Research Article

The Dual and Mirror Images of the Dunwoody 3-Manifolds

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Recently, in 2013, we proved that certain presentations present the Dunwoody 3-manifold groups. Since the Dunwoody 3-manifolds do not have a unique Heegaard diagram, we cannot determine a unique group presentation for the Dunwoody 3-manifolds. It is well known that every $(1, 1)$ -knots in a lens space can be represented by the set $\mathcal{D}$ of the 4-tuples (a, b, c, r) (Cattabriga and Mulazzani (2004); S. H. Kim and Y. Kim (2012, 2013)). In particular, to determine a unique Heegaard diagram of the Dunwoody 3-manifolds, we proved the fact that the certain subset of $\mathcal{D}$ representing all 2-bridge knots of $(1, 1)$ -knots is determined completely by using the dual and mirror $(1, 1)$ -decompositions (S. H. Kim and Y. Kim (2011)). In this paper, we show how to obtain the dual and mirror images of all elements in $\mathcal{D}$ as the generalization of some results by Grasselli and Mulazzani (2001); S. H. Kim and Y. Kim (2011).

1. Introduction

In the study of the Dunwoody 3-manifolds, Heegaard splittings and Heegaard diagrams provide a simple means of understanding the Dunwoody 3-manifolds by changing a 3-dimensional problem into a 2-dimensional problem, for the basic definitions of Heegaard splittings and Heegaard diagrams; see [1, 2]. Even with the help of Heegaard diagrams, because the Dunwoody 3-manifolds do not have a unique Heegaard diagram, generally, the following are still open questions about 3-manifolds: given 3-manifolds M_1 and M_2 , (homeomorphism problem) is $M_1 \cong M_2$? or (isomorphism problem) is $\pi(M_1) \cong \pi(M_2)$? The Dunwoody 3-manifolds have a Heegaard diagram, from which one can obtain presentations for a group; however, not all group presentations arise from Heegaard diagrams of Dunwoody 3-manifolds. The Heegaard diagrams are useful for understanding properties of the Dunwoody 3-manifolds as there is a correspondence between the Heegaard diagrams and the fundamental groups for the Dunwoody 3-manifolds, allowing transformations of a group presentation for the fundamental groups to a simple calculus of the Heegaard diagrams. Thus, transformations of the Heegaard diagrams are corresponding to transformations of a group presentation for the fundamental groups of the Dunwoody 3-manifolds. In [3], we gave some conditions for the answer to the generalized Dunwoody 3-manifolds,

constructed by group presentations for the fundamental group. For an answer to the above problems between the Dunwoody 3-manifolds, we consider the dual and mirror images of the Dunwoody 3-manifolds and give the basic definitions about them in the following.

Let (V_1, V_2) be a Heegaard splitting of a 3-manifold M with genus $n \geq 1$. A properly embedded disc D in the handlebody V_2 is called a meridian disc of V_2 if cutting V_2 along D yields a handlebody of genus $n - 1$. A collection of n mutually disjoint meridian discs $\{D_i\}$ in V_2 is called a complete system of meridian discs of V_2 if cutting V_2 along $\bigcup_i D_i$ gives a 3-ball. Let α_i denote the 1-sphere ∂D_i which lies in the closed orientable surface $\partial V_1 = \partial V_2$ of genus n . The system is said to be a Heegaard diagram of the 3-manifold M and denoted by $(V_1; \alpha_1, \alpha_2, \dots, \alpha_n)$. On the other hand, the system $(V_2; \beta_1, \beta_2, \dots, \beta_n)$ is called a dual Heegaard diagram of the 3-manifold M if $\{D_i\}$ is a complete system of n mutually disjoint meridian discs in V_1 , and β_i is the 1-sphere ∂D_i which lies in the closed orientable surface $\partial V_1 = \partial V_2$ of genus n . In other words, $(V_2; \beta_1, \beta_2, \dots, \beta_n)$ is the dual Heegaard diagram of $(V_1; \alpha_1, \alpha_2, \dots, \alpha_n)$.

For (M, K) , we call K and M each $(1, 1)$ -knot and lens space, and in this paper, all lens spaces are assumed to include $\mathbb{S}^3$, and $\mathbb{S}^1 \times \mathbb{S}^2$. Then, (M, K) admits a $(1, 1)$ -decomposition provided that there exists a Heegaard splitting of genus one $(V_1, K_1) \bigcup_\phi (V_2, K_2)$ of (M, K) such that $(V_1; \alpha_1)$ is a Heegaard

FIGURE 1: The Dunwoody (1, 1)-decomposition $D(a, b, c, r)$.

diagram of M and $K_1 \subset V_1$ and $K_2 \subset V_2$ are properly embedded trivial arcs, where ϕ is an attaching homeomorphism. By the dual (1, 1)-decomposition of (M, K) we mean that $(V_2; \beta_1)$ is the dual Heegaard diagram of M instead of $(V_1; \alpha_1)$.

