Research Article
On Cartesian Products of Orthogonal Double Covers

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Let H be a graph on n vertices and $\mathcal{G}$ a collection of n subgraphs of H, one for each vertex, where $\mathcal{G}$ is an orthogonal double cover (ODC) of H if every edge of H occurs in exactly two members of $\mathcal{G}$ and any two members share an edge whenever the corresponding vertices are adjacent in H and share no edges whenever the corresponding vertices are nonadjacent in H. In this paper, we are concerned with the Cartesian product of symmetric starter vectors of orthogonal double covers of the complete bipartite graphs and using this method to construct ODCs by new disjoint unions of complete bipartite graphs.

1. Introduction

For the definition of an orthogonal double cover (ODC) of the complete graph K_n by a graph G and for a survey on this topic, see [1]. In [2], this concept has been generalized to ODCs of any graph H by a graph G.

While in principle any regular graph is worth considering (e.g., the remarkable case of hypercubes has been investigated in [2]), the choice of $H = K_{n,n}$ is quite natural, and also in view of a technical motivation, ODCs of such graphs are a helpful tool for constructing ODCs of K_n (see [3, page 48]).

In this paper, we assume $H = K_{n,n}$, the complete bipartite graph with partition sets of size n each.

An ODC of $K_{n,n}$ is a collection $\mathcal{G} = \{G_0, G_1, \ldots, G_{n-1}, F_0, F_1, \ldots, F_{n-1}\}$ of $2n$ subgraphs (called *pages*) of $K_{n,n}$ such that

   (i) every edge of $K_{n,n}$ is in exactly one page of $\{G_0, G_1, \ldots, G_{n-1}\}$ and in exactly one page of $\{F_0, F_1, \ldots, F_{n-1}\}$;

   (ii) for $i, j \in \{0, 1, 2, \ldots, n-1\}$ and $i \neq j$, $E(G_i) \cap E(G_j) = E(F_i) \cap E(F_j) = \emptyset$; and $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{0, 1, 2, \ldots, n-1\}$.

If all the pages are isomorphic to a given graph G, then $\mathcal{G}$ is said to be an ODC of $K_{n,n}$ by G.

Denote the vertices of the partition sets of $K_{n,n}$ by $\{0_0, 1_0, \ldots, (n-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$. The length of an edge $x_0 y_1$ of $K_{n,n}$ is defined to be the difference $y - x$, where

$x, y \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Note that sums and differences are calculated in $\mathbb{Z}_n$ (i.e., sums and differences are calculated modulo n).

Throughout the paper we make use of the usual notation: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle on n vertices, K_n for the complete graph on n vertices, K_1 for an isolated vertex, $G \cup H$ for the disjoint union of G and H, and mG for m disjoint copies of G.

An algebraic construction of ODCs via "symmetric starters" (see Section 2) has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$, a few exceptions apart, all graphs G are found this way (see [3, Table 1]). This method has been applied in [3, 4] to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G.

In [5], Scapellato et al. studied the ODCs of Cayley graphs and they proved the following. (i) All 3-regular Cayley graphs, except K_4, have ODCs by P_4. (ii) All 3-regular Cayley graphs on Abelian groups, except K_4, have ODCs by $P_3 \cup K_2$. (iii) All 3-regular Cayley graphs on Abelian groups, except K_4 and the 3-prism (Cartesian product of C_3 and K_2), have ODCs by $3K_2$.

Much research on this subject focused on the detection of ODCs with pages isomorphic to a given graph G. For a summary of results on ODCs, see [1, 4]. The other terminologies not defined here can be found in [6].

2. Symmetric Starters

All graphs here are finite, simple, and undirected. Let $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order n. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{w_i, u_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $w, u \in \Gamma$ and $i, j \in \mathbb{Z}_2$. If there is no chance of confusion, (w, u) will be written instead of $\{w_0, u_1\}$ for the edge between the vertices w_0, u_1.

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph $G + a$ with $E(G + a) = \{(u + a, v + a) : (u, v) \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$.

G is called a *half starter* with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are mutually distinct; that is, $\{d(e) : e \in E(G)\} = \Gamma$. The following three results were established in [3].

