{N_2N_1}(a, b) - \frac{1}{8}M_{AH}(a, b), \\ D_{N_2G-N_2N_1}(a, b) &= \frac{1}{3}M_{N_2G}(a, b) - M_{N_2N_1}(a, b), \\ D_{AG-N_2G}(a, b) &= \frac{1}{4}M_{AG}(a, b) - \frac{1}{3}M_{N_2G}(a, b), \\ D_{AN_2-AG}(a, b) &= M_{AN_2}(a, b) - \frac{1}{4}M_{AG}(a, b), \\ D_{SN_2-SA}(a, b) &= \frac{4}{5}M_{SN_2}(a, b) - M_{SA}(a, b), \\ D_{AN_2-SN_2}(a, b) &= 4M_{AN_2}(a, b) - \frac{4}{5}M_{SN_2}(a, b), \\ D_{SN_1-SH}(a, b) &= 2M_{SN_1}(a, b) - M_{SH}(a, b), \\ D_{SG-SN_1}(a, b) &= \frac{3}{2}M_{SG}(a, b) - 2M_{SN_1}(a, b), \end{aligned} \quad (1.10)$$

$$D_{SN_3-SA}(a, b) = \frac{3}{4}M_{SN_3}(a, b) - M_{SA}(a, b), \quad (1.11)$$

$$D_{SN_1-SN_3}(a, b) = \frac{2}{3}M_{SN_1}(a, b) - \frac{3}{4}M_{SN_3}(a, b).$$

The proof of this theorem will be given in Section 3. Applying this result, in Section 4, we prove some inequalities related to the considered differences of means. Obtained inequalities are refinements of inequalities (1.5)–(1.9).

2. Definitions and Auxiliary Lemmas

The Schur-convex function was introduced by Schur in 1923, and it has many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information-theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields (cf. [4–14]).

In 2003, Zhang first proposed concepts of “Schur-geometrically convex function” which is extension of “Schur-convex function” and established corresponding decision theorem [15]. Since then, Schur-geometric convexity has evoked the interest of many researchers and numerous applications and extensions have appeared in the literature (cf. [16–19]).

In order to prove the main result of this paper we need the following definitions and auxiliary lemmas.

Definition 2.1 (see [4, 20]). Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

- (i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} < \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.
- (ii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for every $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.
- (iii) Let $\Omega \subseteq \mathbb{R}^n$. The function $\varphi: \Omega \rightarrow \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} < \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2.2 (see [15]). Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}_+^n$.

- (i) $\Omega \subseteq \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \dots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$.
- (ii) Let $\Omega \subseteq \mathbb{R}_+^n$. The function $\varphi: \Omega \rightarrow \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \dots, \ln x_n) < (\ln y_1, \dots, \ln y_n)$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

Definition 2.3 (see [4, 20]). (i) The set $\Omega \subseteq \mathbb{R}^n$ is called symmetric set, if $\mathbf{x} \in \Omega$ implies $P\mathbf{x} \in \Omega$ for every $n \times n$ permutation matrix P .

(ii) The function $\varphi : \Omega \rightarrow \mathbb{R}$ is called symmetric if, for every permutation matrix P , $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 2.4 (see [15]). Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω^0 . Let $\varphi : \Omega \rightarrow \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0 . If φ is symmetric on Ω and

$$(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0) \quad (2.1)$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 2.5. For $(a, b) \in R_+^2 = (0, \infty) \times (0, \infty)$ one has

$$1 \geq \frac{a+b}{\sqrt{2(a^2+b^2)}} \geq \frac{1}{2} + \frac{2ab}{(a+b)^2}, \quad (2.2)$$

$$\frac{a+b}{\sqrt{2(a^2+b^2)}} - \frac{ab}{(a+b)^2} \leq \frac{3}{4}, \quad (2.3)$$

$$\frac{3}{2} \geq \frac{\sqrt{a+b}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} \geq \frac{5}{4} + \frac{ab}{(a+b)^2}. \quad (2.4)$$