In 1995, Dunwoody introduced the 6-tuples yielding a family of genus n Heegaard diagrams of closed orientable 3-manifolds called the Dunwoody 3-manifolds [4]. Moreover, the Dunwoody 3-manifolds are determined by the n -fold strongly cyclic coverings of lens spaces M branched over (1, 1)-knots K , defined by the monodromy $C : H_1(M - K) \rightarrow \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order n , $n \geq 2$ [3, 5–8]. In fact, such branched sets in the quotient spaces of the Dunwoody 3-manifolds by a cyclic action of order n are representing the (1, 1)-knots in lens spaces (possibly the 3-sphere) [7, 9], and some classes of such knots represented by the Dunwoody 3-manifolds contain all (1, 1)-knots in $\mathbb{S}^3$ [6].

Now, in view of (1, 1)-knot K , the Dunwoody (1, 1)-decomposition of (M, K) determined by two permutations α and β and 4-tuples (a, b, c, r) of integers as follows

$$\mathcal{D} = \{ (a, b, c, r) \mid a > 0, b \geq 0, c \geq 0, \quad (1) \\ d = 2a + b + c, r \in \mathbb{Z}_d, |\beta\alpha| = 2 \}$$

was introduced in [3, 8, 10] (see Figure 1), where each cycle of α corresponds to the end points of line segments in the Heegaard diagram as in Figure 1, and each cycle of β corresponds to a pair of end points which is identified in forming the handlebody V_1 . For each $(a, b, c, r) \in \mathcal{D}$, we denote the Dunwoody (1, 1)-decomposition of (M, K) by $D(a, b, c, r)$ and the Dunwoody (1, 1)-knot K , represented from $D(a, b, c, r)$, by $K(a, b, c, r)$. Note that for each point x on m^+ , the simple closed curve l_1 on $D(a, b, c, r)$ is determined by the repeated applications of α and β as follows:

$$x, \alpha(x), \beta\alpha(x), \alpha\beta\alpha(x), \dots, \alpha\beta \cdots \alpha(x), \quad (*)$$

which forms exactly an orbit of $\beta\alpha$ [8]. It is well known that all (1, 1)-knots in $\mathbb{S}^3$ are to be the Dunwoody (1, 1)-knots, but the representation of (1, 1)-knots by 4-tuples (a, b, c, r) is not unique. For example, $K(1, 3, 4, 7)$ and $K(2, 1, 4, 4)$ represent the pretzel knot $P(-2, 3, 7)$ as a (1, 1)-knot. However all types of the Dunwoody (1, 1)-decompositions representing all 2-bridge knots were determined completely in [7, 10], and moreover, the types of the Dunwoody (1, 1)-decompositions

representing the certain class of torus knots are given in [6, 11, 12]. For each $n \geq 2$, the 6-tuples (n, a, b, c, r, s) satisfying conditions $ps + q \equiv 0 \pmod{n}$ and $(a, b, c, r) \in \mathcal{D}$ induce the Dunwoody 3-manifolds, denoted by $D_n(a, b, c, r, s)$, as closed orientable 3-manifolds, where s , p , and q are some integers defined in [4, 8]. Thus, the Dunwoody 3-manifold $D_n(a, b, c, r, s)$ is the n -fold strongly cyclic covering space of a lens space M branched over the Dunwoody (1, 1) knot $K(a, b, c, r)$.

The Dunwoody 3-manifolds play an important role in determining which cyclically presented group corresponds to a 3-manifold; see [13, 14]. Indeed, in order to study a 3-manifold with some particular group as the fundamental group, the Dunwoody 3-manifolds have a Heegaard diagram from which one can obtain a presentation for the group. Thus, to find the Dunwoody 3-manifolds, one must seek a cyclically presented group associated with a 3-manifold. Furthermore, as in [15], the branched covering spaces of the spatial Θ -curves containing (1, 1)-knots as the constituent knots are related to the Dunwoody 3-manifolds. Therefore, the concept of the Dunwoody 3-manifolds is important in knots, branched coverings, and graph theories. Moreover, recent manuscripts of the Dunwoody 3-manifolds can be found in [3, 8, 10, 12, 16–19].

Moreover, the (1, 1)-decompositions obtained from the Dunwoody 3-manifolds are very useful for transformations of the Heegaard diagrams. In this paper, we denote the dual decomposition of $D(a, b, c, r)$ by $D'(a, b, c, r)$ or $D(a', b', c', r')$, which has the Heegaard diagram (V_2, β_1) of (V_2, K_2) , and the mirror decomposition of $D(a, b, c, r)$ by $D^*(a, b, c, r)$ or $D(a^*, b^*, c^*, r^*)$. The Dunwoody 3-manifolds obtained from the dual and mirror decompositions of $D(a, b, c, r)$ are called the dual and mirror images of the Dunwoody 3-manifold $D_n(a, b, c, r, s)$, respectively, where $s \in \mathbb{Z}$. For a specific example, see [10].