Theorem 1. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$; that is, $\bigcup_{a \in \Gamma} E(G + a) = E(K_{n,n})$.*

Hereafter, a half starter G will be represented by the vector $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i in G.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be *orthogonal* if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2. *If two half starter vectors $v(G_0)$ and $v(G_1)$ are orthogonal, then $G = \{G_{a,i} : (a, i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{(u_0, v_1) : (v_0, u_1) \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a *symmetric starter* with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 3. *Let n be a positive integer and let G be a half starter represented by the vector $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

The above results on ODCs of graphs motivated us to consider ODCs of $K_{mn,mn}$ if we have the ODCs of $K_{n,n}$ by G and ODCs of $K_{m,m}$ by H where G, H are symmetric starters. In this paper, we have settled the existence problem of ODCs of $K_{mn,mn}$ by few infinite families of graphs presented in the next section.

3. The Main Results

In the following, if there is no danger of ambiguity, if $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_m$ we can write (i, j) as ij.

Theorem 4. *The Cartesian product of any two symmetric starter vectors is a symmetric starter vector with respect to the Cartesian product of the corresponding groups.*

Proof. Let $v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n$ be a symmetric starter vector of an ODC of $K_{n,n}$ by G with respect to $\mathbb{Z}_n$, then

$$\{v_i - v_{-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n. \tag{1}$$

Let $u(H) = (u_0, u_1, \ldots, u_{m-1}) \in \mathbb{Z}_m^m$ be a symmetric starter vector of an ODC of $K_{m,m}$ by H with respect to $\mathbb{Z}_m$, then

$$\{u_j - u_{-j} + j : j \in \mathbb{Z}_m\} = \mathbb{Z}_m. \tag{2}$$

Then $v(G) \times u(H) = (v_0 u_0, v_0 u_1, \ldots, v_i u_j, \ldots, v_{n-1} u_{m-1})$ where $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_m$.

From (1) and (2), we conclude

$$\begin{aligned}
&\{v_i u_j - v_{-i} u_{-j} + ij : ij \in \mathbb{Z}_n \times \mathbb{Z}_m\} \\
&= \{(v_i - v_{-i} + i)(u_j - u_{-j} + j) : i \in \mathbb{Z}_n, \\
&\qquad j \in \mathbb{Z}_m, ij \in \mathbb{Z}_n \times \mathbb{Z}_m\} \\
&= \mathbb{Z}_n \times \mathbb{Z}_m.
\end{aligned} \tag{3}$$

Then $v(G) \times u(H)$ is a symmetric starter vector of an ODC of $K_{mn,mn}$, with respect to $\mathbb{Z}_n \times \mathbb{Z}_m$, by a new graph $G \times H$ which can be described as follows.

Since $E(G) = \{(v_i, v_i + i) : i \in \mathbb{Z}_n\}$ and $E(H) = \{(u_j, u_j + j) : j \in \mathbb{Z}_m\}$, then $E(G \times H) = \{(v_i u_j, v_i u_j + ij) : ij \in \mathbb{Z}_n \times \mathbb{Z}_m\}$. It should be noted that $G \times H$ is not the usual Cartesian product of the graphs G and H that has been studied widely in the literature. □

All our results based on the following two major points:

(1) the cartesian product construction in Theorem 4,

(2) The existence of symmetric starters for a few classes of graphs that can be used as ingredients for cartesian product construction to obtain new symmetric starters. These are as follows.

   (1) $K_{1,n}$ which is a symmetric starter of an ODC of $K_{n,n}$ whose vector is $v(K_{1,n}) = (\overbrace{0, 0, 0, \ldots, 0}^{n \text{ times}}) \in \mathbb{Z}_n^n$, see Corollary 2.2.7 in [7].

   (2) $mK_{2,2}$ which is a symmetric starter of an ODC of $K_{4m,4m}$ whose vector is $v(mK_{2,2}) = (0, 1, 2, \ldots, 2m - 1, 0, 1, 2, \ldots, 2m - 1) \in \mathbb{Z}_{4m}^{4m}$, see [7, Lemma 2.2.13].

   (3) $K_{1,2} \cup K_{1,2(n-1)}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is $v(K_{1,2} \cup K_{1,2(n-1)}) = (0, \overbrace{n, n, \ldots, n, n}^{(n-1) \text{ times}}, 0, \overbrace{n, n, \ldots, n, n}^{(n-1) \text{ times}}) \in \mathbb{Z}_{2n}^{2n}$, $n \geq 2$, and it is easily checked that $v_i(K_{1,2} \cup K_{1,2(n-1)}) = v_{-i}(K_{1,2} \cup K_{1,2(n-1)})$, and hence $\{v_i - v_{-i} + i : i \in \mathbb{Z}_{2n}\} = \mathbb{Z}_{2n}$.