Proof. It is easy to see that the left-hand inequality in (2.2) is equivalent to $(a-b)^2 \geq 0$, and the right-hand inequality in (2.2) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \leq \frac{(a+b)^2 - 4ab}{2(a+b)^2}, \quad (2.5)$$

that is,

$$\frac{(a-b)^2}{2(a^2+b^2) + \sqrt{2(a^2+b^2)}(a+b)} \leq \frac{(a-b)^2}{2(a+b)^2}. \quad (2.6)$$

Indeed, from the left-hand inequality in (2.2) we have

$$2(a^2+b^2) + \sqrt{2(a^2+b^2)}(a+b) \geq 2(a^2+b^2) + (a+b)^2 \geq 2(a+b)^2, \quad (2.7)$$

so the right-hand inequality in (2.2) holds.

The inequality in (2.3) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \geq \frac{(a-b)^2}{4(a+b)^2}. \quad (2.8)$$

Since

$$\begin{aligned}\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} &= \frac{2(a^2+b^2) - (a+b)^2}{\sqrt{2(a^2+b^2)}(\sqrt{2(a^2+b^2)} + (a+b))} \\ &= \frac{(a-b)^2}{2(a^2+b^2) + (a+b)\sqrt{2(a^2+b^2)}},\end{aligned}\quad (2.9)$$

so it is sufficient prove that

$$2(a^2+b^2) + (a+b)\sqrt{2(a^2+b^2)} \leq 4(a+b)^2, \quad (2.10)$$

that is,

$$(a+b)\sqrt{2(a^2+b^2)} \leq 2(a^2+b^2+4ab), \quad (2.11)$$

and, from the left-hand inequalities in (2.2), we have

$$(a+b)\sqrt{2(a^2+b^2)} \leq 2(a^2+b^2) \leq 2(a^2+b^2+4ab), \quad (2.12)$$

so the inequality in (2.3) holds.

Notice that the functions in the inequalities (2.4) are homogeneous. So, without loss of generality, we may assume $\sqrt{a} + \sqrt{b} = 1$, and set $t = \sqrt{ab}$. Then $0 < t \leq 1/4$ and (2.4) reduces to

$$\frac{3}{2} \geq \frac{\sqrt{1-2t}}{\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1-2t}} \geq \frac{5}{4} + \frac{t^2}{(1-2t)^2}. \quad (2.13)$$

Squaring every side in the above inequalities yields

$$\frac{9}{4} \geq \frac{1-2t}{2} + \frac{1}{2-4t} + 1 \geq \frac{25}{16} + \frac{t^4}{(1-2t)^4} + \frac{5t^2}{2(1-2t)^2}. \quad (2.14)$$

Reducing to common denominator and rearranging, the right-hand inequality in (2.14) reduces to

$$\frac{(1-2t)(16t^2(2t-1)^2 + (1/8)(16t-7)^2 + (7/8))}{16(2t-1)^4} \geq 0, \quad (2.15)$$

and the left-hand inequality in (2.14) reduces to

$$\frac{2(1-2t)^2 + 2 - 5(1-2t)}{2(1-2t)} = -\frac{1+2t}{2} \leq 0, \quad (2.16)$$

so two inequalities in (2.4) hold. $\square$

Lemma 2.6 (see [16]). *Let $a \leq b$, $u(t) = ta + (1-t)b$, $v(t) = tb + (1-t)a$. If $1/2 \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1/2$, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) < (u(t_2), v(t_2)) < (u(t_1), v(t_1)) < (a, b). \quad (2.17)$$

3. Proof of Main Result

Proof of Theorem I. Let $(a, b) \in R_+^2$.

