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Research Article

On BRK-Algebras

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The notion of BRK-algebra is introduced which is a generalization of BCK/BCI/BCH/Q/QS/BM-algebras. The concepts of G-part, p-radical, medial of a BRK-algebra are introduced and studied their properties. We proved that the variety of all medial BRK-algebras is congruence permutable and showed that every associative BRK-algebra is a group.

1. Introduction

In 1996, Imai and Iséki [1] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [2] introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2001, Neggers et al. [3] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002, Neggers and Kim [4] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [5] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [6], C. B. Kim and H. S. Kim introduced BG-algebra as a generalization of B-algebra. We introduce a new notion, called a BRK-algebra, which is a generalization of BCK/BCI/BCH/Q/QS/BM-algebras. The concept of G-part, p-radical, and medial of a BRK-algebra are introduced and studied their properties.

2. Preliminaries

First, we recall certain definitions from [2–5, 7, 8] that are required in the paper.

Definition 2.1. A BCI-algebra is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following conditions:

(B_1) $(x * y) * (x * z) \leq (z * y)$,

(B_2) $x * (x * y) \leq y$,

(B_3) $x \leq x$,

(B_4) $x \leq y$ and $y \leq x$ imply $x = y$,

(B_5) $x \leq 0$ implies $x = 0$, where $x \leq y$ is defined by $x * y = 0$, for all $x, y, z \in X$.

If (B_5) is replaced by (B_6): $0 \leq x$, then the algebra is called a BCK-algebra. It is known that every BCK-algebra is a BCI-algebra but not conversely.

Definition 2.2. A BCH-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_4), and (B_7): $(x * y) * z = (x * z) * y$.

It is shown that every BCI-algebra is a BCH-algebra but not conversely.

Definition 2.3. A Q-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_7), and (B_8): $x * 0 = x$.

A Q-algebra is said to be a QS-algebra if it satisfies the additional relation:

(B_9) $(x * y) * (x * z) = z * y$,

for any $x, y, z \in X$. It is shown that every BCH-algebra is a Q-algebra but not conversely.

Definition 2.4. A B-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_8), and (B_{10}): $(x * y) * z = x * (z * (0 * y))$.

A B-algebra is said to be 0-commutative if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$. In [3], it is shown that Q-algebras and B-algebras are different notions.

Definition 2.5. A BF-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_8), and (B_{11}): $0 * (x * y) = (y * x)$.

It is shown that every B-algebra is BF-algebra but not conversely.

Definition 2.6. A BM-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_8) and (B_{12}): $(x * y) * (x * z) = z * y$.

Definition 2.7. A BH-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_4), and (B_8).

Definition 2.8. A BG-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_8), and (BG): $(x * y) * (0 * y) = x$.

3. BRK-Algebras

In this section, we define the notion of BRK-algebra and observe that the axioms in the definition are independent.

Definition 3.1. A BRK-algebra is a nonempty set A with a constant 0 and a binary operation $*$ satisfying axioms:

(B_8) $x * 0 = x$,

(B_{13}) $(x * y) * x = 0 * y$ for any $x, y \in A$.

For brevity, we also call A a BRK-algebra. In A, we can define a binary relation " $\leq$" by $x \leq y$ if and only if $x * y = 0$.

Example 3.2. Let $A := \mathbb{R} - \{-n\}, 0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define a binary operation $*$ on A by

$$x * y = \frac{n(x - y)}{n + y}, \quad (3.1)$$

then $(A, *, 0)$ is an BRK-algebra.

Example 3.3. Let $A = \{0, 1, 2\}$ in which $*$ is defined by

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

(3.2)

Then $(A, *, 0)$ is a BRK-algebra.

We know that every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a Q-algebra. We can observe that every Q-algebra is a BRK-algebra but converse needs not be true.

