Research Article

Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

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The object of this paper is to study invariant submanifolds M of Sasakian manifolds $\tilde{M}$ admitting a semisymmetric nonmetric connection, and it is shown that M admits semisymmetric nonmetric connection. Further it is proved that the second fundamental forms σ and $\bar{\sigma}$ with respect to Levi-Civita connection and semi-symmetric nonmetric connection coincide. It is shown that if the second fundamental form σ is recurrent, 2-recurrent, generalized 2-recurrent, semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel and M has parallel third fundamental form with respect to semisymmetric nonmetric connection, then M is totally geodesic with respect to Levi-Civita connection.

1. Semisymmetric Nonmetric Connection

The geometry of invariant submanifolds M of Sasakian manifolds $\tilde{M}$ is carried out from 1970's by M. Kon [1], D. Chinea [2], K. Yano and M. Kon [3] and B.S. Anitha and C.S. Bagewadi [4]. The author [1] has proved that invariant submanifold of Sasakian structure also carries Sasakian structure. In this paper we extend the results to invariant submanifolds M of Sasakian manifolds admitting Semisymmetric Nonmetric connection.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla g = 0$; otherwise it is Nonmetric. Further it is said to be Semisymmetric if its torsion tensor $T(X, Y) = 0$; that is, $T(X, Y) = w(Y)X - w(X)Y$, where w is a 1-form. A study of Semisymmetric connection on a Riemannian manifold was initiated by Yano [5]. In 1992, Agashe and Chafle [6] introduced the notion of Semisymmetric Nonmetric connection. If $\bar{\nabla}$ denotes Semisymmetric Nonmetric connection on a contact metric manifold,

then it is given by [6]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X, \quad (1.1)$$

where $\eta(Y) = g(Y, \xi)$.

The covariant differential of the p th order, $p \geq 1$ of a $(0, k)$ -tensor field T , $k \geq 1$ denoted by $\nabla^p T$, defined on a Riemannian manifold (M, g) with the Levi-Civita connection ∇ . The tensor T is said to be *recurrent* [7], if the following condition holds on M :

$$(\nabla T)(X_1, \dots, X_k; X)T(Y_1, \dots, Y_k) = (\nabla T)(Y_1, \dots, Y_k; X)T(X_1, \dots, X_k), \quad (1.2)$$

respectively.

Consider

$$\left(\nabla^2 T\right)(X_1, \dots, X_k; X, Y)T(Y_1, \dots, Y_k) = \left(\nabla^2 T\right)(Y_1, \dots, Y_k; X, Y)T(X_1, \dots, X_k), \quad (1.3)$$

where $X, Y, X_1, Y_1, \dots, X_k, Y_k \in TM$. From (1.2) it follows that at a point $x \in M$, if the tensor T is nonzero, then there exists a unique 1-form ϕ , respectively, a $(0, 2)$ -tensor ψ , defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|), \quad (1.4)$$

respectively.

The following

$$\nabla^2 T = T \otimes \psi \quad (1.5)$$

holds on U , where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T, T)$. The tensor T is said to be *generalized 2-recurrent* if

$$\begin{aligned} & \left(\left(\nabla^2 T \right)(X_1, \dots, X_k; X, Y) - (\nabla T \otimes \phi)(X_1, \dots, X_k; X, Y) \right) T(Y_1, \dots, Y_k) \\ &= \left(\left(\nabla^2 T \right)(Y_1, \dots, Y_k; X, Y) - (\nabla T \otimes \phi)(Y_1, \dots, Y_k; X, Y) \right) T(X_1, \dots, X_k) \end{aligned} \quad (1.6)$$

holds on M , where ϕ is a 1-form on M . From this it follows that at a point $x \in M$ if the tensor

T is nonzero, then there exists a unique $(0, 2)$ -tensor ψ , defined on a neighborhood U of x , such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi \quad (1.7)$$

holds on U .

2. Isometric Immersion

Let $f : (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ be an isometric immersion from an n -dimensional Riemannian manifold (M, g) into $(n + d)$ -dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$, respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (2.1)$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.2)$$

for any tangent vector fields X, Y and the normal vector field N on M , where σ , A , and $\nabla^\perp$ are the second fundamental form, the shape operator, and the normal connection, respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N is related by

$$\widetilde{g}(\sigma(X, Y), N) = g(A_N X, Y), \quad (2.3)$$

for tangent vector fields X, Y . The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (2.4)$$

$$\begin{aligned} (\widetilde{\nabla}^2 \sigma)(Z, W, X, Y) &= (\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(Z, W) \\ &= \nabla_X^\perp((\widetilde{\nabla}_Y \sigma)(Z, W)) - (\widetilde{\nabla}_Y \sigma)(\nabla_X Z, W) \\ &\quad - (\widetilde{\nabla}_X \sigma)(Z, \nabla_Y W) - (\widetilde{\nabla}_{\nabla_X Y} \sigma)(Z, W), \end{aligned} \quad (2.5)$$

respectively, where $\widetilde{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [8]. If $\widetilde{\nabla} \sigma = 0$,

then M is said to have *parallel second fundamental form* [8]. We next define endomorphisms $R(X, Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \\ (X \wedge_B Y)Z &= B(Y, Z)X - B(X, Z)Y, \end{aligned} \quad (2.6)$$

respectively, where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0, 2)$ -tensor.

Now, for a $(0, k)$ -tensor field T , $k \geq 1$ and a $(0, 2)$ -tensor field B on (M, g) , we define the tensor $Q(B, T)$ by

$$Q(B, T)(X_1, \dots, X_k; X, Y) = -(T(X \wedge_B Y)X_1, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}(X \wedge_B Y)X_k). \quad (2.7)$$

Putting into consideration the previous formula " $B = g, S$ and $T = \sigma$," we obtain the tensors $Q(g, \sigma)$ and $Q(S, \sigma)$.