In Section 2, we show that the defined regions in the Dunwoody (1, 1)-decompositions are divided by inessential and essential subregions. Moreover, we show that the regions in the Dunwoody (1, 1)-decompositions representing all lens spaces containing $\mathbb{S}^2 \times \mathbb{S}^1$ are connected. In Section 3, we introduce an algorithm to obtain $D'(a, b, c, r)$ from $D(a, b, c, r)$ by using matrices related to the certain sequences of $D(a, b, c, r)$. Moreover, we introduce an algorithm to obtain the mirror (1, 1)-decomposition $D^*(a, b, c, r)$ as the mirror image of $D(a, b, c, r)$ under an orientation-reversing autohomeomorphism by using matrices. Thus, our results generalize some results in [7, 10], classifying the types of the Dunwoody (1, 1)-decompositions representing 2-bridge knots.

2. On the Regions of $D(a, b, c, r)$

We assume that l_1 is the simple closed curve of $D(a, b, c, r)$, and that $R = \text{cl}(\partial V_1 - l_1)$ is the region of $D(a, b, c, r)$. Then, $R = \text{cl}(\partial V_1 - l_1)$ is a disconnected region if l_1 is the inessential loop in the surface ∂V_1 , and otherwise, $R = \text{cl}(\partial V_1 - l_1)$ is a connected region. Therefore, there are three kinds of loops on the ∂V_1 : (1) l_1 is the inessential loop, (2) l_1 is the meridian loop, and (3) l_1 is the loop that is isotopic to torus

knot. In case of (1), M is denoted by $L(0, 0)$; of (2), M is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and denoted $L(0, 1)$; and of (3), it is homeomorphic to the lens space or the 3-sphere and denoted by $L(|p|, *)$, where p is the integer defined in [4]. In fact, M is homeomorphic to the 3-sphere if and only if R is connected and $|p| = 1$.

Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$ -decomposition of $(M, K) = (V_1, K_1) \cup_\phi (V_2, K_2)$. Since $|\beta\alpha| = 2$, the number of simple closed curves on ∂V_1 is one, denoted by $L = 1$, and thus, M is to be a lens space [3]. Since d and p have the same parity, the Dunwoody $(1, 1)$ -decomposition $D(a, b, c, r)$ with even $d = 2a + b + c$ is representing the lens space containing $\mathbb{S}^2 \times \mathbb{S}^1$ (for $p = 0$), but it is not representing 3-sphere even if R is connected; see Corollary 3.6 in [3]. For example, the region R of $D(1, 0, 0, 2)$ with $p = 0$ and $q = 0$ is disconnected and both $D(1, 0, 0, 1)$ and $D(1, 1, 1, 2)$ with $p = 0$ have a connected region and represent $\mathbb{S}^2 \times \mathbb{S}^1$.

Lemma 1. *If the region R is disconnected, both $d = 2a + b + c$ and r of $D(a, b, c, r)$ have even numbers.*

Proof. The Dunwoody $(1, 1)$ -decomposition $D(a, b, c, r)$ with disconnected R is obtained from $D(1, 0, 0, 2)$ by isotopic moves of inessential loop l_1 on ∂V_1 . Thus, $d = 2a + b + c$ and r are the result of an increase by an even number from $d = 2$ and $r = 2$ of $D(1, 0, 0, 2)$. Thus, both $d = 2a + b + c$ and r of $D(a, b, c, r)$ have even numbers. $\square$

Recall that each cycle of α corresponds to the end points of line segments in the Heegaard diagram as in Figure 1 and each cycle of β corresponds to a pair of end points which is identified with m^+ and m^- . In Figure 1, we consider a cyclic sequence $\vec{m}$ on m^+ with starting point 1 and a cyclic sequence $\vec{l}$ of points with starting point d that l_1 and m^+ cross by repeated applications of α and β as follows:

$$\begin{aligned} \vec{m}: 1 &\longrightarrow 2 \longrightarrow \cdots \longrightarrow r \longrightarrow r+1 \cdots \longrightarrow d-1 \longrightarrow d \longrightarrow, \\ \vec{l}: d &\longrightarrow \cdots \longrightarrow r+1 \longrightarrow r \longrightarrow \cdots \longrightarrow 1 \longrightarrow, \\ d &\longrightarrow \cdots \longrightarrow r \longrightarrow r+1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow. \end{aligned} \quad (2)$$

A subregion of R is said to be inessential in ∂V_1 if the subregion is bounded by $r \rightarrow r+1$ of $\vec{m}$ and $r \rightarrow r+1$ of $\vec{l}$ as Figure 2, and otherwise, the subregion is said to be essential in ∂V_1 .

Theorem 2. *Let R be the region of $D(a, b, c, r)$. Then, R has the inessential subregion if and only if R is disconnected.*

Proof. Suppose that the region R of $D(a, b, c, r)$ has the inessential subregion. Then, the simple closed curve l_1 is isotopic to an inessential loop on ∂V_1 . By Lemma 1, R is disconnected. Now assume that the region R of $D(a, b, c, r)$ is disconnected. By Lemma 1, d and r are even numbers which are obtained by increasing as even numbers from $d = 2$ and $r = 2$ of $D(1, 0, 0, 2)$. Thus, R has the inessential subregion. $\square$

FIGURE 2: The inessential subregion.

FIGURE 3: The partition of the Dunwoody $(1, 1)$ -decompositions.