Example 3.4. Let $A = \{0, 1, 2, 3\}$ in which $*$ is defined by

$*$	0	1	2	3
0	0	1	0	1
1	1	0	1	0
2	2	1	0	1
3	3	2	3	0

(3.3)

Then $(A, *, 0)$ is a BRK-algebra, which is not a BCK/BCI/BCH/Q-algebra.

We know that every QS-algebra is a BM-algebra and we can observe that every BM-algebra is a BRK-algebra but converses need not be true.

Example 3.5. Let $A = \{0, 1, 2, 3\}$ in which $*$ is defined by

$*$	0	1	2	3
0	0	2	2	0
1	1	0	0	2
2	2	0	0	2
3	3	1	1	0

(3.4)

Then $(A, *, 0)$ is a BRK-algebra, which is not a QS/BM-algebra.

It is easy to see that B/BG/BF/BH-algebra and BRK-algebras are different notions. For example, Example 3.3 is a BRK-algebra which is not a BH-algebra and Example 3.4 is an BRK-algebra which is not B/BG/BF-algebra. Consider the following example. Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

(3.5)

Then $(A, *, 0)$ is a B/BF/BG/BH-algebra which is not an BRK-algebra.

We observe that the two axioms (B_8) and (B_{13}) are independent. Let $A = \{0, 1, 2\}$ be a set with the following left table:

$*$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

$*$	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

(3.6)

Then the axiom (B_8) holds but not (B_{13}), since $(1*2)*1 = 2*1 = 1 \neq 2 = 0*2$. Similarly, the set $A = \{0, 1, 2\}$ with the above right table satisfies the axiom (B_{13}) but not (B_8), since $2*0 = 0 \neq 2$.

Proposition 3.6. *If $(A, *, 0)$ is a BRK-algebra, then, for any $x, y \in A$, the following conditions hold:*

(1) $x * x = 0$,

(2) $x * y = 0 \Rightarrow 0 * x = 0 * y$.

Proof. Let $(A, *, 0)$ be a BRK-algebra and $x, y \in A$. Then

(1) $x * x = (x * 0) * x = 0 * 0 = 0$ (by B_8 and B_{13}),

(2) $x * y = 0 \Rightarrow (x * y) * x = 0 * x \Rightarrow 0 * y = 0 * x$. □

Proposition 3.7. *Every BRK-algebra A satisfies the following property:*

$$0 * (x * y) = (0 * x) * (0 * y), \tag{3.7}$$

for any $x, y \in A$.

Proof. Let $x, y \in A$. Then

$$\begin{aligned} 0 * (x * y) &= ((0 * y) * (x * y)) * (0 * y) \quad (\text{by } B_{13}) \\ &= [((x * y) * x) * (x * y)] * (0 * y) \quad (\text{by } B_{13}) \\ &= (0 * x) * (0 * y). \end{aligned} \tag{3.8}$$

□

Theorem 3.8. *Every BRK-algebra A satisfying* $x * (x * y) = x * y$ *for all* $x, y \in A$ *is a trivial algebra.*

Proof. Putting $x = y$ in the equation $x * (x * y) = x * y$, we obtain $x * 0 = 0 \Rightarrow x = 0$. Hence, A is a trivial algebra. □

Theorem 3.9. *Every BRK-algebra A satisfying* $(x * y) * (x * z) = z * y$ *for all* $x, y, z \in A$ *is a BCI-algebra.*

Proof. Let $(A, *, 0)$ be a BRK-algebra and $(x * y) * (x * z) = z * y$ for all $x, y, z \in A$. Then

(1) $(x * y) * (x * z) * (z * y) = (z * y) * (z * y) = 0$,

(2) $(x * (x * y)) * y = ((x * 0) * (x * y)) * y = (y * 0) * y = y * y = 0$,

(3) $x * x = 0$,

(4) Let $x * y = 0 = y * x$. Then $x = x * 0 = x * (x * y) = (x * 0) * (x * y) = y * 0 = y$,

(5) $x * 0 = 0 \Rightarrow x = 0$. □

Theorem 3.10. *Every 0-commutative B-algebra is a BRK-algebra.*

Proof. Let A be a 0-commutative B-algebra. Then $x * (x * y) = y$ for all $x, y \in A$. Hence, $(x * y) * x = x * (x * (0 * y)) = 0 * y$. □

The following theorem can be proved easily.