3. Sasakian Manifolds

An n -dimensional differential manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1, 1)$, a vector field ξ , and 1-form η on M , respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \quad (3.1)$$

Thus a manifold M equipped with this structure is called an almost contact manifold and is denoted by (M, ϕ, ξ, η) . If g is a Riemannian metric on an almost contact manifold M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (3.2)$$

where X, Y are vector fields defined on M , then M is said to have an almost contact metric structure (ϕ, ξ, η, g) , and M with this structure is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g) .

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies

$$\Phi(X, Y) = d\eta(X, Y) = g(X, \phi Y), \quad (3.3)$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and together with manifold M is called contact metric manifold and Φ is a 2-form. The contact metric structure (M, ϕ, ξ, η, g) is said to be normal if

$$[\phi, \phi](X, Y) + 2d\eta \otimes \xi = 0. \quad (3.4)$$

If the contact metric structure is normal, then it is called a Sasakian structure and M is called a Sasakian manifold. Note that an almost contact metric manifold defines Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (3.5)$$

$$\nabla_X \xi = -\phi X. \quad (3.6)$$

Example of Sasakian Manifold

Consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \quad (3.7)$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1. \end{aligned} \quad (3.8)$$

The (ϕ, ξ, η) is given by

$$\begin{aligned} \eta &= 2ydx + dz, & \xi &= E_3 = \frac{\partial}{\partial z}, \\ \phi E_1 &= E_2, & \phi E_2 &= -E_1, & \phi E_3 &= 0. \end{aligned} \quad (3.9)$$

The linearity property of ϕ and g yields

$$\begin{aligned} \eta(E_3) &= 1, & \phi^2 U &= -U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) - \eta(U)\eta(W), & g(U, \xi) &= \eta(U), \end{aligned} \quad (3.10)$$

for any vector fields U, W on M . By definition of Lie bracket, we have

$$[E_1, E_2] = 2E_3. \quad (3.11)$$

Let ∇ be the Levi-Civita connection with respect to previously mentioned metric g and be given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \quad (3.12)$$

Then, we have

$$\begin{aligned}\nabla_{E_1}E_1 &= 0, & \nabla_{E_1}E_2 &= E_3, & \nabla_{E_1}E_3 &= -E_2, \\ \nabla_{E_2}E_1 &= -E_3, & \nabla_{E_2}E_2 &= 0, & \nabla_{E_2}E_3 &= E_1, \\ \nabla_{E_3}E_1 &= -E_2, & \nabla_{E_3}E_2 &= E_1, & \nabla_{E_3}E_3 &= 0.\end{aligned}\tag{3.13}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3 ; that is, $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy (3.1), (3.2), (3.5), and (3.6). Thus M is a Sasakian manifold. Further the following relations hold:

$$\begin{aligned}R(X, Y)Z &= \{g(Y, Z)X - g(X, Z)Y\}, \\ R(X, Y)\xi &= \{\eta(Y)X - \eta(X)Y\},\end{aligned}\tag{3.14}$$

$$\begin{aligned}R(\xi, X)Y &= \{g(X, Y)\xi - \eta(Y)X\}, \\ R(\xi, X)\xi &= \{\eta(X)\xi - X\},\end{aligned}\tag{3.15}$$

$$S(X, \xi) = (n-1)\eta(X),\tag{3.16}$$

$$Q\xi = (n-1)\xi,\tag{3.17}$$

for all vector fields, X, Y, Z and where ∇ denotes the operator of covariant differentiation with respect to g , ϕ is a $(1,1)$ tensor field, S is the Ricci tensor of type $(0,2)$, and R is the Riemannian curvature tensor of the manifold.

4. Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

If $\widetilde{M}$ is a Sasakian manifold with structure tensors $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$, then we know that its invariant submanifold M has the induced Sasakian structure (ϕ, ξ, η, g) .

A submanifold M of a Sasakian manifold $\widetilde{M}$ with a Semisymmetric Nonmetric connection is called an invariant submanifold of $\widetilde{M}$ with a Semisymmetric Nonmetric connection, if for each $x \in M$, $\phi(T_x M) \subset T_x M$. As a consequence, ξ becomes tangent to M . For an invariant submanifold of a Sasakian manifold with a Semisymmetric Nonmetric connection we have

$$\sigma(X, \xi) = 0,\tag{4.1}$$

for any vector X tangent to M .

Let $\widetilde{M}$ be a Sasakian manifold admitting a Semisymmetric Nonmetric connection $\widetilde{\nabla}$.

Lemma 4.1. *Let M be an invariant submanifold of contact metric manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection $\widetilde{\nabla}$, and let σ and $\overline{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and Semisymmetric Nonmetric connection; then (1) M admits Semisymmetric Nonmetric connection and (2) the second fundamental forms with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$ are equal.*

Proof. We know that the contact metric structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1.1), we get

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}_X Y + \eta(Y)X. \quad (4.2)$$

By using (2.1) in (4.2), we get

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) + \eta(Y)X. \quad (4.3)$$

Now Gauss formula (2.1) with respect to Semisymmetric Nonmetric connection is given by

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X, Y). \quad (4.4)$$

Equating (4.3) and (4.4), we get (1.1) and

$$\overline{\sigma}(X, Y) = \sigma(X, Y). \quad (4.5)$$

□

Now we introduce the definitions of semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection.

Definition 4.2. An immersion is said to be semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection, respectively, if the following conditions hold for all vector fields X, Y tangent to M :

$$\begin{aligned} \widetilde{R} \cdot \sigma &= 0, \\ \widetilde{R} \cdot \sigma &= L_1 Q(g, \sigma), \\ \widetilde{R} \cdot \sigma &= L_2 Q(S, \sigma), \end{aligned} \quad (4.6)$$

where $\widetilde{R}$ denotes the curvature tensor with respect to connection $\widetilde{\nabla}$. Here L_1 and L_2 are functions depending on σ .