Corollary 3. *Let R be the region of $D(a, b, c, r)$. Then, R has the essential subregion if and only if R is connected.*

Corollary 4. *Let p and q be the integers defined in [4] and R the region of $D(a, b, c, r)$. If R is a disconnected region, then both p and q are equal to zero.*

For counterexample, the region of $D(3, 2, 6, 7)$ has $p = 0$ and $q = 0$, but it has no inessential subregion; that is, it is the connected region.

Therefore, we have the partition of the Dunwoody $(1, 1)$ -decompositions as the following diagram in Figure 3. In Figure 3, A is a set of the Dunwoody $(1, 1)$ -decompositions having a disconnected region, $A \cup B$ is a set of the Dunwoody $(1, 1)$ -decompositions satisfying $p = 0$ and $q = 0$, and $B \cup C$ is a set of the Dunwoody $(1, 1)$ -decompositions having a connected region. Note that $D(1, 6, 4, 2)$, $D(3, 2, 6, 7)$, and $D(2, 3, 3, 2)$ are examples for A , B , and C , respectively.

In the remainder of this paper, we work in the connected region R unless otherwise specified. In the connected region R , we assume two cyclic sequences $\vec{m}$ and $\vec{l}$ as follows:

$$\begin{aligned} \vec{m}: 1 &\longrightarrow 2 \longrightarrow \cdots \longrightarrow r \longrightarrow r+1 \cdots \longrightarrow d-1 \longrightarrow d \longrightarrow, \\ \vec{l}: d &\longrightarrow \cdots \longrightarrow r+1 \longrightarrow r \longrightarrow \cdots \longrightarrow 1 \longrightarrow. \end{aligned} \quad (**)$$

FIGURE 4: $x-1 \rightarrow x \rightarrow x+1$ and $y_1 \rightarrow x \rightarrow y_2$ for each $x \in X^+$.

3. The Dual and Mirror Images of $D(a, b, c, r)$

We introduce an algorithm to obtain $(V_2; \beta_1)$ from $(V_1; \alpha_1)$ as follows. Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$ -decomposition of $(M, K) = (V_1, K_1) \cup_{\phi} (V_2, K_2)$. Suppose that $D(a, b, c, r)$ is the $(1, 1)$ -decomposition in view of (V_1, K_1) as Figure 1 and that $Du(a, b, c, r) = D(a', b', c', r')$ is the dual decomposition in view of (V_2, K_2) . Let m and l_1 be the meridian disk and the oriented simple closed curve on $D(a, b, c, r)$. Then, it is understood easily from the attaching homeomorphism ϕ that the images of l_1 and m on $D(a, b, c, r)$ are defined as the meridian curve m' and the oriented simple closed curve l'_1 on $D(a', b', c', r')$, respectively.

We consider a cyclic sequence $\vec{m}$ on m^+ with starting point 1 and a cyclic sequence $\vec{l}$ of points with starting point $d - l_1$ and m^+ cross by repeated applications of two permutations α and β defined in [3, 8, 10] as $(**)$. Let $X^+ = \{1, 2, \dots, d\}$ and $X^- = \{-1, -2, \dots, -d\}$ be sets of d vertices in m^+ and m^- (see Figure 1), respectively. For each $x \in X^+$, let $x-1 \rightarrow x \rightarrow x+1$ be a consecutive 3-series in $\vec{m}$ and $y_1 \rightarrow x \rightarrow y_2$ a consequence 3-series in $\vec{l}$. Then, x has two 2×1 matrices such that if x is the beginning point of a curve corresponding to α (resp., β); then, we define $x-1$ (resp., y_1) as $(1, 1)$ -entries in 2×1 matrix and $x+1$ (resp., y_2) as $(2, 1)$ -entries, and otherwise (i.e., x is the final point), we define $x-1$ (resp., y_1) as $(2, 1)$ -entries and $x+1$ (resp., y_2) as $(1, 1)$ -entries; see Figure 4 for each of (1) and (2). In fact, for each point x , in case of (1), two permutations α and β induce 2×1 matrices denoted by

$$\alpha \begin{bmatrix} x \\ x-1 \\ x+1 \end{bmatrix} \quad \beta \begin{bmatrix} x \\ y_2 \\ y_1 \end{bmatrix}, \quad (3)$$

and in case of (2), two permutations α and β induce

$$\alpha \begin{bmatrix} x \\ x+1 \\ x-1 \end{bmatrix} \quad \beta \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix}. \quad (4)$$

In particular, 1 has 2×1 matrix such that d and 2 are $(1, 1)$ -entries and $(2, 1)$ -entries, respectively, and d has 2×1 matrix

such that $d-1$ and 1 are $(2, 1)$ -entries and $(1, 1)$ -entries, respectively. The 2×1 matrices are denoted by

$$\alpha \begin{bmatrix} 1 \\ d \end{bmatrix} \quad \alpha \begin{bmatrix} d \\ d-1 \end{bmatrix}. \quad (5)$$