Theorem 3.11. *Let* $(A, *, 0)$ *be a BRK-algebra. Then, for any* $x, y \in A$*, the following conditions hold.*

(1) *If* $(x * y) * (0 * (0 * y)) = (x * y) * y$*, then* $0 * (0 * (0 * y)) = 0 * y$.

(2) *If* $(x * y) * (0 * y) = (x * y) * y$*, then* $0 * (0 * y) = 0 * y$.

(3) *If* $x * (y * x) = x * (0 * (x * y))$*, then* $0 * (y * x) = 0 * (0 * (x * y))$.

4. G-Part of BRK-Algebras

In this section, we define G-part, p-radical and medial of a BRK-algebra. We give a necessary and sufficient condition for a BRK-algebra to become a medial BRK-algebra and investigate the properties of G-part in BRK-algebras.

Definition 4.1. A nonempty subset I of a BRK-algebra A is called a subalgebra of A if $x * y \in I$ whenever $x, y \in I$.

Definition 4.2. A nonempty subset I of a BRK-algebra A is called an ideal of A if for any $x, y \in A$:

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and A are ideals of A. We call $\{0\}$ and A the zero ideal and the trivial ideal of A, respectively. An ideal I is said to be proper if $I \neq A$.

Definition 4.3. An ideal I of a BRK-algebra A is called a closed ideal of A *if* $0 * x \in I$ for all $x \in I$.

Example 4.4. Let $A = \{0, 1, 2\}$ in which $*$ is defined by

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

. (4.1)

Then $(A, *, 0)$ is a BRK-algebra and the set $I = \{0, 2\}$ is a subalgebra, an ideal, and a closed ideal of A.

Definition 4.5. Let A be a BRK-algebra. For any subset S *of* A, we define

$$G(S) = \{x \in S \mid 0 * x = x\}. \tag{4.2}$$

In particular, if $S = A$, then we say that $G(A)$ is the G-part of a BRK-algebra.

For any BRK-algebra A, the set:

$$B(A) = \{x \in A \mid 0 * x = 0\} \tag{4.3}$$

is called a p-radical of A. Clearly, $B(A)$ is a subalgebra and an ideal of A.

A BRK-algebra A is said to be p-semisimple if $B(A) = \{0\}$.

The following property is obvious:

$$G(A) \cap B(A) = \{0\}. \tag{4.4}$$

Lemma 4.6. *If $(A, *, 0)$ is a BRK-algebra and $a * b = a * c$ for $a, b, c \in A$, then $0 * b = 0 * c$.*

Proof. Let $(A, *, 0)$ be a BRK-algebra and $a, b, c \in A$. Then by (B_{13}), $a * b = a * c \Rightarrow (a * b) * a = (a * c) * a \Rightarrow 0 * b = 0 * c$. □

Theorem 4.7. *Let $(A, *, 0)$ be a BRK-algebra. Then a left cancellation law holds in $G(A)$.*

Proof. Let $a, b, c \in G(A)$ with $a * b = a * c$. Then, by Lemma 4.6, $0 * b = 0 * c$. Since $b, c \in G(A)$, we obtain $b = c$. □

Proposition 4.8. *Let $(A, *, 0)$ be a BRK-algebra. If $x \in G(A)$, then $0 * x \in G(A)$.*

Proof. Let $x \in G(A)$. Then $0 * x = x$ and hence $0 * (0 * x) = 0 * x$. Therefore, $0 * x \in G(A)$. □

Converse of the above proposition needs not be true. From Example 4.4, we can see that $0 * 1 = 2 \in \{0, 2\} = G(A)$ but $1 \notin G(A)$.