Lemma 4.3. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection. Then Gauss and Weingarten formulae with respect to Semisymmetric Nonmetric connection are given by*

$$\begin{aligned} \tan(\widetilde{\widetilde{R}}(X, Y)Z) &= R(X, Y)Z + \eta(\nabla_Y Z)X + \eta(Z)\nabla_X Y + \eta(Z)\eta(Y)X \\ &\quad - \eta(\nabla_X Z)Y - \eta(Z)\nabla_Y X - \eta(Z)\eta(X)Y - \eta(Z)[X, Y] \\ &\quad + \tan\left\{\widetilde{\widetilde{\nabla}}_X\{\sigma(Y, Z)\} - \widetilde{\widetilde{\nabla}}_Y\{\sigma(X, Z)\} - \widetilde{\widetilde{\nabla}}_Y\eta(Z)X + \widetilde{\widetilde{\nabla}}_X\eta(Z)Y\right\}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \text{nor}(\widetilde{\widetilde{R}}(X, Y)Z) &= \sigma(X, \nabla_Y Z) + \eta(Z)\sigma(X, Y) - \sigma(Y, \nabla_X Z) - \eta(Z)\sigma(Y, X) - \sigma([X, Y], Z) \\ &\quad + \text{nor}\left\{\widetilde{\widetilde{\nabla}}_X\{\sigma(Y, Z)\} - \widetilde{\widetilde{\nabla}}_Y\{\sigma(X, Z)\} - \widetilde{\widetilde{\nabla}}_Y\eta(Z)X + \widetilde{\widetilde{\nabla}}_X\eta(Z)Y\right\}. \end{aligned} \quad (4.8)$$

Proof. The Riemannian curvature tensor $\widetilde{R}$ on $\widetilde{M}$ with respect to Semisymmetric Nonmetric connection is given by

$$\widetilde{R}(X, Y)Z = \widetilde{\widetilde{\nabla}}_X\widetilde{\widetilde{\nabla}}_Y Z - \widetilde{\widetilde{\nabla}}_Y\widetilde{\widetilde{\nabla}}_X Z - \widetilde{\widetilde{\nabla}}_{[X, Y]}Z. \quad (4.9)$$

Using (1.1) and (2.1) in (4.9), we get

$$\begin{aligned} \widetilde{\widetilde{R}}(X, Y)Z &= R(X, Y)Z + \sigma(X, \nabla_Y Z) + \eta(\nabla_Y Z)X + \widetilde{\widetilde{\nabla}}_X\{\sigma(Y, Z)\} + \widetilde{\widetilde{\nabla}}_X\eta(Z)Y \\ &\quad + \eta(Z)\nabla_X Y + \eta(Z)\sigma(X, Y) + \eta(Z)\eta(Y)X - \sigma(Y, \nabla_X Z) - \eta(\nabla_X Z)Y \\ &\quad - \widetilde{\widetilde{\nabla}}_Y\{\sigma(X, Z)\} - \widetilde{\widetilde{\nabla}}_Y\eta(Z)X - \eta(Z)\nabla_Y X \\ &\quad - \eta(Z)\sigma(Y, X) - \eta(Z)\eta(X)Y - \sigma([X, Y], Z) - \eta(Z)[X, Y]. \end{aligned} \quad (4.10)$$

Comparing tangential and normal part of (4.10), we obtain Gauss and Weingarten formulae (4.7) and (4.8). $\square$

Lemma 4.4. *Let M be an invariant submanifold of contact manifold $\widetilde{M}$ which admits Semisymmetric Nonmetric connection. If σ is semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel with*

respect to Semisymmetric Nonmetric connection, then we have

$$\begin{aligned}
 \left(\bar{\bar{R}}(X, Y) \cdot \sigma \right)(U, V) &= R^\perp(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\
 &\quad - \nabla_X A_{\sigma(U, V)}Y + \nabla_Y A_{\sigma(U, V)}X - A_{\nabla_Y^\perp \sigma(U, V)}X + A_{\nabla_X^\perp \sigma(U, V)}Y \\
 &\quad + A_{\sigma(U, V)}[X, Y] - \sigma(X, A_{\sigma(U, V)}Y) + \sigma(Y, A_{\sigma(U, V)}X) \\
 &\quad - \eta(A_{\sigma(U, V)}Y)X + \eta(A_{\sigma(U, V)}X)Y - \eta(\nabla_Y U)\sigma(X, V) \\
 &\quad - \eta(U)\sigma(\nabla_X Y, V) - \eta(U)\eta(Y)\sigma(X, V) + \eta(\nabla_X U)\sigma(Y, V) \\
 &\quad + \eta(U)\sigma(\nabla_Y X, V) + \eta(U)\eta(X)\sigma(Y, V) + \eta(U)\sigma([X, Y], V) \\
 &\quad - \sigma\left(\bar{\bar{\nabla}}_X \eta(U)Y, V\right) + \sigma\left(\bar{\bar{\nabla}}_Y \eta(U)X, V\right) - \sigma\left(\bar{\bar{\nabla}}_X \{\sigma(Y, U)\}, V\right) \\
 &\quad + \sigma\left(\bar{\bar{\nabla}}_Y \{\sigma(X, U)\}, V\right) - \sigma(\sigma(X, \nabla_Y U), V) - \eta(U)\sigma(\sigma(X, Y), V) \\
 &\quad + \sigma(\sigma(Y, \nabla_X U), V) + \eta(U)\sigma(\sigma(Y, X), V) + \sigma(\sigma([X, Y], U), V) \\
 &\quad - \eta(\nabla_Y V)\sigma(U, X) - \eta(V)\sigma(U, \nabla_X Y) - \eta(V)\eta(Y)\sigma(U, X) \\
 &\quad + \eta(\nabla_X V)\sigma(U, Y) + \eta(V)\sigma(U, \nabla_Y X) + \eta(V)\eta(X)\sigma(U, Y) \\
 &\quad + \eta(V)\sigma(U, [X, Y]) - \sigma\left(U, \bar{\bar{\nabla}}_X \eta(V)Y\right) + \sigma\left(U, \bar{\bar{\nabla}}_Y \eta(V)X\right) \\
 &\quad - \sigma\left(U, \bar{\bar{\nabla}}_X \{\sigma(Y, V)\}\right) + \sigma\left(U, \bar{\bar{\nabla}}_Y \{\sigma(X, V)\}\right) - \sigma(U, \sigma(X, \nabla_Y V)) \\
 &\quad - \eta(V)\sigma(U, \sigma(X, Y)) + \sigma(U, \sigma(Y, \nabla_X V)) \\
 &\quad + \eta(V)\sigma(U, \sigma(Y, X)) + \sigma(U, \sigma([X, Y], V)),
 \end{aligned} \tag{4.11}$$