(1) For

$$D_{SH-SA}(a, b) = \frac{1}{3}M_{SH}(a, b) - M_{SA}(a, b) = \frac{a+b}{2} - \frac{2ab}{3(a+b)} - \frac{2}{3}\sqrt{\frac{a^2+b^2}{2}}, \quad (3.1)$$

we have

$$\begin{aligned} \frac{\partial D_{SH-SA}(a, b)}{\partial a} &= \frac{1}{2} - \frac{2b^2}{3(a+b)^2} - \frac{2}{3} \frac{a}{\sqrt{2(a^2+b^2)}}, \\ \frac{\partial D_{SH-SA}(a, b)}{\partial b} &= \frac{1}{2} - \frac{2a^2}{3(a+b)^2} - \frac{2}{3} \frac{b}{\sqrt{2(a^2+b^2)}}, \end{aligned} \quad (3.2)$$

whence

$$\begin{aligned} \Lambda &:= (\ln a - \ln b) \left(a \frac{\partial D_{SH-SA}(a, b)}{\partial a} - b \frac{\partial D_{SH-SA}(a, b)}{\partial b} \right) \\ &= (a-b)(\ln a - \ln b) \left(\frac{1}{2} + \frac{2ab}{3(a+b)^2} - \frac{2}{3} \frac{a+b}{\sqrt{2(a^2+b^2)}} \right). \end{aligned} \quad (3.3)$$

From (2.3) we have

$$\frac{1}{2} + \frac{2ab}{3(a+b)^2} - \frac{2}{3} \frac{a+b}{\sqrt{2(a^2+b^2)}} \geq 0, \quad (3.4)$$

which implies $\Lambda \geq 0$ and, by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2 .

(2) For

$$D_{AH-SH}(a, b) = \frac{1}{2}M_{AH}(a, b) - \frac{1}{3}M_{SH}(a, b) = \frac{a+b}{4} - \frac{ab}{3(a+b)} - \frac{1}{3}\sqrt{\frac{a^2+b^2}{2}}. \quad (3.5)$$

To prove that the function D_{AH-SH} is Schur-geometrically convex in R_+^2 it is enough to notice that $D_{AH-SH}(a, b) = (1/2)D_{SH-SA}(a, b)$.

(3) For

$$D_{SG-AH}(a, b) = M_{SG}(a, b) - M_{AH}(a, b) = \sqrt{\frac{a^2 + b^2}{2}} - \sqrt{ab} - \frac{a+b}{2} + \frac{2ab}{a+b}, \quad (3.6)$$

we have

$$\begin{aligned} \frac{\partial D_{SG-AH}(a, b)}{\partial a} &= \frac{a}{\sqrt{2(a^2 + b^2)}} - \frac{b}{2\sqrt{ab}} - \frac{1}{2} + \frac{2b^2}{(a+b)^2}, \\ \frac{\partial D_{SG-AH}(a, b)}{\partial b} &= \frac{b}{\sqrt{2(a^2 + b^2)}} - \frac{a}{2\sqrt{ab}} - \frac{1}{2} + \frac{2a^2}{(a+b)^2}, \end{aligned} \quad (3.7)$$

and then

$$\begin{aligned} \Lambda &:= (\ln a - \ln b) \left(a \frac{\partial D_{SH-SA}(a, b)}{\partial a} - b \frac{\partial D_{SH-SA}(a, b)}{\partial b} \right) \\ &= (a - b)(\ln a - \ln b) \left(\frac{a+b}{\sqrt{2(a^2 + b^2)}} - \frac{1}{2} - \frac{2ab}{(a+b)^2} \right). \end{aligned} \quad (3.8)$$

From (2.2) we have $\Lambda \geq 0$, so by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2 .