Theorem 4.9. *If $x, y \in G(A)$, then $x * y \in G(A)$.*

Proof. Let $x, y \in G(A)$. Then $0 * x = x$ and $0 * y = y$. Hence, $0 * (x * y) = (0 * x) * (0 * y) = x * y$. Therefore, $x * y \in G(A)$. □

Proposition 4.10. *If $(A, *, 0)$ is a BRK-algebra and $x, y \in A$, then*

$$y \in B(A) \Longleftrightarrow (x * y) * x = 0. \tag{4.5}$$

Proof. Let $(A, *, 0)$ be a BRK-algebra and $x, y \in A$. Then, by (B_{13}), $y \in B(A) \Leftrightarrow 0 * y = 0 \Leftrightarrow (x * y) * x = 0$. □

Theorem 4.11. *If S is a subalgebra of a BRK-algebra $(A, *, 0)$, then $G(A) \cap S = G(S)$.*

Proof. Clearly, $G(A) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 * x = x$ and $x \in S \subseteq A$. Hence, $x \in G(A)$. Therefore, $x \in G(A) \cap S$. Thus, $G(A) \cap S = G(S)$. □

Theorem 4.12. *Let $(A, *, 0)$ be a BRK-algebra. If $G(A) = A$, then A is p-semisimple.*

Proof. Assume that $G(A) = A$. Then $\{0\} = G(A) \cap B(A) = A \cap B(A) = B(A)$. Hence, A is p-semisimple. □

Theorem 4.13. *Every closed ideal of a BRK-algebra is a subalgebra.*

Proof. Let I be a closed ideal of a BRK-algebra $(A, *, 0)$ and $x, y \in I$. Then $0 * y \in I$. By (B_{13}), $(x * y) * x = 0 * y \in I$. Since I is an ideal and $x \in I$, we have $x * y \in I$. So I is a subalgebra of A. □

Note that the converse of the above theorem is not true. In Example 3.4, the set $\{0, 1, 2\}$ is a subalgebra but not a closed ideal.

Theorem 4.14. *Let I be a subset of a BRK-algebra A. Then I is a closed ideal of A if and only if it satisfies (i) $0 \in I$ (ii) $x * z \in I$, $y * z \in I$ and $z \in I$ imply $x * y \in I$, for all $x, y, z \in A$.*

Proof. Let I be a closed ideal of A. Clearly, $0 \in I$. Assume that $x * z,\ y * z,\ z \in I$. Since I is an ideal, we have $x, y \in I$ which implies that $x * y \in I$ because I is a closed ideal and hence a subalgebra of A. Conversely, assume that I satisfies (i) and (ii). Let $x * y, y \in I$. Since $0 * 0,\ y * 0,\ 0 \in I$, by (ii) we have $0 * y \in I$. From (ii), again it follows that $x = x * 0 \in I$ so that I is an ideal of A. Now suppose that $x \in I$. Since $0 * 0,\ x * 0,\ 0 \in I$, we obtain $0 * x \in I$ by (ii). This completes the proof. □

Definition 4.15. A BRK-algebra $(A, *, 0)$ is said to be positive implicative if

$$((x * y) * y) * (0 * y) = x * y \tag{4.6}$$

for all $x, y \in A$.

The BRK-algebra in Example 3.3 is positive implicative.

Definition 4.16. Let $(A, *, 0)$ be a BRK-algebra. For a fixed $a \in A$. The map $R_a : A \to A$ given by $R_a(y) = y * a$ for all $y \in A$ is called right translation of A. Similarly the map $L_a : A \to A$ given by $L_a(y) = a * y$ for all $y \in A$ is called a left translation of A.