for all vector fields X, Y, U , and V tangent to M , where

$$R^\perp(X, Y) = \left[\nabla_X^\perp, \nabla_Y^\perp \right] - \nabla_{[X, Y]}^\perp. \tag{4.12}$$

Proof. We know, from tensor algebra, that

$$\left(\bar{\bar{R}}(X, Y) \cdot \sigma \right)(U, V) = \bar{\bar{R}}(X, Y)\sigma(U, V) - \sigma\left(\bar{\bar{R}}(X, Y)U, V\right) - \sigma\left(U, \bar{\bar{R}}(X, Y)V\right). \tag{4.13}$$

Replacing Z by $\sigma(U, V)$ in (4.9), we get

$$\bar{\bar{R}}(X, Y)\sigma(U, V) = \bar{\bar{\nabla}}_X \bar{\bar{\nabla}}_Y \sigma(U, V) - \bar{\bar{\nabla}}_Y \bar{\bar{\nabla}}_X \sigma(U, V) - \bar{\bar{\nabla}}_{[X, Y]} \sigma(U, V). \tag{4.14}$$

In view of (1.1), (2.1), and (2.2), we have the following equalities:

$$\begin{aligned}\bar{\nabla}_X \bar{\nabla}_Y \sigma(U, V) &= \bar{\nabla}_X \left(-A_{\sigma(U, V)} Y + \nabla_Y^\perp \sigma(U, V) \right), \\ &= -\nabla_X A_{\sigma(U, V)} Y - \eta(A_{\sigma(U, V)} Y) X - \sigma(X, A_{\sigma(U, V)} Y) \\ &\quad - A_{\nabla_Y^\perp \sigma(U, V)} X + \nabla_X^\perp \nabla_Y^\perp \sigma(U, V).\end{aligned}\quad (4.15)$$

Similarly

$$\begin{aligned}\bar{\nabla}_Y \bar{\nabla}_X \sigma(U, V) &= -\nabla_Y A_{\sigma(U, V)} X - \eta(A_{\sigma(U, V)} X) Y - \sigma(Y, A_{\sigma(U, V)} X) - A_{\nabla_X^\perp \sigma(U, V)} Y + \nabla_Y^\perp \nabla_X^\perp \sigma(U, V),\end{aligned}\quad (4.16)$$

$$\bar{\nabla}_{[X, Y]} \sigma(U, V) = -A_{\sigma(U, V)} [X, Y] + \nabla_{[X, Y]}^\perp \sigma(U, V). \quad (4.17)$$

Substituting (4.15), (4.16) and (4.17) into (4.14), we get

$$\begin{aligned}\bar{R}(X, Y) \sigma(U, V) &= R^\perp(X, Y) \sigma(U, V) - \nabla_X A_{\sigma(U, V)} Y + \nabla_Y A_{\sigma(U, V)} X - A_{\nabla_Y^\perp \sigma(U, V)} X \\ &\quad + A_{\nabla_X^\perp \sigma(U, V)} Y + A_{\sigma(U, V)} [X, Y] - \sigma(X, A_{\sigma(U, V)} Y) + \sigma(Y, A_{\sigma(U, V)} X) \\ &\quad - \eta(A_{\sigma(U, V)} Y) X + \eta(A_{\sigma(U, V)} X) Y.\end{aligned}\quad (4.18)$$

By virtue of (4.10) in $\sigma(\bar{R}(X, Y)U, V)$ and $\sigma(U, \bar{R}(X, Y)V)$, we get

$$\begin{aligned}\sigma\left(\bar{R}(X, Y)U, V\right) &= \sigma(R(X, Y)U, V) + \eta(\nabla_Y U) \sigma(X, V) + \eta(U) \sigma(\nabla_X Y, V) \\ &\quad + \eta(U) \eta(Y) \sigma(X, V) - \eta(\nabla_X U) \sigma(Y, V) - \eta(U) \sigma(\nabla_Y X, V) \\ &\quad - \eta(U) \eta(X) \sigma(Y, V) - \eta(U) \sigma([X, Y], V) + \sigma\left(\bar{\nabla}_X \eta(U) Y, V\right) \\ &\quad - \sigma\left(\bar{\nabla}_Y \eta(U) X, V\right) + \sigma\left(\bar{\nabla}_X \{\sigma(Y, U)\}, V\right) - \sigma\left(\bar{\nabla}_Y \{\sigma(X, U)\}, V\right) \\ &\quad + \sigma(\sigma(X, \nabla_Y U), V) + \eta(U) \sigma(\sigma(X, Y), V) - \sigma(\sigma(Y, \nabla_X U), V) \\ &\quad - \eta(U) \sigma(\sigma(Y, X), V) - \sigma(\sigma([X, Y], U), V),\end{aligned}\quad (4.19)$$