Therefore, we have $2 \times d$ matrix M_m corresponding to a cyclic sequence $\vec{m}$ along α as follows:

$$M_m = \begin{bmatrix} 1 & \cdots & x & \cdots & d \\ d & \cdots & x-1 & \cdots & 1 \\ 2 & \cdots & x+1 & \cdots & d-1 \end{bmatrix}, \quad (6)$$

$$M_m = \begin{bmatrix} 1 & \cdots & x & \cdots & d \\ d & \cdots & x+1 & \cdots & 1 \\ 2 & \cdots & x-1 & \cdots & d-1 \end{bmatrix}.$$

Considering a sequence $\vec{l}$ and β , $r+1$ has 2×1 matrix such that r is $(2, 1)$ -entries, and r has 2×1 matrix such that $r+1$ is $(2, 1)$ -entries. Therefore, we have $2 \times d$ matrix M_l corresponding to β for a cyclic sequence $\vec{l}$ as follows:

$$M_l = \begin{bmatrix} d & \cdots & r+1 & r & \cdots & y & \cdots & 1 \\ 1 & \cdots & * & * & \cdots & y_1 & \cdots & d \\ * & \cdots & r & r+1 & \cdots & y_2 & \cdots & * \end{bmatrix}, \quad (7)$$

$$M_l = \begin{bmatrix} d & \cdots & r+1 & r & \cdots & y & \cdots & 1 \\ 1 & \cdots & * & * & \cdots & y_2 & \cdots & d \\ * & \cdots & r & r+1 & \cdots & y_1 & \cdots & * \end{bmatrix}.$$

Theorem 5. Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$ -decomposition and $d = 2a + b + c$. Then, there exist two $2 \times d$ matrices M_m and M_l such that

- (1) the number of 2×2 submatrices $\begin{bmatrix} y & y' \\ y' & y \end{bmatrix}$ and $\begin{bmatrix} y & y' \\ * & * \end{bmatrix}$ in M_l corresponding to consecutive 2-series $y \rightarrow y'$ of $\vec{l}$ is equal to $2a$,
- (2) let $x = r - a$ be a point of l meeting m^- and $y_1 \rightarrow x \rightarrow y_2$ consecutive 3-series in $\vec{l}$; then, $b = y_2 - a$ if $\begin{bmatrix} x \\ y_1 \end{bmatrix}$, and otherwise, $b = y_1 - a$,
- (3) $c = d - 2a - b$.

Proof. (1) Let M_m and M_l be $2 \times d$ matrices corresponding to $\vec{m}$ and l along α and β , respectively. Then, 2×2 submatrices $\begin{bmatrix} y & y' \\ y' & y \end{bmatrix}$ and $\begin{bmatrix} y & y' \\ * & * \end{bmatrix}$ of M_l are representing a curves of l meeting m^+ and m^- . Thus, the number of such submatrices of M_l is $2a$.

(2) Given r and a , the point $x = r - a$ in m^- is corresponding to the point $a + b$ in m^+ by α . If the corresponding submatrix is $\begin{bmatrix} x \\ y_1 \end{bmatrix}$, then $y_1 \rightarrow x \rightarrow a + b$ and x is the beginning point of β . If the corresponding submatrix is $\begin{bmatrix} x \\ y_2 \end{bmatrix}$, then $a + b \rightarrow x \rightarrow y_2$, and x is the terminal point of β . These are determined by an orientation of the curve l . For each of the cases, $a + b$ is the point in m^+ , so $(a + b) - a$ means the number of curves of l meeting m^+ .

- (3) It is proved from the fact that $d = 2a + b + c$. $\square$

Let m and l be the meridian disk and the oriented simple closed curve on $D(a, b, c, r)$ corresponding to $(V_1; \alpha_1)$, respectively. Then, $\phi(l)$ and $\phi(m)$ are the meridian disk and the orientable closed curve on $Du(a, b, c, r)$ corresponding to $(V_2; \beta_1)$, respectively, where ϕ is an attaching homeomorphism for the $(1, 1)$ -decomposition $(V_1, K_1) \cup_{\phi} (V_2, K_2)$. We obtain $D(a', b', c', r')$ from $D(a, b, c, r)$ by the following facts and the properties of Theorem 5.

By the definition of the dual process, the role of $\{m, l\}$ will be changed into $\{\phi(m) = l', \phi(l) = m'\}$ on $Du(a, b, c, r)$.

Thus, we obtain two cyclic sequences $\overrightarrow{m'}$ and $\overrightarrow{l'}$ from $\overrightarrow{l}$ and $\overrightarrow{m}$, respectively, for $Du(a, b, c, r)$ as in the following:

$$\begin{aligned} \overrightarrow{m'}: d \longrightarrow \cdots \longrightarrow r+1 \longrightarrow r \longrightarrow \cdots \longrightarrow 1 \longrightarrow, \\ \overrightarrow{l'}: 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow r \longrightarrow r+1 \cdots \longrightarrow d-1 \longrightarrow d \longrightarrow. \end{aligned} \quad (8)$$

Moreover, two matrices $M_{m'}$ and $M_{l'}$ for $\overrightarrow{m'}$ and $\overrightarrow{l'}$ can be obtained from M_l and M_m , respectively. If 1 in $\overrightarrow{m}$, in the bigon bounded by $(1, d)$, is a starting point in $D(a, b, c, r)$, then d in $\overrightarrow{m'}$ will be starting point in $Du(a, b, c, r)$. Moreover, the number r' is equal to the number situated on $r+1$ after converting d on the sequence $\overrightarrow{m'}$ to 1. Thus, we can rearrange terms of $\overrightarrow{m'}$ and $\overrightarrow{l'}$ as follows.