Definition 4.17. Let $(A, *, 0)$ be a BRK-algebra. For a fixed $a \in A$. The map $T_a : A \to A$ given by $T_a(y) = (y * a) * (0 * a)$ for all $y \in A$ is called a weak right translation of A. Similarly, the map $M_a : A \to A$ given by $M_a(y) = (a * y) * (0 * y)$ for all $y \in A$ is called a weak left translation of A.

Theorem 4.18. *A BRK-algebra $(A, *, 0)$ is positive implicative if and only if $R_z = T_z \circ R_z$ for all $z \in A$.*

Proof. Let A be a BRK-algebra and $R_z = T_z \circ R_z$ for $z \in A$. Then

$$\begin{aligned} y * z = R_z(y) &= (T_z \circ R_z)(y) = T_z(R_z(y)) \\ &= T_z(y * z) = ((y * z) * z) * (0 * z), \quad \forall y, z \in A. \end{aligned} \tag{4.7}$$

Hence, A is positive implicative BRK-algebra. Conversely, assume that A is positive implicative BRK-algebra. Let $x, y \in A$. Then

$$\begin{aligned} R_x(y) = y * x &= ((y * x) * x) * (0 * x) = (R_x(y) * x) * (0 * x) \\ &= T_x(R_x(y)) = (T_x \circ R_x)(y). \end{aligned} \tag{4.8}$$

Hence, $R_x = T_x \circ R_x$. □

Definition 4.19. A BRK-algebra $(A, *, 0)$ satisfying

$$(x * y) * (z * u) = (x * z) * (y * u) \tag{4.9}$$

for any x, y, z and $u \in A$, is called a medial BRK-algebra.

Example 4.20. Let $A := \mathbb{R} - \{-n\}$, $0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define a binary operation $*$ on A by

$$x * y = \frac{n(x-y)}{n+y}, \tag{4.10}$$

then $(A, *, 0)$ is a medial BRK-algebra.

Theorem 4.21. *If A is a medial BRK-algebra, then, for any $x, y, z \in A$, the following hold:*

(i) $x * (y * z) = (x * y) * (0 * z)$,

(ii) $(x * y) * z = (x * z) * y$.

Proof. Let A be a medial BRK-algebra and $x, y, z \in A$. Then

(i) $(x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z)$,

(ii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y$.

□

By the above theorem, the following corollary follows.

Corollary 4.22. *Every medial BRK-algebra is a Q-algebra.*

Theorem 4.23. *Let A be a medial BRK-algebra. Then the right cancellation law holds in $G(A)$.*

Proof. Let $a, b, x \in G(A)$ with $a * x = b * x$. Then, for any $y \in G(A)$, $x * y = (0 * x) * y = (0 * y) * x = y * x$. Therefore,

$$a = 0 * a = (x * a) * x = (a * x) * x = (b * x) * x = (x * b) * x = 0 * b = b. \tag{4.11}$$

□

Now, we give a necessary and sufficient condition for a BRK-algebra to become a medial BRK-algebra.

Theorem 4.24. *A BRK-algebra A is medial if and only if it satisfies:*

(i) $x * y = 0 * (y * x)$ *for all* $x, y \in A$,

(ii) $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in A$.

Proof. Suppose $(A, *, 0)$ is medial and $x, y, z \in A$. Then

(i) $0 * (y * x) = (x * x) * (y * x) = (x * y) * (x * x) = (x * y) * 0 = x * y$,

(ii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y$.

