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Research Article

On Almost Orthogonal Frames

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Almost orthogonal frames have been introduced and studied. It has been proved that a bounded almost orthogonal frame satisfies Feichtinger conjecture. Also, we prove that a bounded almost orthogonal frame contains a Riesz basis.

1. Introduction

Frames were formally introduced in 1952 by Duffin and Schaeffer [1]. In 1985, frames were resurfaced in the book by Young [2]. The theory of frames began to be more widely studied only after the landmark paper of Daubechies et al. [3] in 1986. For an introduction to frames, one may refer to [4–6].

Feichtinger in his work on time frequency analysis noted that all Gabor frames (which he was using for his work) had the property that they could be divided into a finite number of subsets which were Riesz basis sequences. This observation led to the following conjecture, called the Feichtinger conjecture “Every bounded frame can be written as a finite union of Riesz basic sequences.”

Feichtinger conjecture is connected to the famous Kadison-Singer conjecture. It was shown in [7] that Kadison-Singer conjecture implies Feichtinger conjecture. For literature related to Feichtinger conjecture, one may refer to [7, 8].

In the present paper, we introduce and study almost orthogonal frames in Hilbert spaces and prove that a bounded almost orthogonal frame satisfies Feichtinger conjecture. Also, we prove that a bounded almost orthogonal frame contains a Riesz basis.

2. Preliminaries

Throughout the paper, H will denote an infinite-dimensional Hilbert space, $\{n_k\}$ an infinite-increasing sequence in $\mathbb{N}$, $[x_n]$ the closed linear span of $\{x_n\}$, and for any set D, $|D|$ will denote cardinality of D.

Definition 2.1. A sequence $\{x_n\}$ in a Hilbert space H is said to be a frame for H if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad x \in H. \tag{2.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{x_n\}$. The inequality (2.1) is called the *frame inequality* for the frame $\{x_n\}$.

A frame $\{x_n\}$ in H is called *tight* if it is possible to choose A, B satisfying inequality (2.1) with $A = B$ as frame bounds and is called normalized tight if $A = B = 1$. A frame $\{x_n\}$ in H is called *exact* if removal of any x_n renders the collection $\{x_n\}$ no longer a frame for H. A sequence $\{x_n\} \in H$ is called a *Bessel sequence* if it satisfies upper frame inequality in (2.1).

Definition 2.2. A sequence $\{x_n\}$ in H is called a *Riesz basic sequence* if there exist positive constants A and B such that for all finite sequence of scalars $\{\alpha_k\}$, we have

$$A\sum_k |\alpha_k|^2 \leq \left\|\sum_k \alpha_k x_k\right\|^2 \leq B\sum_k |\alpha_k|^2. \tag{2.2}$$

In case, the Riesz basic sequence $\{x_n\}$ is complete in H, it is called a Riesz basis for H.

Definition 2.3. A sequence $\{y_n\}$ in a Hilbert space H is said to be a *block sequence* with respect to a given sequence $\{x_n\}$ in H, if it is of the form

$$y_n = \sum_{i\in D_n} \alpha_i x_i \neq 0, \quad n \in \mathbb{N}, \tag{2.3}$$

where D_n's are finite subsets of $\mathbb{N}$ with $D_n \cap D_m = \emptyset$, $n \neq m$, $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{N}$ and α_i's are any scalars.

It has been observed in [9] that a block sequence with respect to a frame in a Hilbert space may not be a frame for H. Also, a block sequence with respect to a sequence in H which is not even a frame for H may be a frame for H.

3. Main Results

We begin with a sufficient condition for a bounded frame to satisfy the Feichtinger conjecture.

Theorem 3.1. *Let $\{x_n\}$ be a bounded frame for H. If there exists a sequence of finite subsets $\{D_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ with $D_i \cap D_j = \emptyset$, for all $i \neq j$, $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$ and $\sup_n\{|D_n|\} < \infty$ such that $H = \bigoplus_{n\in\mathbb{N}} V_n$, where $V_n = [x_n]_{i\in D_n}$, then $\{x_n\}$ can be decomposed into a finite union of a Riesz basic sequences.*

Proof. Suppose the problem has an affirmative answer. Let $\{D_n\}_{n\in\mathbb{N}}$ be sequence of finite subsets of $\mathbb{N}$ with $D_n \cap D_n = \emptyset$, $n \neq m$ and $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{N}$ such that $H = \bigoplus_{n\in\mathbb{N}} V_n$, where $V_n = [x_i]_{i\in D_n}$ and $\{x_n\}$ is a bounded frame for H. Let $\{G_n\}$ be a sequence of sets given by

$$G_n = \{x_i\}_{i\in D_n}, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Now, for each $j \in \mathbb{N}$, choose a sequence $\{y_i^j\}_{i\in\mathbb{N}}$ such that

$$y_i^j = \begin{cases} j^{\text{th}} \text{ element of } G_i, & \text{if } G_i \text{ contains } j\text{th element}, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{3.2}$$