- (1) We replace each term of $\overrightarrow{m'}$ with terms of an arithmetic sequence with first item 1, common difference 1, and final term d and denote it by $\overrightarrow{m''}$.
- (2) Furthermore, we rearrange terms of $\overrightarrow{l'}$, $M_{m'}$, and $M_{l'}$ by new terms along the rule of (1) and denote each of them by $\overrightarrow{l''}$, $M_{m''}$ and $M_{l''}$.
- (3) r' th term of $\overrightarrow{m''}$ will be the number $r+1$ of $\overrightarrow{m'}$.

The following corollary follows from the properties of Theorem 5.

Corollary 6. Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$ -decomposition and $d = 2a + b + c$. Then, the dual $(1, 1)$ -decomposition of $D(a, b, c, r)$ and $D(a', b', c', r')$ is obtained by the following:

- (1) the number of 2×2 submatrices $\begin{bmatrix} y & y' \\ y' & y \end{bmatrix}$ or $\begin{bmatrix} y & y' \\ * & * \end{bmatrix}$ in $M_{l''}$ corresponding to consecutive 2-series $y \rightarrow y'$ of $\overrightarrow{l''}$ is equal to $2a'$;
- (2) let $x = r' - a'$ and $y_1 \rightarrow x \rightarrow y_2$ consecutive 3-series in $\overrightarrow{l''}$. Then, $b' = y_2 - a'$ if $\begin{bmatrix} x \\ y_1 \end{bmatrix}$ and otherwise $b' = y_1 - a'$;
- (3) $c' = d - 2a' - b'$.

On the contrary, one can obtain $D(a, b, c, r)$ from $D(a', b', c', r')$ by the dual process of the above.

We denote by $D^*(a, b, c, r) = D(a^*, b^*, c^*, r^*)$ the mirror image of $D(a, b, c, r)$ under an orientation-reversing auto-homeomorphism g , where we orient $D^*(a, b, c, r)$ with orientation induced from $D(a, b, c, r)$ by g . Then, the orientations of l , m^+ , and m^- may be changed the other way in $D(a^*, b^*, c^*, r^*)$, and so the role of 1 in m^+ will be changed into -1 in m^- . Considering a cyclic sequence $\overleftarrow{m}$ on m^- with starting point -1 and a cyclic sequence $\overleftarrow{l}$ of points with starting point $-d$; then, l^* and m^- cross by repeated applications of α and β of $(*)$ as follows

$$\begin{aligned} \overleftarrow{m}: -1 \longrightarrow \cdots \longrightarrow -r \longrightarrow -(r+1) \longrightarrow \cdots \longrightarrow -d \longrightarrow \\ \overleftarrow{l}: -d \longrightarrow \cdots \longrightarrow -(r+1) \longrightarrow -r \longrightarrow \cdots \longrightarrow -1 \longrightarrow. \end{aligned} \quad (9)$$

By definition of β , it is equivalent to

$$\begin{aligned} \overleftarrow{m^*}: \beta(r+1) \longrightarrow \cdots \longrightarrow \beta(d) \longrightarrow \beta(1) \longrightarrow \cdots \longrightarrow \beta(r) \longrightarrow \\ \overleftarrow{l^*}: \beta(r) \longrightarrow \cdots \longrightarrow \beta(1) \longrightarrow \beta(d) \longrightarrow \cdots \longrightarrow \beta(r+1) \longrightarrow. \end{aligned} \quad (10)$$

Putting $\beta(r+1) = 1$ as a starting point in $\overleftarrow{m^*}$, we can easily find r^* such that $r^* = \beta(d) = d - r$.

Corollary 7. Let $D(a, b, c, r)$ be the Dunwoody $(1, 1)$ -decomposition and $d = 2a + b + c$. Then, the mirror image of $D(a, b, c, r)$ is $D(a, c, b, d - r)$, which is obtained from the corresponding matrices as follows:

$$\begin{aligned} M_{m^*} = \begin{bmatrix} r+1 & \cdots & d & 1 & \cdots & r \\ r & \cdots & d-1 & 2 & \cdots & r+1 \\ r+2 & \cdots & 1 & d & \cdots & r-1 \end{bmatrix}, \\ M_{l^*} = \begin{bmatrix} r & \cdots & 1 & d & \cdots & r+1 \\ r+1 & \cdots & * & * & \cdots & r \\ * & \cdots & d & 1 & \cdots & * \end{bmatrix}, \end{aligned} \quad (11)$$

where $a^* = a$, $b^* = c$, and $c^* = b$.

Remark that another proof which represents the mirror image was obtained in [17]. Here, we give the canonical examples in the following.