$$\begin{aligned}
\sigma\left(\mathcal{U}, \widetilde{R}(X, Y)V\right) &= \sigma(\mathcal{U}, R(X, Y)V) + \eta(\nabla_Y V)\sigma(\mathcal{U}, X) + \eta(V)\sigma(\mathcal{U}, \nabla_X Y) \\
&\quad + \eta(V)\eta(Y)\sigma(\mathcal{U}, X) - \eta(\nabla_X V)\sigma(\mathcal{U}, Y) - \eta(V)\sigma(\mathcal{U}, \nabla_Y X) \\
&\quad - \eta(V)\eta(X)\sigma(\mathcal{U}, Y) - \eta(V)\sigma(\mathcal{U}, [X, Y]) + \sigma\left(\mathcal{U}, \widetilde{\nabla}_X \eta(V)Y\right) \\
&\quad - \sigma\left(\mathcal{U}, \widetilde{\nabla}_Y \eta(V)X\right) + \sigma\left(\mathcal{U}, \widetilde{\nabla}_X \{\sigma(Y, V)\}\right) - \sigma\left(\mathcal{U}, \widetilde{\nabla}_Y \{\sigma(X, V)\}\right) \\
&\quad + \sigma(\mathcal{U}, \sigma(X, \nabla_Y V)) + \eta(V)\sigma(\mathcal{U}, \sigma(X, Y)) - \sigma(\mathcal{U}, \sigma(Y, \nabla_X V)) \\
&\quad - \eta(V)\sigma(\mathcal{U}, \sigma(Y, X)) - \sigma(\mathcal{U}, \sigma([X, Y], V)).
\end{aligned} \tag{4.20}$$

Substituting (4.18), (4.19) and (4.20) into (4.13), we get (4.11). $\square$

5. Recurrent Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

We consider invariant submanifolds of a Sasakian manifold when σ is recurrent, 2-recurrent, and generalized 2-recurrent and M has parallel third fundamental form with respect to Semisymmetric Nonmetric connection. We write (2.4) and (2.5) with respect to Semisymmetric Nonmetric connection, and they are given by

$$\left(\widetilde{\nabla}_X \sigma\right)(Y, Z) = \overline{\nabla}_X^\perp(\sigma(Y, Z)) - \sigma(\overline{\nabla}_X Y, Z) - \sigma(Y, \overline{\nabla}_X Z), \tag{5.1}$$

$$\begin{aligned}
\left(\widetilde{\nabla}^2 \sigma\right)(Z, W, X, Y) &= \left(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma\right)(Z, W) \\
&= \overline{\nabla}_X^\perp\left(\left(\widetilde{\nabla}_Y \sigma\right)(Z, W)\right) - \left(\widetilde{\nabla}_Y \sigma\right)(\overline{\nabla}_X Z, W) \\
&\quad - \left(\widetilde{\nabla}_X \sigma\right)(Z, \overline{\nabla}_Y W) - \left(\widetilde{\nabla}_{\overline{\nabla}_X Y} \sigma\right)(Z, W).
\end{aligned} \tag{5.2}$$

We prove the following theorems.

Theorem 5.1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be recurrent with respect to Semisymmetric Nonmetric connection; from (1.4) we get

$$\left(\widetilde{\nabla}_X \sigma\right)(Y, Z) = \phi(X)\sigma(Y, Z), \tag{5.3}$$

where ϕ is a 1-form on M ; in view of (5.1) and putting $Z = \xi$ in the above equation, we have

$$\bar{\nabla}_X^\perp \sigma(Y, \xi) - \sigma(\bar{\nabla}_X Y, \xi) - \sigma(Y, \bar{\nabla}_X \xi) = \phi(X) \sigma(Y, \xi). \quad (5.4)$$

By virtue of (4.1) in (5.4), we get

$$-\sigma(\bar{\nabla}_X Y, \xi) - \sigma(Y, \bar{\nabla}_X \xi) = 0. \quad (5.5)$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.5), we get

$$\sigma(Y, \phi X) - \sigma(Y, X) = 0. \quad (5.6)$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.6), we get

$$-\sigma(Y, X) - \sigma(Y, \phi X) = 0. \quad (5.7)$$

Adding (5.6) and (5.7), we obtain $\sigma(X, Y) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. $\square$

Theorem 5.2. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then M has parallel third fundamental form with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M have parallel third fundamental form with respect to Semisymmetric Nonmetric connection. Then we have

$$\left(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma \right) (Z, W) = 0. \quad (5.8)$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\bar{\nabla}_X^\perp \left(\left(\widetilde{\nabla}_Y \sigma \right) (Z, \xi) \right) - \left(\widetilde{\nabla}_Y \sigma \right) (\bar{\nabla}_X Z, \xi) - \left(\widetilde{\nabla}_X \sigma \right) (Z, \bar{\nabla}_Y \xi) - \left(\widetilde{\nabla}_{\bar{\nabla}_X Y} \sigma \right) (Z, \xi) = 0. \quad (5.9)$$

