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*Research Article*

# A Nice Separation of Some Seiffert-Type Means by Power Means

**Iulia Costin[1] and Gheorghe Toader[2]**

[1] *Department of Computer Sciences, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania*
[2] *Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania*

Correspondence should be addressed to Gheorghe Toader, gheorghe.toader@math.utcluj.ro

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Seiffert has defined two well-known trigonometric means denoted by $\mathcal{P}$ and $\mathcal{T}$. In a similar way it was defined by Carlson the logarithmic mean $\mathcal{L}$ as a hyperbolic mean. Neuman and Sándor completed the list of such means by another hyperbolic mean $\mathcal{M}$. There are more known inequalities between the means $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{L}$ and some power means $\mathcal{A}_p$. We add to these inequalities two new results obtaining the following nice chain of inequalities $\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/2} < \mathcal{P} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} < \mathcal{A}_2$, where the power means are evenly spaced with respect to their order.

## 1. Means

A *mean* is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad \forall a,b > 0. \tag{1.1}$$

Each mean is *reflexive*; that is,

$$M(a,a) = a, \quad \forall a > 0. \tag{1.2}$$

This is also used as the definition of $M(a,a)$.

We will refer here to the following means:

(i) the power means $\mathcal{A}_p$, defined by

$$\mathcal{A}_p(a,b) = \left[\frac{a^p + b^p}{2}\right]^{1/p}, \quad p \neq 0; \tag{1.3}$$

(ii) the geometric mean $\mathcal{G}$, defined as $\mathcal{G}(a,b) = \sqrt{ab}$, but verifying also the property

$$\lim_{p\to 0}\mathcal{A}_p(a,b) = \mathcal{A}_0(a,b) = \mathcal{G}(a,b); \tag{1.4}$$

(iii) the first Seiffert mean $\mathcal{P}$, defined in [1] by

$$\mathcal{P}(a,b) = \frac{a-b}{2\sin^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.5}$$

(iv) the second Seiffert mean $\mathcal{T}$, defined in [2] by

$$\mathcal{T}(a,b) = \frac{a-b}{2\tan^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.6}$$

(v) the Neuman-Sándor mean $\mathcal{M}$, defined in [3] by

$$\mathcal{M}(a,b) = \frac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, \quad a \neq b; \tag{1.7}$$

(vi) the Stolarsky means $\mathcal{S}_{p,q}$ defined in [4] as follows:

$$\mathcal{S}_{p,q}(a,b) = \begin{cases} \left[\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right]^{1/(p-q)}, & pq(p-q) \neq 0 \\ \dfrac{1}{e^p}\left(\dfrac{a^{a^p}}{b^{b^p}}\right)^{1/(a^p-b^p)}, & p = q \neq 0 \\ \left[\dfrac{a^p-b^p}{p(\ln a - \ln b)}\right]^{1/p}, & p \neq 0,\ q = 0 \\ \sqrt{ab}, & p = q = 0. \end{cases} \tag{1.8}$$

The mean $\mathcal{A}_1 = \mathcal{A}$ is the arithmetic mean and the mean $\mathcal{S}_{1,0} = \mathcal{L}$ is the logarithmic mean. As Carlson remarked in [5], the logarithmic mean can be represented also by

$$\mathcal{L}(a,b) = \frac{a-b}{2\tanh^{-1}((a-b)/(a+b))}; \tag{1.9}$$

thus the means $\mathcal{P}, \mathcal{T}, \mathcal{M}$, and $\mathcal{L}$ are very similar. In [3] it is also proven that these means can be defined using the nonsymmetric Schwab-Borchardt mean $\mathcal{SB}$ given by

$$\mathcal{SB}(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & \text{if } a < b \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & \text{if } a > b \end{cases} \tag{1.10}$$

(see [6, 7]). It has been established in [3] that

$$\mathcal{L} = \mathcal{SB}(\mathcal{A},\mathcal{G}), \qquad \mathcal{P} = \mathcal{SB}(\mathcal{G},\mathcal{A}), \qquad \mathcal{T} = \mathcal{SB}(\mathcal{A},\mathcal{A}_2), \qquad \mathcal{M} = \mathcal{SB}(\mathcal{A}_2,\mathcal{A}). \tag{1.11}$$

## 2. Interlacing Property of Power Means

Given two means $M$ and $N$, we will write $M < N$ if

$$M(a,b) < N(a,b), \quad \text{for } a \neq b. \tag{2.1}$$

It is known that the family of power means is an increasing family of means, thus

$$\mathcal{A}_p < \mathcal{A}_q, \quad \text{if } p < q. \tag{2.2}$$

Of course, it is more difficult to compare two Stolarsky means, each depending on two parameters. To present the comparison theorem given in [8, 9], we have to give the definitions of the following two auxiliary functions:

$$\begin{gathered} k(x,y) = \begin{cases} \dfrac{|x|-|y|}{x-y}, & x \neq y \\ \operatorname{sign}(x), & x = y, \end{cases} \\ l(x,y) = \begin{cases} \mathcal{L}(x,y), & x > 0,\ y > 0 \\ 0, & x \geq 0,\ y \geq 0,\ xy = 0. \end{cases} \end{gathered} \tag{2.3}$$