Then, for each $j \in \mathbb{N}$, $\{y_i^j\}_{i\in\mathbb{N}}$ is a sequence of orthogonal vectors which are norm bounded. So, $\{y_i^j\}_{i\in\mathbb{N}}$ is a Riesz basic sequence for H, for each $j \in \mathbb{N}$. Also, note that

$$\{x_n\} = \bigcup_j \left\{ y_i^j \right\}. \tag{3.3}$$

Since D_n's are finite, j varies on a finite set. Hence $\{x_n\}$ is decomposed into finite number of Riesz basic sequences. □

We will now introduce a concept which is more general than *orthogonal frame* and call it *almost orthogonal frame*. We give the following definition of almost orthogonal frame.

Definition 3.2. A frame $\{x_n\}$ in a Hilbert space H is called an almost orthogonal frame of order K ($K \in \mathbb{N}$) if K is the smallest natural number for which there exists a permutation $\{\sigma_n\}$ of $\mathbb{N}$ such that

$$\langle x_{\sigma_n}, \ x_{\sigma_m} \rangle = 0, \quad \forall\, \sigma_n, \sigma_m \text{ such that } |\sigma_n - \sigma_m| \geq K. \tag{3.4}$$

Note 1. We use $\langle x_n \rangle$ instead of $\langle x_{\sigma_n} \rangle$ for convenience.

Example 3.3. (I) An orthogonal basis is an almost orthogonal frame of order 1.

(II) $\{e_1, e_1, e_2, e_2, \ldots, e_n, e_n, \ldots\}$ is an almost orthogonal frame of order 2.

(III) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \ldots\}$ is not an almost orthogonal frame of any order.

(IV) $\{e_1, e_1 + e_2, e_2 + e_3, e_3, e_3, e_3 + e_4, \ldots\}$ is an almost orthogonal frame of order 3.

(V) $\{e_1, e_2 + e_1/4, e_3 + e_1/8, \ldots\}$ is not an almost orthogonal frame of any order.

(VI) $\{e_i + (1/i)e_{i+1}\}_{i=1}^{\infty}$ is an almost orthogonal frame of order 2.

(VII) $\{e_1, (1/2)e_2, (1-(1/2^2))^{1/2}e_2, (1/3)e_3, (1-(1/3^2))^{1/2}e_3, \ldots\} = \{(1/n)e_n\} \cup \{(1-(1/2^2))^{1/2}e_n\}$ is a tight frame with $A = B = 1$, which is almost orthogonal of order 2 and is not bounded below.

Observations

(I) A bounded frame may or may not be an almost orthogonal frame. (See Example I and Example V.)

(II) An almost orthogonal frame of some finite ($\neq 1$) order may or may not be a Riesz basis. (See Example II and Example V.)

(III) A Riesz basis may or may not be an almost orthogonal frame. (See Example I and Example V.)

Theorem 3.4. *A bounded almost orthogonal frame satisfies Feichtinger conjecture.*

Proof. Let $\{x_n\}$ be a bounded almost orthogonal frame of order K. Define a sequence $\{G_n\}$ of subspaces as follows:

$$\begin{aligned} G_1 &= [x_1, x_2, \dots, x_K], \\ G_2 &= [x_{K+1}, \dots, x_{2K}], \\ &\vdots \\ G_n &= \left[x_{(n-1)K+1}, x_{(n-1)K+2}, \dots, x_{nK}\right], \quad n \in \mathbb{N}. \end{aligned} \tag{3.5}$$

Now, since $\{x_n\}$ is an almost orthogonal frame of degree K. This gives

$$\langle x_n, x_m \rangle = 0 \quad \forall n, m \in \mathbb{N} \text{ such that } |n-m| \geq K. \tag{3.6}$$

Let $x \in G_n$ and $y \in G_{n+2}$, for any $n \in \mathbb{N}$. Then

$$x = \sum_{(n-1)K+1}^{nK} \alpha_i x_i, \qquad y = \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j. \tag{3.7}$$