(4) For

$$D_{AG-SG}(a, b) = M_{AG}(a, b) - \frac{1}{2}M_{SG}(a, b) = \frac{1}{2} \left(a + b - \sqrt{ab} - \sqrt{\frac{a^2 + b^2}{2}} \right), \quad (3.9)$$

we have

$$\begin{aligned} \frac{\partial D_{AG-SG}(a, b)}{\partial a} &= \frac{1}{2} \left(1 - \frac{b}{2\sqrt{ab}} - \frac{a}{\sqrt{2(a^2 + b^2)}} \right), \\ \frac{\partial D_{AG-SG}(a, b)}{\partial b} &= \frac{1}{2} \left(1 - \frac{a}{2\sqrt{ab}} - \frac{b}{\sqrt{2(a^2 + b^2)}} \right), \end{aligned} \quad (3.10)$$

and then

$$\begin{aligned} \Lambda &:= (\ln a - \ln b) \left(a \frac{\partial D_{SH-SA}(a, b)}{\partial a} - b \frac{\partial D_{SH-SA}(a, b)}{\partial b} \right) \\ &= (a - b)(\ln a - \ln b) \left(1 - \frac{a+b}{\sqrt{2(a^2 + b^2)}} \right). \end{aligned} \quad (3.11)$$

By (2.2) we infer that

$$1 - \frac{a+b}{\sqrt{2(a^2+b^2)}} \geq 0, \quad (3.12)$$

so $\Lambda \geq 0$. By Lemma 2.4, we get that D_{AG-SG} is Schur-geometrically convex in R_+^2 .

(5) For

$$\begin{aligned} D_{N_2N_1-AH}(a,b) &= M_{N_2N_1}(a,b) - \frac{1}{8}M_{AH}(a,b) \\ &= \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right) - \frac{1}{4}(a+b) - \frac{1}{2}\sqrt{ab} - \frac{1}{8}\left(\frac{a+b}{2} - \frac{2ab}{a+b}\right), \end{aligned} \quad (3.13)$$

we have

$$\begin{aligned} \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a} &= \frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}} + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} \\ &\quad - \frac{1}{4} - \frac{b}{4\sqrt{ab}} - \frac{1}{8}\left(\frac{1}{2} - \frac{2b^2}{(a+b)^2}\right), \\ \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b} &= \frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}} + \frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} \\ &\quad - \frac{1}{4} - \frac{a}{4\sqrt{ab}} - \frac{1}{8}\left(\frac{1}{2} - \frac{2a^2}{(a+b)^2}\right), \end{aligned} \quad (3.14)$$

and then

$$\begin{aligned} \Lambda &= (\ln a - \ln b)\left(a\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a} - b\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}\right) \\ &= \frac{1}{4}(a-b)(\ln a - \ln b)\left(\frac{\sqrt{a+b}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} - \frac{5}{4} - \frac{ab}{(a+b)^2}\right). \end{aligned} \quad (3.15)$$

From (2.4) we have

$$\frac{\sqrt{a+b}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} - \frac{5}{4} - \frac{ab}{(a+b)^2} \geq 0, \quad (3.16)$$

so $\Lambda \geq 0$; it follows that $D_{N_2N_1-AH}$ is Schur-geometrically convex in R_+^2 .

(6) For

$$\begin{aligned} D_{N_2G-N_2N_1}(a, b) &= \frac{1}{3}M_{N_2G}(a, b) - M_{N_2N_1}(a, b) \\ &= \frac{a+b}{4} + \frac{\sqrt{ab}}{6} - \frac{2}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \end{aligned} \quad (3.17)$$

we have

$$\begin{aligned} \frac{\partial D_{N_2G-N_2N_1}(a, b)}{\partial a} &= \frac{1}{4} + \frac{b}{12\sqrt{ab}} - \frac{1}{6\sqrt{a}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}, \\ \frac{\partial D_{N_2G-N_2N_1}(a, b)}{\partial b} &= \frac{1}{4} + \frac{a}{12\sqrt{ab}} - \frac{1}{6\sqrt{b}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}, \end{aligned} \quad (3.18)$$