**Theorem 2.1.** *Let $p,q,r,s \in \mathbb{R}$. Then the comparison inequality*

$$\mathcal{S}_{p,q} \leq \mathcal{S}_{r,s} \tag{2.4}$$

*holds true if and only if $p+q \leq r+s$, and (1) $l(p,q) \leq l(r,s)$ if $0 \leq \min(p,q,r,s)$, (2) $k(p,q) \leq k(r,s)$ if $\min(p,q,r,s) < 0 < \max(p,q,r,s)$, or (3) $-l(-p,-q) \leq -l(-r,-s)$ if $\max(p,q,r,s) \leq 0$.*

We need also in what follows an important double-sided inequality proved in [3] for the Schwab-Borchardt mean:

$$\sqrt[3]{ab^2} < \mathcal{SB}(a,b) < \frac{a+2b}{3}, \quad a \neq b. \tag{2.5}$$

Being rather complicated, the Seiffert-type means were evaluated by simpler means, first of all by power means. The *evaluation* of a given mean $M$ by power means assumes the determination of some real indices $p$ and $q$ such that $\mathcal{A}_p < M < \mathcal{A}_q$. The evaluation is *optimal* if $p$ is the the greatest and $q$ is the smallest index with this property. This means that $M$ cannot be compared with $\mathcal{A}_r$ if $p < r < q$.

For the logarithmic mean in [10], it was determined the optimal evaluation

$$\mathcal{A}_0 < L < \mathcal{A}_{1/3}. \tag{2.6}$$

For the Seiffert means, there are known the evaluations

$$\mathcal{A}_{1/3} < P < \mathcal{A}_{2/3}, \tag{2.7}$$

proved in [11] and

$$\mathcal{A}_1 < T < \mathcal{A}_2, \tag{2.8}$$

given in [2]. It is also known that

$$\mathcal{A}_1 < M < \mathcal{T}, \tag{2.9}$$

as it was shown in [3]. Moreover in [12] it was determined the optimal evaluation

$$\mathcal{A}_{\ln 2/\ln \pi} < P < \mathcal{A}_{2/3}. \tag{2.10}$$

Using these results we deduce the following chain of inequalities:

$$\mathcal{A}_0 < L < \mathcal{A}_{1/2} < P < \mathcal{A}_1 < M < \mathcal{T} < \mathcal{A}_2. \tag{2.11}$$

To prove the full interlacing property of power means, our aim is to show that $\mathcal{A}_{3/2}$ can be put between $\mathcal{M}$ and $\mathcal{T}$. We thus obtain a nice separation of these Seiffert-type means by power means which are evenly spaced with respect to their order.

## 3. Main Results

We add to the inequalities (2.11) the next results.

**Theorem 3.1.** *The following inequalities*

$$\mathcal{M} < \mathcal{A}_{3/2} < T \tag{3.1}$$

*are satisfied.*

*Proof.* First of all, let us remark that $\mathcal{A}_{3/2} = \mathcal{S}_{3,3/2}$. So, for the first inequality in (3.1), it is sufficient to prove that the following chain of inequalities

$$\mathcal{M} < \frac{\mathcal{A}_2 + 2\mathcal{A}}{3} < \mathcal{S}_{3,1} < \mathcal{S}_{3,3/2} \tag{3.2}$$

is valid. The first inequality in (3.2) is a simple consequence of the property of the mean $\mathcal{M}$ given in (1.11) and the second inequality from (2.5). The second inequality can be proved by direct computation or by taking $a = 1 + t,\ b = 1 - t,\ (0 < t < 1)$ which gives

$$\frac{\sqrt{1+t^2}+2}{3} < \sqrt{\frac{3+t^2}{3}}, \tag{3.3}$$

which is easy to prove. The last inequality in (3.2) is given by the comparison theorem of the Stolarsky means. In a similar way, the second inequality in (3.1) is given by the relations

$$\mathcal{S}_{3,3/2} < \mathcal{S}_{4,1} = \sqrt[3]{\mathcal{A}\mathcal{A}_2^2} < T. \tag{3.4}$$

The first inequality is again given by the comparison theorem of the Stolarsky means. The equality in (3.4) is shown by elementary computations, and the last inequality is a simple consequence of the property of the mean $\mathcal{T}$ given in (1.11) and the first inequality from (2.5). □

**Corollary 3.2.** *The following two-sided inequality*

$$\frac{x}{\sinh^{-1}x} < \mathcal{A}_{3/2}(1-x, 1+x) < \frac{x}{\tan^{-1}x}, \tag{3.5}$$

*is valid for all* $0 < x < 1$.

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