Conversely, assume that the conditions hold. Then

$$\begin{aligned}(x*y)*(z*u) &= 0*((z*u)*(x*y)) \quad (\text{by (i)})\\ &= 0*((z*(x*y))*u) \quad (\text{by (ii)})\\ &= (0*(z*(x*y)))*(0*u) \quad (\text{by Proposition 3.7})\\ &= ((x*y)*z)*(0*u) \quad (\text{by (i)})\\ &= ((x*z)*y)*(0*u) \quad (\text{by (ii)})\\ &= ((x*z)*(0*u))*y \quad (\text{by (ii)})\\ &= (0*((0*u)*(x*z)))*y \quad (\text{by (i)})\\ &= (0*((z*x)*u))*y \quad (\text{by (ii) \& (i)})\\ &= (u*(z*x))*y \quad (\text{by (i)})\\ &= (u*y)*(z*x) \quad (\text{by (ii)})\\ &= 0*((z*x)*(u*y)) \quad (\text{by (i)})\\ &= (x*z)*(y*u) \quad (\text{by Proposition 3.7 and (i)})\end{aligned} \tag{4.12}$$

Therefore, A is medial. □

Corollary 4.25. *A BRK-algebra A is medial if and only if it is a medial QS-algebra.*

The following theorem can be proved easily.

Theorem 4.26. *An algebra $(A,*,0)$ of type $(2,0)$ is a medial BRK-algebra if and only if it satisfies:*

(i) $x*(y*z)=z*(y*x)$,

(ii) $x*0=x$,

(iii) $x*x=0$.

Corollary 4.27. *If A is a medial BRK-algebra, then $x*(x*y)=y$ for all $x,y\in A$.*

Corollary 4.28. *The class of all of medial BRK-algebras forms a variety, written $\nu(MR)$.*

Proposition 4.29. *A variety ν is congruence-permutable if and only if there is a term $p(x,y,z)$ such that*

$$\nu\models p(x,x,y)\approx y, \qquad \nu\models p(x,y,y)\approx x. \tag{4.13}$$

Corollary 4.30. *The variety $\nu(MR)$ is congruence permutable.*

Proof. Let $p(x,y,z)=x*(y*z)$. Then by Corollary 4.25 and (B_8), we have $p(x,x,y)=y$ and $p(x,y,y)=x$, and so the variety $\nu(MR)$ is congruence permutable. □

The following example shows that a BRK-algebra may not satisfy the associative law.

Example 4.31. Let $A = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	2
1	1	0	0
2	2	0	0

(4.14)

Then $(A, *, 0)$ is a BRK-algebra, but associativity does not hold since $(1*2)*1 = 0*1 = 2 \neq 1 = 1*0 = 1*(2*1)$.

Theorem 4.32. *If A is an associative BRK-algebra, then, for any $x \in B(A)$, $x = 0$.*

Proof. Let $x \in B(A)$. Then $0 = 0 * x = (x * x) * x = x * (x * x) = x * 0 = x$. □

Theorem 4.33. *If A is an associative BRK-algebra, then $G(A) = A$.*

Proof. Let A be an associative BRK-algebra. Clearly, $G(A) \subseteq A$. Let $x \in A$. Then $0 * x = (x * x) * x = x * (x * x) = x * 0 = x$. Hence, $x \in G(A)$. Therefore, $G(A) = A$. □

Now, we prove that every associative BRK-algebra is a group.

Theorem 4.34. *Every BRK-algebra $(A, *, 0)$ satisfying the associative law is a group under the operation "$*$".*

Proof. Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (B_3) and (B_8), we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of A. By (B_3), every element x of A has as its inverse the element x itself. Therefore, $(A, *)$ is a group. □

5. Conclusion and Future Research

In this paper, we have introduced the concept of BRK-algebra and studied their properties. In addition, we have defined G-part, p-radical, and medial of BRK-algebra and proved that the variety of medial algebras is congruence permutable. Finally, we proved that every associative BRK-algebra is a group.

In our future work, we introduce the concept of fuzzy BRK-algebra, interval-valued fuzzy BRK-algebra, intuitionistic fuzzy structure of BRK-algebra, intuitionistic fuzzy ideals of BRK-algebra, and intuitionistic (T,S)-normed fuzzy subalgebras of BRK-algebras, intuitionistic L-fuzzy ideals of BRK-algebra.

I hope this work would serve as a foundation for further studies on the structure of BRK-algebras.

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