(4) $K_{1,2} \cup K_{2,n-1}$ which is a symmetric starter of an ODC of $K_{2n,2n}$ whose vector is $v(K_{1,2} \cup K_{2,n-1}) = (0, \overset{(n-1)\,\text{times}}{\overbrace{1, 1, 1, \ldots, 1, 1}}, n, \overset{(n-1)\,\text{times}}{\overbrace{n+1, n+1, \ldots, n+1}}) \in \mathbb{Z}_{2n}^{2n}$, $n \geq 2$, for this vector, and it is easily checked that

$$v_i - v_{-i} + i = \begin{cases} i & \text{if } i = 0, n, \text{ or} \\ n + i & \text{otherwise} \end{cases} \tag{4}$$

and hence $\{v_i - v_{-i} + i : i \in \mathbb{Z}_{2n}\} = \mathbb{Z}_{2n}$.

(5) $K_{2,3}$ which is a symmetric starter of an ODC of $K_{6,6}$ whose vector is $v(K_{2,3}) = (0, 0, 3, 3, 3, 0) \in \mathbb{Z}_6^6$, see [4, Theorem 2.2.7].

These known symmetric starters will be used as ingredients for the cartesian product construction to obtain new symmetric starters.

Theorem 5. *For all positive integers m, n with $\gcd(m, 3) = 1$, there exists an ODC of $K_{4mn,4mn}$ by $mK_{2,2n} \cup 2m(3n - 1)K_1$.*

Proof. Since $v(mK_{2,2})$ and $v(K_{1,n})$ are symmetric starter vectors, then $v(mK_{2,2}) \times v(K_{1,n})$ is a symmetric starter vector with respect to $\mathbb{Z}_{4m} \times \mathbb{Z}_n$ (Theorem 4). The resulting symmetric starter graph has the following edges set:

$$E\left(mK_{2,2n}\right)$$
$$= \bigcup_{i=0}^{m-1} \left\{ \begin{array}{l} (0i, \beta(2i)), (0i, 2\beta(m+i)), (0(m+i), \beta(2i)), \\ (0(m+i), 2\beta(m+i) : 0 \leq \beta \leq n-1 \end{array} \right\}. \tag{5}$$

Lemma 6. *For any positive integer m with $\gcd(m, 3) = 1$, there exists an ODC of $K_{16m,16m}$ by $mK_{4,4} \cup 24mK_1$.*

Proof. Since $v(K_{2,2})$ and $v(mK_{2,2})$ are symmetric starter vectors, then $v(K_{2,2}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_4 \times \mathbb{Z}_{4m}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(mK_{4,4}\right)$$
$$= \bigcup_{l=0}^{m-1} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \{(0l, 0l + ij) : ij \in \{0l, 0(l+2m), 2l, 2(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (0(l+m), 0(l+m) + ij) : \\ ij \in \{0(l+m), 0(l+3m), 2(l+m), 2(l+3m)\} \end{array} \right\} \end{array} \right\} \cup \\ \left\{ \begin{array}{l} \{(1l, 1l + ij) : ij \in \{1l, 1(l+2m), 3l, 3(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (1(l+m), 1(l+m) + ij) : \\ ij \in \{1(l+m), 1(l+3m), 3(l+m), 3(l+3m)\} \end{array} \right\} \end{array} \right\} \end{array} \right\}. \tag{6}$$

Lemma 7. *For any positive integer m with $\gcd(m, 3) = 1$, there exists an ODC of $K_{32m,32m}$ by $2mK_{4,4} \cup 48mK_1$.*

Proof. Since $v(2K_{2,2})$ and $v(mK_{2,2})$ are symmetric starter vectors, then $v(2K_{2,2}) \times v(mK_{2,2})$ is a symmetric starter vector