and then

$$\begin{aligned} \Lambda &= (\ln a - \ln b) \left(a \frac{\partial D_{N_2G-N_2N_1}(a, b)}{\partial a} - b \frac{\partial D_{N_2G-N_2N_1}(a, b)}{\partial b} \right) \\ &= (\ln a - \ln b) \left(\frac{1}{4}(a-b) - \frac{\sqrt{a}-\sqrt{b}}{6}\sqrt{\frac{a+b}{2}} - \frac{(a-b)(\sqrt{a}+\sqrt{b})}{12}\left(\frac{a+b}{2}\right)^{-1/2} \right) \\ &= \frac{1}{6}(a-b)(\ln a - \ln b) \left(\frac{3}{2} - \frac{\sqrt{a+b}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} \right). \end{aligned} \quad (3.19)$$

By (2.4) we infer that $\Lambda \geq 0$, which proves that $D_{N_2G-N_2N_1}$ is Schur-geometrically convex in R_+^2 .

(7) For

$$\begin{aligned} D_{AG-N_2G}(a, b) &= \frac{1}{4}M_{AG}(a, b) - \frac{1}{3}M_{N_2G}(a, b) \\ &= \frac{a+b}{8} + \frac{1}{12}\sqrt{ab} - \frac{1}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \end{aligned} \quad (3.20)$$

we have

$$\begin{aligned} \frac{\partial D_{AG-N_2G}(a, b)}{\partial a} &= \frac{1}{8} + \frac{b}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2a}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}}, \\ \frac{\partial D_{AG-N_2G}(a, b)}{\partial b} &= \frac{1}{8} + \frac{a}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2b}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}}, \end{aligned} \quad (3.21)$$

and then

$$\begin{aligned}
 \Lambda &= (\ln a - \ln b) \left(a \frac{\partial D_{AG-N_2G}(a, b)}{\partial a} - b \frac{\partial D_{AG-N_2G}(a, b)}{\partial b} \right) \\
 &= (\ln a - \ln b) \left(\frac{a-b}{8} - \frac{\sqrt{a+b}(\sqrt{a}-\sqrt{b})}{12\sqrt{2}} - \frac{(a-b)(\sqrt{a}+\sqrt{b})}{12\sqrt{2}(a+b)} \right) \\
 &= \frac{(a-b)(\ln a - \ln b)}{8} \left(1 - \frac{2}{3} \left(\frac{\sqrt{a+b}}{\sqrt{2}(\sqrt{a}+\sqrt{b})} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} \right) \right).
 \end{aligned} \tag{3.22}$$

From (2.4) we have $\Lambda \geq 0$, and, consequently, by Lemma 2.4, we obtain that D_{AG-N_2G} is Schur-geometrically convex in R_+^2 .

(8) In order to prove that the function $D_{AN_2-AG}(a, b)$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{AN_2-AG}(a, b) = M_{AN_2}(a, b) - \frac{1}{4}M_{AG}(a, b) = 3D_{AG-N_2G}(a, b). \tag{3.23}$$

(9) For

$$\begin{aligned}
 D_{SN_2-SA}(a, b) &= \frac{4}{5}M_{SN_2}(a, b) - M_{SA}(a, b) \\
 &= \frac{a+b}{2} - \frac{1}{5}\sqrt{\frac{a^2+b^2}{2}} - \frac{1}{5}(\sqrt{a}+\sqrt{b})\sqrt{2(a+b)},
 \end{aligned} \tag{3.24}$$

we have

$$\begin{aligned}
 \frac{\partial D_{SN_2-SA}(a, b)}{\partial a} &= \frac{1}{2} - \frac{a}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2a}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}}, \\
 \frac{\partial D_{SN_2-SA}(a, b)}{\partial b} &= \frac{1}{2} - \frac{b}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2b}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}},
 \end{aligned} \tag{3.25}$$

and then

$$\begin{aligned}
 \Lambda &= (\ln a - \ln b) \left(\frac{\partial D_{SN_2-SA}(a, b)}{\partial a} - \frac{\partial D_{SN_2-SA}(a, b)}{\partial b} \right) \\
 &= (\ln a - \ln b) \left(\frac{a-b}{2} - \frac{a^2-b^2}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5} \left(\sqrt{\frac{a(a+b)}{2}} - \sqrt{\frac{b(a+b)}{2}} \right) - \frac{(\sqrt{a} + \sqrt{b})(a-b)}{5\sqrt{2(a+b)}} \right) \\
 &= \frac{(a-b)(\ln a - \ln b)}{5\sqrt{2}} \left(\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a} + \sqrt{b}} - \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a+b}} \right).
 \end{aligned} \tag{3.26}$$