In view of (4.1) and by virtue of (5.1) in (5.9), we get

$$\begin{aligned} 0 = & -\bar{\nabla}_X^\perp \left\{ \sigma(\bar{\nabla}_Y Z, \xi) + \sigma(Z, \bar{\nabla}_Y \xi) \right\} - \bar{\nabla}_Y^\perp \sigma(\bar{\nabla}_X Z, \xi) + \sigma(\bar{\nabla}_Y \bar{\nabla}_X Z, \xi) \\ & + 2\sigma(\bar{\nabla}_X Z, \bar{\nabla}_Y \xi) - \bar{\nabla}_X^\perp \sigma(Z, \bar{\nabla}_Y \xi) + \sigma(Z, \bar{\nabla}_X \bar{\nabla}_Y \xi) + \sigma(\bar{\nabla}_{\bar{\nabla}_X Y} Z, \xi) + \sigma(Z, \bar{\nabla}_{\bar{\nabla}_X Y} \xi). \end{aligned} \quad (5.10)$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.10), we get

$$\begin{aligned} 0 = & 2\bar{\nabla}_X^\perp \sigma(Z, \phi Y) - 2\bar{\nabla}_X^\perp \sigma(Z, Y) - 2\eta(Z)\sigma(X, \phi Y) + 2\sigma(\nabla_X Z, Y) \\ & + 2\eta(Z)\sigma(X, Y) - \sigma(Z, \nabla_X \phi Y) - \sigma(Z, \phi \nabla_X Y) - \eta(Y)\sigma(Z, \phi X) \\ & + 2\sigma(Z, \nabla_X Y) + 2\eta(Y)\sigma(Z, X) - 2\sigma(\nabla_X Z, \phi Y). \end{aligned} \quad (5.11)$$

Putting $Y = \xi$ and using (3.1), (3.6), and (4.1) in (5.11), we get

$$0 = \sigma(Z, X) - 3\sigma(Z, \phi X). \quad (5.12)$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.12), we get

$$0 = \sigma(Z, \phi X) + 3\sigma(Z, X). \quad (5.13)$$

Multiplying (5.12) by 1 and (5.13) by 3 and adding these two equations, we obtain $\sigma(X, Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. $\square$

Corollary 5.3. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is 2-recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be 2-recurrent with respect to Semisymmetric Nonmetric connection; from (1.5), we have

$$\left(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma \right) (Z, W) = \sigma(Z, W) \phi(X, Y). \quad (5.14)$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\begin{aligned} & \bar{\nabla}_X^\perp \left(\left(\widetilde{\nabla}_Y \sigma \right) (Z, \xi) \right) - \left(\widetilde{\nabla}_Y \sigma \right) (\bar{\nabla}_X Z, \xi) - \left(\widetilde{\nabla}_X \sigma \right) (Z, \bar{\nabla}_Y \xi) - \left(\widetilde{\nabla}_{\bar{\nabla}_X Y} \sigma \right) (Z, \xi) \\ & = \sigma(Z, \xi) \phi(X, Y). \end{aligned} \quad (5.15)$$

In view of (4.1) and by virtue of (5.1) in (5.15), we get

$$\begin{aligned} 0 = & -\bar{\nabla}_X^\perp \left\{ \sigma(\bar{\nabla}_Y Z, \xi) + \sigma(Z, \bar{\nabla}_Y \xi) \right\} - \bar{\nabla}_Y^\perp \sigma(\bar{\nabla}_X Z, \xi) + \sigma(\bar{\nabla}_Y \bar{\nabla}_X Z, \xi) \\ & + 2\sigma(\bar{\nabla}_X Z, \bar{\nabla}_Y \xi) - \bar{\nabla}_X^\perp \sigma(Z, \bar{\nabla}_Y \xi) + \sigma(Z, \bar{\nabla}_X \bar{\nabla}_Y \xi) + \sigma(\bar{\nabla}_{\bar{\nabla}_X Y} Z, \xi) + \sigma(Z, \bar{\nabla}_{\bar{\nabla}_X Y} \xi). \end{aligned} \quad (5.16)$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.16), we get

$$\begin{aligned} 0 &= 2\bar{\nabla}_X^\perp \sigma(Z, \phi Y) - 2\bar{\nabla}_X^\perp \sigma(Z, Y) - 2\eta(Z)\sigma(X, \phi Y) + 2\sigma(\nabla_X Z, Y) \\ &\quad + 2\eta(Z)\sigma(X, Y) - \sigma(Z, \nabla_X \phi Y) - \sigma(Z, \phi \nabla_X Y) - \eta(Y)\sigma(Z, \phi X) \\ &\quad + 2\sigma(Z, \nabla_X Y) + 2\eta(Y)\sigma(Z, X) - 2\sigma(\nabla_X Z, \phi Y). \end{aligned} \quad (5.17)$$

Putting $Y = \xi$ and using (3.1), (3.6), (4.1) in (5.17), we get

$$0 = \sigma(Z, X) - 3\sigma(Z, \phi X). \quad (5.18)$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.18), we get

$$0 = \sigma(Z, \phi X) + 3\sigma(Z, X). \quad (5.19)$$

Multiplying (5.18) by 1 and (5.19) by 3 and adding these two equations, we obtain $\sigma(X, Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. $\square$

Theorem 5.4. *Let M be an invariant submanifold of a Sasakian manifold $\tilde{M}$ admitting a Semisymmetric Nonmetric connection. Then σ is generalized 2-recurrent with respect to Semisymmetric Nonmetric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Letting σ be generalized 2-recurrent with respect to Semisymmetric Nonmetric connection, from (1.7), we have

$$\left(\bar{\nabla}_X \bar{\nabla}_Y \sigma \right) (Z, W) = \psi(X, Y)\sigma(Z, W) + \phi(X) \left(\bar{\nabla}_Y \sigma \right) (Z, W), \quad (5.20)$$

where ψ and ϕ are 2-recurrent and 1-form, respectively. Taking $W = \xi$ in (5.20) and using (4.1), we get

$$\left(\bar{\nabla}_X \bar{\nabla}_Y \sigma \right) (Z, \xi) = \phi(X) \left(\bar{\nabla}_Y \sigma \right) (Z, \xi). \quad (5.21)$$