Therefore, we have

$$\begin{aligned} \langle x, y \rangle &= \left\langle \sum_{(n-1)K+1}^{nK} \alpha_i x_i, \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j \right\rangle = \sum_{(n-1)K+1}^{nK} \alpha_i \left\langle x_i, \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j \right\rangle \\ &= \sum_{(n-1)K+1}^{nK} \alpha_i \left(\sum_{(n+1)K+1}^{(n+2)K} \overline{\beta}_j \langle x_i, x_j \rangle \right) = 0. \\ &\Rightarrow G_n \cap G_{n+2} = \phi, \quad \forall n \in \mathbb{N}, \\ &\Rightarrow \text{span}\,\overline{\{G_n, G_{n+2}\}} = G_n \oplus G_{n+2} \quad \forall n, \\ &\Rightarrow \overline{\text{span}\{G_1, G_3, G_5, \dots\}} = G_1 \oplus G_3 \oplus G_5 \oplus \cdots = \bigoplus_{n\in\mathbb{N}} G_{2n-1} = H_1. \end{aligned} \tag{3.8}$$

Also, we have

$$\overline{\text{span}\{G_2, G_4, G_6, \dots\}} = \bigoplus_{n\in\mathbb{N}} G_{2n} = H_2. \tag{3.9}$$

So, by Theorem 3.1

$$\{x_1, x_2, \ldots, x_K, x_{2K+1}, x_{2K+2}, \ldots, x_{3K}, \ldots\} \tag{3.10}$$

can be written as finite union of Riesz basic sequences.

Similarly, using Theorem 3.1,

$$\{x_{K+1}, x_{K+2}, \ldots, x_{2K}, x_{3K+1}, \ldots, x_{4K}, \ldots\} \tag{3.11}$$

can be written as finite union of Riesz basic sequences.

Hence, $\{x_n\}$ can be written as finite union of Riesz basic sequences. □

Remark 3.5. Almost orthogonal frames produce fusion frames (nonorthogonal) and fusion frame systems. Indeed, let $\{x_n\}$ be an almost orthogonal frame of order K. Proceeding as in Theorem 3.4, we get a sequence of subspaces $\{G_n\}$ satisfying

$$\begin{aligned} \overline{\operatorname{span}\{G_1, G_3, G_5, \ldots\}} &= G_1 \oplus G_3 \oplus G_5 \oplus \cdots = H_1, \\ \overline{\operatorname{span}\{G_2, G_4, G_6, \ldots\}} &= G_2 \oplus G_4 \oplus G_6 \oplus \cdots = H_2. \end{aligned} \tag{3.12}$$

Now, define a sequence of projections $\{v_i\}$ $(v_i : H \to G_i)$. Then, we can easily verify that $\{v_{2i-1}, G_{2i-1}\}_{i\in\mathbb{N}}$ is a fusion frame for H_1 and $\{v_{2i}, G_{2i}\}_{i\in\mathbb{N}}$ is a fusion frame for H_2. So, $\{v_i, G_i\}_{i\in\mathbb{N}}$ is a fusion frame for H.

Finally, we prove that for any bounded almost orthogonal frame, there exists a block sequence with respect to the almost orthogonal frame such that the block sequence is a Riesz basis. More precisely, we have the following.

Theorem 3.6. *A bounded almost orthogonal frame contains a Riesz basis.*

Proof. Let $\{x_n\}$ be an almost orthogonal frame of order K. Consider $\{x_1, x_2, \ldots, x_K, x_{K+1}, \ldots, x_{2K}, x_{2K+1}, \ldots\}$. Then, following the steps in Theorem 3.4, we get a sequence of subspaces $\{G_n\}$ which are finite dimensional. So, we can extract a Riesz basis for G_n out of $\{x_{(n-1)K+1}, x_{(n-1)K+2}, \ldots, x_{nK}\}$ and let it be $\{x_i^n\}$. Then $\bigcup_{n\in\mathbb{N}}\{x_i^{2n-1}\}$ is a Riesz basis for H_1 and $\bigcup_{n\in\mathbb{N}}\{x_i^{2n}\}$ is a Riesz basis for H_2, where H_1 and H_2 are as in Theorem 3.4. Write $F_n = G_n \cap G_{n+1}$ for all $n \in \mathbb{N}$, then, for each $n \in \mathbb{N}$, F_n is a finite-dimensional subspace of G_n. Let $\{x_i^{n'}\}$ be an extracted Riesz basis for F_n which is extracted from $\{x_{(n-1)K+1}, x_{(n-1)K+2}, \ldots, x_{nK}\}$ or $\{x_{nK+1}, \ldots, x_{(n+1)K}\}$. Then, $\bigcup_n\{x_i^n\} \sim \bigcup_n\{x_i^{n'}\}$ is the desired Riesz basis for H. □

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