From (2.2) and (2.4) we obtain that

$$\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a} + \sqrt{b}} - \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a+b}} \geq \frac{5}{\sqrt{2}} - \sqrt{2} - \frac{3}{\sqrt{2}} = 0, \tag{3.27}$$

so $\Lambda \geq 0$, which proves that the function $D_{SN_2-SA}(a, b)$ is Schur-geometrically convex in R_+^2 .

(10) One can easily check that

$$D_{AN_A N_2-SN_2}(a, b) = 4D_{SN_2-SA}(a, b), \tag{3.28}$$

and, consequently, the function $D_{AN_2-SN_2}$ is Schur-geometrically convex in R_+^2 .

(11) To prove that the function

$$D_{SN_1-SH}(a, b) = 2M_{SN_1}(a, b) - M_{SH}(a, b) = \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} - \sqrt{ab} + \frac{2ab}{a+b} \tag{3.29}$$

is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SH}(a, b) = D_{SG-AH}(a, b). \tag{3.30}$$

(12) For

$$\begin{aligned}
 D_{SG-SN_1}(a, b) &= \frac{3}{2}M_{SG}(a, b) - 2M_{SN_1}(a, b) \\
 &= \frac{1}{2} \left(a+b - \sqrt{ab} - \sqrt{\frac{a^2+b^2}{2}} \right),
 \end{aligned} \tag{3.31}$$

we have

$$\begin{aligned}\frac{\partial D_{SG-SN_1}(a, b)}{\partial a} &= \frac{1}{2} \left(1 - \frac{b}{2\sqrt{ab}} - \frac{a}{\sqrt{2(a^2 + b^2)}} \right), \\ \frac{\partial D_{SG-SN_1}(a, b)}{\partial b} &= \frac{1}{2} \left(1 - \frac{a}{2\sqrt{ab}} - \frac{b}{\sqrt{2(a^2 + b^2)}} \right),\end{aligned}\tag{3.32}$$

and then

$$\begin{aligned}\Lambda &= (\ln a - \ln b) \left(a \frac{\partial D_{SG-SN_1}(a, b)}{\partial a} - b \frac{\partial D_{SG-SN_1}(a, b)}{\partial b} \right) \\ &= \frac{(a - b)(\ln a - \ln b)}{2} \left(1 - \frac{a + b}{\sqrt{2(a^2 + b^2)}} \right).\end{aligned}\tag{3.33}$$

By the inequality (2.2) we get that $\Lambda \geq 0$, which proves that D_{SG-SN_1} is Schur-geometrically convex in R_+^2 .

(13) It is easy to check that

$$D_{SN_3-SA}(a, b) = \frac{1}{2} D_{AG-SG}(a, b),\tag{3.34}$$

which means that the function D_{SN_3-SA} is Schur-geometrically convex in R_+^2 .

(14) To prove that the function $D_{SN_1-SN_3}$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SN_3}(a, b) = \frac{1}{6} D_{AG-SG}(a, b).\tag{3.35}$$

The proof of Theorem I is complete. $\square$

4. Applications

Applying Theorem I, Lemma 2.6, and Definition 2.2 one can easily prove the following.