Using (4.1) and (5.2) in above equation, we get

$$\begin{aligned} &\bar{\nabla}_X^\perp \left(\left(\bar{\nabla}_Y \sigma \right) (Z, \xi) \right) - \left(\bar{\nabla}_Y \sigma \right) (\bar{\nabla}_X Z, \xi) - \left(\bar{\nabla}_X \sigma \right) (Z, \bar{\nabla}_Y \xi) - \left(\bar{\nabla}_{\bar{\nabla}_X Y} \sigma \right) (Z, \xi) \\ &= -\phi(X) \left\{ \sigma(\bar{\nabla}_Y Z, \xi) + \sigma(Z, \bar{\nabla}_Y \xi) \right\}. \end{aligned} \quad (5.22)$$

In view of (4.1) and by virtue of (5.1) in (5.22), we get

$$\begin{aligned}
 & -\bar{\nabla}_X^\perp \left\{ \sigma(\bar{\nabla}_Y Z, \xi) + \sigma(Z, \bar{\nabla}_Y \xi) \right\} - \bar{\nabla}_Y^\perp \sigma(\bar{\nabla}_X Z, \xi) + \sigma(\bar{\nabla}_Y \bar{\nabla}_X Z, \xi) \\
 & + 2\sigma(\bar{\nabla}_X Z, \bar{\nabla}_Y \xi) - \bar{\nabla}_X^\perp \sigma(Z, \bar{\nabla}_Y \xi) + \sigma(Z, \bar{\nabla}_X \bar{\nabla}_Y \xi) + \sigma(\bar{\nabla}_{\bar{\nabla}_X Y} Z, \xi) + \sigma(Z, \bar{\nabla}_{\bar{\nabla}_X Y} \xi) \quad (5.23) \\
 & = -\phi(X) \left\{ \sigma(\bar{\nabla}_Y Z, \xi) + \sigma(Z, \bar{\nabla}_Y \xi) \right\}.
 \end{aligned}$$

Using (1.1), (3.1), (3.6), and (4.1) in (5.23), we get

$$\begin{aligned}
 0 &= 2\bar{\nabla}_X^\perp \sigma(Z, \phi Y) - 2\bar{\nabla}_X^\perp \sigma(Z, Y) - 2\eta(Z)\sigma(X, \phi Y) + 2\sigma(\nabla_X Z, Y) + 2\eta(Z)\sigma(X, Y) \\
 & - \sigma(Z, \nabla_X \phi Y) - \sigma(Z, \phi \nabla_X Y) - \eta(Y)\sigma(Z, \phi X) \quad (5.24) \\
 & + 2\sigma(Z, \nabla_X Y) + 2\eta(Y)\sigma(Z, X) - 2\sigma(\nabla_X Z, \phi Y) \\
 & = -\phi(X) \{-\sigma(Z, \phi Y) + \sigma(Z, Y)\}.
 \end{aligned}$$

Putting $Y = \xi$ and using (3.1), (3.6), (4.1) in (5.24), we get

$$0 = \sigma(Z, X) - 3\sigma(Z, \phi X). \quad (5.25)$$

Replacing X by ϕX and by virtue of (3.1) and (4.1) in (5.25), we get

$$0 = \sigma(Z, \phi X) + 3\sigma(Z, X). \quad (5.26)$$

Multiplying (5.25) by 1 and (5.26) by 3 and adding these two equations, we obtain $\sigma(X, Z) = 0$. Thus M is totally geodesic. The converse statement is trivial. This proves the theorem. $\square$

6. Semiparallel, Pseudoparallel, and Ricci-Generalized Pseudoparallel Invariant Submanifolds of Sasakian Manifolds Admitting Semisymmetric Nonmetric Connection

We consider invariant submanifolds of Sasakian manifolds admitting Semisymmetric Nonmetric connection satisfying the conditions $\bar{R} \cdot \sigma = 0$, $\bar{R} \cdot \sigma = L_1 Q(g, \sigma)$, $\bar{R} \cdot \sigma = L_2 Q(S, \sigma)$.

Theorem 6.1. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is semiparallel with respect to Semisymmetric Nonmetric connection if and only if $6 = 2\bar{\phi} + \xi$.*

Proof. Let M be semiparallel $\widetilde{R} \cdot \sigma = 0$. Putting $X = V = \xi$ and by virtue of (3.1), (3.6), and (4.1) in (4.11), we get

$$\begin{aligned} 0 = & -\sigma(U, R(\xi, Y)\xi) - \sigma\left(\widetilde{\nabla}_\xi \eta(U)Y, \xi\right) + \sigma\left(\widetilde{\nabla}_Y \eta(U)\xi, \xi\right) - \sigma\left(\widetilde{\nabla}_\xi \sigma(Y, U), \xi\right) \\ & - \sigma(U, \nabla_\xi Y) + \sigma(U, \nabla_Y \xi) + \sigma(U, [\xi, Y]) - \sigma\left(U, \widetilde{\nabla}_\xi Y\right) + \sigma\left(U, \widetilde{\nabla}_Y \xi\right) + \sigma(U, Y). \end{aligned} \quad (6.1)$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.1), we get

$$0 = 3\sigma(U, Y) - \sigma\left(\widetilde{\nabla}_\xi \eta(U)Y, \xi\right) - \sigma(U, \phi Y) - \sigma(U, \nabla_\xi Y). \quad (6.2)$$

By definition σ is a vector-valued covariant tensor, and so $\sigma(U, Y)$ is a vector. Therefore $\widetilde{\nabla}_\xi \sigma(Y, U)$ is a vector, and hence by (4.1), we have

$$\sigma\left(\widetilde{\nabla}_\xi \sigma(Y, U), \xi\right) = 0. \quad (6.3)$$

Then from (6.2), we get

$$3\sigma(U, Y) = \bar{\phi}\sigma(U, Y) + \sigma(U, \nabla_\xi Y). \quad (6.4)$$

Interchanging Y and U in (6.4), we get

$$3\sigma(Y, U) = \bar{\phi}\sigma(Y, U) + \sigma(U, \nabla_\xi Y). \quad (6.5)$$

Adding these two equations, (6.4) and (6.5), we get

$$6 = 2\bar{\phi} + \xi. \quad (6.6)$$

□

Theorem 6.2. Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is pseudoparallel with respect to Semisymmetric Nonmetric connection if and only if $L_1 = \bar{\phi} + \xi/2 - 3$.