Example 8. Let $D(1, 1, 6, 6)$ be the Dunwoody $(1, 1)$ -decomposition, in fact, it is representing the torus knot $T(3, 7)$ in $\mathbb{S}^3$. Then, the following are cyclic sequences corresponding to m and l , respectively, for $D(1, 1, 6, 6)$:

$$\begin{aligned} \overrightarrow{m}: 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8 \longrightarrow 9 \longrightarrow, \\ \overrightarrow{l}: 9 \longrightarrow 4 \longrightarrow 8 \longrightarrow 3 \longrightarrow 7 \longrightarrow 6 \longrightarrow 2 \longrightarrow 5 \longrightarrow 1 \longrightarrow. \end{aligned} \quad (12)$$

If we replace $\vec{l}$ and $\vec{m}$ with $\vec{m}'$ and $\vec{l}'$, respectively, then

$$\begin{aligned}\vec{m}' : 9 \rightarrow 4 \rightarrow 8 \rightarrow 3 \rightarrow 7 \rightarrow 6 \rightarrow 2 \rightarrow 5 \rightarrow 1 \rightarrow, \\ \vec{l}' : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow\end{aligned}\quad (13)$$

are cyclic sequences for $Du(1, 1, 6, 6)$. Replacing 9 by 1 as a starting point in $\vec{m}'$, we can rearrange $\vec{m}'$ and $\vec{l}'$ as follows:

$$\begin{aligned}\vec{m}'' : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow, \\ \vec{l}'' : 9 \rightarrow 7 \rightarrow 4 \rightarrow 2 \rightarrow 8 \rightarrow 6 \rightarrow 5 \rightarrow 3 \rightarrow 1 \rightarrow.\end{aligned}\quad (14)$$

The corresponding matrices of them are

$$\begin{aligned}M_{m''} &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 2 & 3 & 4 & 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 & 6 & 5 & 6 & 7 & 8 \end{bmatrix}, \\ M_{l''} &= \begin{bmatrix} 9 & 7 & 4 & 2 & 8 & 6 & 5 & 3 & 1 \\ 1 & 9 & 2 & 8 & 2 & 8 & 3 & 1 & 9 \\ 7 & 4 & 7 & 4 & 6 & 5 & 6 & 5 & 3 \end{bmatrix}.\end{aligned}\quad (15)$$

Since $M_{l''}$ has the four submatrices as follows:

$$\begin{bmatrix} 1 & 9 \\ 9 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 8 \\ 8 & 2 \end{bmatrix}, \quad \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}, \quad \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}, \quad (16)$$

we have $a' = 2$ by Corollary 6. The following is a result of Corollary 6. Since 7 of $\vec{m}'$ corresponds to 5 of $\vec{m}''$; thus, $r' = 5$. From $a' = 2$ and $r' = 5$, we have $x = 5 - 2 = 3$. For $5 \rightarrow 3 \rightarrow 1$ in $M_{l''}$, we have $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, so $b' = 5 - 2 = 3$. Thus, $c' = 2$, and we have $Du(1, 1, 6, 6) = D(2, 3, 2, 5)$.

Example 9. To obtain the mirror image of $D(1, 1, 6, 6)$, we consider two cyclic sequences

$$\begin{aligned}\overleftarrow{m}^* : 7 \rightarrow 8 \rightarrow 9 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow, \\ \overleftarrow{l}^* : 6 \rightarrow 2 \rightarrow 5 \rightarrow 1 \rightarrow 9 \rightarrow 4 \rightarrow 8 \rightarrow 3 \rightarrow 7 \rightarrow,\end{aligned}\quad (17)$$

which are images of β of cyclic sequences $\overleftarrow{m}$ and $\overleftarrow{l}$ for crossing of $\overleftarrow{m}^*$ and $\overleftarrow{l}^*$, where each number x in cyclic sequences is equal to $\beta(x)$. Then, the corresponding matrices of $\overleftarrow{m}^*$ and $\overleftarrow{l}^*$ are

$$\begin{aligned}M_{m^*} &= \begin{bmatrix} 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 2 & 3 & 2 & 3 & 6 & 7 \\ 8 & 9 & 1 & 9 & 1 & 4 & 5 & 4 & 5 \end{bmatrix}, \\ M_{l^*} &= \begin{bmatrix} 6 & 2 & 5 & 1 & 9 & 4 & 8 & 3 & 7 \\ 7 & 6 & 2 & 5 & 4 & 8 & 3 & 7 & 6 \\ 2 & 5 & 1 & 9 & 1 & 9 & 4 & 8 & 3 \end{bmatrix}.\end{aligned}\quad (18)$$

TABLE 1: $D(a, b, c, r)$ and $D(a', b', c', r')$.