Theorem II. Let $0 < a \leq b$. $1/2 \leq t \leq 1$ or $0 \leq t \leq 1/2$, $u = a^t b^{1-t}$ and $v = b^t a^{1-t}$. Then

$$\begin{aligned}
 M_{SA}(a, b) &\leq \frac{1}{3}M_{SH}(a, b) - \left(\frac{1}{3}M_{SH}(u, v) - M_{SA}(u, v) \right) \leq \frac{1}{3}M_{SH}(a, b) \\
 &\leq \frac{1}{2}M_{AH}(a, b) - \left(\frac{1}{2}M_{AH}(u, v) - \frac{1}{3}M_{SH}(u, v) \right) \leq \frac{1}{2}M_{AH}(a, b) \\
 &\leq \frac{1}{2}M_{SG}(a, b) - \left(\frac{1}{2}M_{SG}(u, v) - \frac{1}{2}M_{AH}(u, v) \right) \leq \frac{1}{2}M_{SG}(a, b) \\
 &\leq M_{AG}(a, b) - \left(M_{AG}(u, v) - \frac{1}{2}M_{SG}(u, v) \right) \leq M_{AG}(a, b),
 \end{aligned} \tag{4.1}$$

$$\begin{aligned}
 \frac{1}{8}M_{AH}(a, b) &\leq M_{N_2N_1}(a, b) - \left(M_{N_2N_1}(u, v) - \frac{1}{8}M_{AH}(u, v) \right) \leq M_{N_2N_1}(a, b) \\
 &\leq \frac{1}{3}M_{N_2G}(a, b) - \left(\frac{1}{3}M_{N_2G}(u, v) - M_{N_2N_1}(u, v) \right) \leq \frac{1}{3}M_{N_2G}(a, b) \\
 &\leq \frac{1}{4}M_{AG}(a, b) - \left(\frac{1}{4}M_{AG}(u, v) - \frac{1}{3}M_{N_2G}(u, v) \right) \leq \frac{1}{4}M_{AG}(a, b) \\
 &\leq M_{AN_2}(a, b) - \left(M_{AN_2}(u, v) - \frac{1}{4}M_{AG}(u, v) \right) \leq M_{AN_2}(a, b),
 \end{aligned} \tag{4.2}$$

$$\begin{aligned}
 M_{SA}(a, b) &\leq \frac{4}{5}M_{SN_2}(a, b) - \left(\frac{4}{5}M_{SN_2}(u, v) - \frac{4}{5}M_{SN_2}(u, v) \right) \leq \frac{4}{5}M_{SN_2}(a, b) \\
 &\leq 4M_{AN_2}(a, b) - \left(4M_{AN_2}(u, v) - \frac{4}{5}M_{SN_2}(u, v) \right) \leq 4M_{AN_2}(a, b),
 \end{aligned} \tag{4.3}$$

$$\begin{aligned}
 M_{SH}(a, b) &\leq 2M_{SN_1}(a, b) - (2M_{SN_1}(u, v) - M_{SH}(u, v)) \leq 2M_{SN_1}(a, b) \\
 &\leq \frac{3}{2}M_{SG}(a, b) - \left(\frac{3}{2}M_{SG}(u, v) - \frac{3}{2}M_{SG}(u, v) \right) \leq \frac{3}{2}M_{SG}(a, b),
 \end{aligned} \tag{4.4}$$

$$\begin{aligned}
 M_{SA}(a, b) &\leq \frac{3}{4}M_{SN_3}(a, b) - \left(\frac{3}{4}M_{SN_3}(u, v) - M_{SA}(u, v) \right) \leq \frac{3}{4}M_{SN_3}(a, b) \\
 &\leq \frac{2}{3}M_{SN_1}(a, b) - \left(\frac{2}{3}M_{SN_1}(u, v) - \frac{3}{4}M_{SN_3}(u, v) \right) \leq \frac{2}{3}M_{SN_1}(a, b).
 \end{aligned} \tag{4.5}$$

Remark 4.1. Equation (4.1), (4.2), (4.3), (4.4), and (4.5) are a refinement of (1.5), (1.6), (1.7), (1.8), and (1.9), respectively.

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