Proof. Let M be pseudoparallel $\widetilde{R} \cdot \sigma = L_1 Q(g, \sigma)$. Putting $X = V = \xi$ and by virtue of (3.1), (3.6), and (4.1) in (2.7), (4.11), we get

$$\begin{aligned} & -\sigma(U, R(\xi, Y)\xi) - \sigma\left(\widetilde{\nabla}_\xi \eta(U)Y, \xi\right) + \sigma\left(\widetilde{\nabla}_Y \eta(U)\xi, \xi\right) - \sigma\left(\widetilde{\nabla}_\xi \sigma(Y, U), \xi\right) - \sigma(U, \nabla_\xi Y) \\ & + \sigma(U, \nabla_Y \xi) + \sigma(U, [\xi, Y]) - \sigma\left(U, \widetilde{\nabla}_\xi Y\right) + \sigma\left(U, \widetilde{\nabla}_Y \xi\right) + \sigma(U, Y) = -L_1 \sigma(U, Y). \end{aligned} \quad (6.7)$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.7), we get

$$3\sigma(U, Y) - \sigma\left(\bar{\nabla}_{\xi}\eta(U)Y, \xi\right) - \sigma(U, \phi Y) - \sigma(U, \nabla_{\xi}Y) = -L_1\sigma(U, Y). \quad (6.8)$$

Now by using (6.3) in (6.8), we get

$$(3 + L_1)\sigma(U, Y) = \bar{\phi}\sigma(U, Y) + \sigma(U, \nabla_{\xi}Y). \quad (6.9)$$

Interchanging Y and U in (6.9), we get

$$(3 + L_1)\sigma(Y, U) = \bar{\phi}\sigma(Y, U) + \sigma(Y, \nabla_{\xi}U). \quad (6.10)$$

Adding (6.9) and (6.10), we get

$$L_1 = \bar{\phi} + \frac{\xi}{2} - 3. \quad (6.11)$$

□

Theorem 6.3. Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then we prove that M is Ricci-generalized pseudoparallel with respect to Semisymmetric Nonmetric connection if and only if $L_2 = (1/(n-1))[\bar{\phi} + \xi/2 - 3]$.

Proof. Let M be Ricci-generalized pseudoparallel $\bar{R} \cdot \sigma = L_2 Q(S, \sigma)$. Putting $X = V = \xi$ and by virtue of (3.1), (3.6), (3.16), and (4.1) in (2.7), (4.11), we get

$$\begin{aligned} & -\sigma(U, R(\xi, Y)\xi) - \sigma\left(\bar{\nabla}_{\xi}\eta(U)Y, \xi\right) + \sigma\left(\bar{\nabla}_Y\eta(U)\xi, \xi\right) - \sigma\left(\bar{\nabla}_{\xi}\sigma(Y, U), \xi\right) - \sigma(U, \nabla_{\xi}Y) \\ & + \sigma(U, \nabla_Y\xi) + \sigma(U, [\xi, Y]) - \sigma\left(U, \bar{\nabla}_{\xi}Y\right) + \sigma\left(U, \bar{\nabla}_Y\xi\right) + \sigma(U, Y) = -L_2(n-1)\sigma(U, Y). \end{aligned} \quad (6.12)$$

Using (1.1), (2.1), (3.6), (3.15), (4.1), and (5.1) in (6.12), we get

$$3\sigma(U, Y) - \sigma\left(\bar{\nabla}_{\xi}\eta(U)Y, \xi\right) - \sigma(U, \phi Y) - \sigma(U, \nabla_{\xi}Y) = -L_2(n-1)\sigma(U, Y). \quad (6.13)$$

Now by using (6.3) in (6.13), we get

$$(3 + L_2(n-1))\sigma(U, Y) = \bar{\phi}\sigma(U, Y) + \sigma(U, \nabla_{\xi}Y). \quad (6.14)$$

Interchanging Y and U in (6.14), we get

$$(3 + L_2(n-1))\sigma(Y, U) = \bar{\phi}\sigma(Y, U) + \sigma(Y, \nabla_{\xi}U). \quad (6.15)$$

Adding (6.14) and (6.15), we get

$$2(3 + L_2(n - 1))\sigma(U, Y) = 2\bar{\phi}\sigma(U, Y) + \nabla_{\xi}\sigma(U, Y). \quad (6.16)$$

Writing the above equation, we have

$$L_2 = \frac{1}{(n - 1)} \left[\bar{\phi} + \frac{\xi}{2} - 3 \right]. \quad (6.17)$$

□

Remark 6.4. Let M be an invariant submanifold of a Sasakian manifold which admits Semisymmetric Nonmetric connection. If M is semiparallel, pseudoparallel, and Ricci-generalized pseudoparallel, then we have obtained conditions connecting ϕ , ξ , L_1 , and L_2 . These conditions need further investigation and are to be interpreted geometrically.

Using Theorems 5.1 to 5.4 and corollary 5.3, we have the following result.

Corollary 6.5. *Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}$ admitting a Semisymmetric Nonmetric connection. Then the following statements are equivalent:*

- (1) σ is recurrent,
- (2) σ is 2-recurrent,
- (3) σ is generalized 2-recurrent,
- (4) M has parallel third fundamental form.

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