d	$D(a, b, c, r)$	$D(a', b', c', r')$	$(1, 1)$ -knot	$D_2(a, b, c, r)$
3	(1, 0, 1, 1)	(1, 0, 1, 2)	$b(3, 1)$	$b(3, 1)$
5	(1, 0, 3, 2)	(2, 0, 1, 2)	$b(5, 1)$	$b(5, 1)$
5	(1, 1, 2, 2)	(1, 2, 1, 3)	$t(3, 4)$	$t(3, 4)$
5	(2, 0, 1, 1)	(2, 0, 1, 4)	$t(5, 2)$	$t(5, 2)$
7	(1, 0, 5, 5)	(3, 0, 1, 4)	$b(7, 1)$	$b(7, 1)$
7	(1, 1, 4, 5)	(2, 1, 2, 3)	$t(3, 5)$	$t(3, 5)$
7	(1, 2, 3, 3)	(1, 3, 2, 4)	$t(4, 5)$	$p(-2, 5, 5)$
7	(2, 0, 3, 4)	(3, 0, 1, 5)	$b(7, 2)$	$b(7, 2)$
7	(3, 0, 1, 1)	(3, 0, 1, 6)	$b(7, 3)$	$b(7, 3)$
9	(1, 0, 7, 2)	(4, 0, 1, 4)	$b(9, 1)$	$b(9, 1)$
9	(1, 1, 6, 6)	(2, 3, 2, 5)	$t(3, 7)$	$p(-2, 3, 7)$
9	(1, 3, 4, 7)	(3, 1, 2, 4)	$p(-2, 3, 7)$	$p(-2, 3, 7)$
9	(2, 1, 4, 4)	(2, 1, 4, 6)	$p(-2, 3, 7)$	$t(3, 7)$
9	(1, 3, 4, 4)	(1, 4, 3, 5)	$t(5, 6)$	$p(-2, 5, 5)$
9	(2, 0, 5, 4)	(4, 0, 1, 2)	$b(9, 2)$	$b(9, 2)$
9	(4, 0, 1, 1)	(4, 0, 1, 8)	$b(9, 4)$	$b(9, 4)$
9	(2, 2, 3, 5)	(3, 1, 2, 2)	9_{42}	9_{42}
9	(3, 0, 3, 5)	(3, 0, 3, 7)	8_{20}	8_{20}

If we replace 7 with 1 in order to rearrange follows

$$\begin{aligned}M_{m^*} &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 2 & 5 & 6 & 5 & 6 & 9 & 1 \\ 2 & 3 & 4 & 3 & 4 & 7 & 8 & 7 & 8 \end{bmatrix}, \\ M_{l^*} &= \begin{bmatrix} 9 & 5 & 8 & 4 & 3 & 7 & 2 & 6 & 1 \\ 1 & 9 & 5 & 8 & 7 & 2 & 6 & 1 & 9 \\ 5 & 8 & 4 & 3 & 4 & 3 & 7 & 2 & 6 \end{bmatrix},\end{aligned}\quad (19)$$

then 9 is to be 3, and so $r^* = 3$; that is, $r^* = \beta(9)$. Using (1), (2), and (3) of Corollary 6 for M_{m^*} and M_{l^*} , we can determine that $a^* = 1$, $b^* = 6$, and $c^* = 1$; that is, $D^*(1, 1, 6, 6) = D(1, 6, 1, 3)$.

Remark 10. As the same problem for $D(3, 1, 2, 4)$, we have $Du(3, 1, 2, 4) = D(1, 3, 4, 7)$ and $D^*(3, 1, 2, 4) = D(3, 2, 1, 5)$.

Summarizing, we have the following.

Corollary 11. For any $(1, 1)$ -knot K in the lens space, the Dunwoody $(1, 1)$ -decompositions for K commute as follows:

$$\begin{aligned}D(a, b, c, r) &\longleftrightarrow D(a', b', c', r') \\ \updownarrow &\quad \updownarrow \\ D(a, c, b, d - r) &\longleftrightarrow D(a', c', b', d - r') \\ &\text{or } Du(a, c, b, d - r).\end{aligned}\quad (20)$$

Moreover, the Dunwoody 3-manifolds obtained from the Dunwoody $(1, 1)$ -decompositions commute, for some $s \in \mathbb{Z}$,

$$\begin{aligned}D_n(a, b, c, r, s) &\longleftrightarrow D_n(a', b', c', r', s') \\ \updownarrow &\quad \updownarrow \\ D_n(a, c, b, (d - r), (-s - 1)) &\longleftrightarrow D_n(a', c', b', (d - r'), (-s - 1)'),\end{aligned}\quad (21)$$

where $n \geq 1$.

Lastly, we consider two presentations to be equivalent if they differ by an isomorphism. Then, the isomorphism problem is hard because the map from the presentation to the Heegaard diagram is not one-to-one. So the isomorphism problem is obviously not solvable, but we can determine that at least four presentations in Corollary 11 are equivalent; that is, they are group presentations of the fundamental group for a closed orientable 3-manifold. Moreover, results in this paper show that there are at least four Dunwoody $(1, 1)$ -decompositions representing a given $(1, 1)$ -knot. Table 1 shows the Dunwoody $(1, 1)$ -decompositions and their dual decompositions of $(1, 1)$ -knots in $\mathbb{S}^3$ by $d \leq 9$, and $b(\cdot, \cdot)$, $t(\cdot, \cdot)$, $p(\cdot, \cdot, \cdot)$, and $\{8_{20}, 9_{42}\}$ means a 2-bridge knot, a torus knot, a pretzel knot, and knots classified by Rolfsen table, respectively. The last column in Table 1 represents a 3-bridge knot induced by the standard involution on $D_2(a, b, c, r, s)$ for some $s \in \mathbb{Z}$ [15].

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