with respect to $\mathbb{Z}_8 \times \mathbb{Z}_{4m}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(2mK_{4,4}\right)$$
$$= \bigcup_{l=0}^{m-1} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \{(0l, 0l + ij) : ij \in \{0l, 0(l+2m), 2l, 2(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (0(l+m), 0(l+m) + ij) : \\ ij \in \{0(l+m), 0(l+3m), 2(l+m), 2(l+3m)\} \end{array} \right\} \end{array} \right\} \cup \\ \left\{ \begin{array}{l} \{(2l, 2l + ij) : ij \in \{1l, 1(l+2m), 7l, 7(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (2(l+m), 2(l+m) + ij) : \\ ij \in \{1(l+m), 1(l+3m), 7(l+m), 7(l+3m)\} \end{array} \right\} \end{array} \right\} \cup \\ \left\{ \begin{array}{l} \{(4l, 4l + ij) : ij \in \{4l, 4(l+2m), 6l, 6(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (4(l+m), 4(l+m) + ij) : \\ ij \in \{4(l+m), 4(l+3m), 6(l+m), 6(l+3m)\} \end{array} \right\} \end{array} \right\} \cup \\ \left\{ \begin{array}{l} \{(6l, 6l + ij) : ij \in \{3l, 3(l+2m), 5l, 5(l+2m)\}\} \cup \\ \left\{ \begin{array}{l} (6(l+m), 6(l+m) + ij) : \\ ij \in \{3(l+m), 3(l+3m), 5(l+m), 5(l+3m)\} \end{array} \right\} \end{array} \right\} \end{array} \right\}. \tag{7}$$

The following conjecture generalizes Lemmas 6 and 7.

Conjecture 8. *For all positive integers m, n with $\gcd(m, 3) = 1$ and $\gcd(n, 3) = 1$, there exists an ODC of $K_{16mn,16mn}$ by $nmK_{4,4} \cup 24mnK_1$.*

Theorem 9. *For all positive integers $m, n \geq 2$, there exists an ODC of $K_{2mn,2mn}$ by $K_{1,2m} \cup K_{1,2m(n-1)} \cup 2(mn - 1)K_1$.*

Proof. Since $v(K_{1,m})$ and $v(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors, then $v(K_{1,m}) \times v(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(K_{1,2m} \cup K_{1,2m(n-1)}\right)$$
$$= \bigcup_{\alpha=0}^{m-1} \{(00, \alpha\beta), (0n, \alpha\gamma) : \beta \in \{0, n\}, \gamma \in \mathbb{Z}_{2n} \setminus \{n\}\}. \tag{8}$$

Theorem 10. *For all positive integers $m, n \geq 2$, there exists an ODC of $K_{4mn,4mn}$ by $K_{1,4} \cup K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)} \cup 4(mn - 1)K_1$.*

Proof. Since $v(K_{1,2} \cup K_{1,2(m-1)})$ and $v(K_{1,2} \cup K_{1,2(n-1)})$ are symmetric starter vectors, then $v(K_{1,2} \cup K_{1,2(m-1)}) \times v(K_{1,2} \cup K_{1,2(n-1)})$ is a symmetric starter vector with respect to $\mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(K_{1,4} \cup K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}\right)$$
$$= \bigcup_{k=1, \, k \neq m}^{2m-1} \left\{ \begin{array}{l} (00, \alpha\beta), (0n, \alpha\gamma), (m0, k\beta), (mn, k\gamma) : \\ \alpha \in \{0, m\}, \beta \in \{0, n\}, \gamma \in \mathbb{Z}_{2n} \setminus \{0, n\} \end{array} \right\}. \tag{9}$$

Theorem 11. *For all positive integers $m, n \geq 2$ with $\gcd(m, 3) = 1$, there exists an ODC of $K_{8mn,8mn}$ by $mK_{2,4} \cup mK_{2,4(n-1)} \cup 4m(3n - 1)K_1$.*

Proof. Since $v(K_{1,2} \cup K_{1,2(n-1)})$ and $v(mK_{2,2})$ are symmetric starter vectors, then $v(K_{1,2} \cup K_{1,2(n-1)}) \times v(mK_{2,2})$ is a symmetric starter vector with respect to $\mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(mK_{2,4} \cup mK_{2,4(n-1)}\right)$$

$$= \bigcup_{l=0}^{m-1} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \{(0l, 0l + ij) : ij \in \{0l, 0\,(l+2m), 2l, 2\,(l+2m)\}\} \cup \\ \{ (0\,(l+m), 0\,(l+m) + ij) : \\ ij \in \{0\,(l+m), 0\,(l+3m), 2\,(l+m), 2\,(l+3m)\} \} \end{array} \right\} \cup \\ \left\{ \begin{array}{l} \{(nl, nl + ij) : ij \in \{1l, 1\,(l+2m), 3l, 3\,(l+2m)\}\} \cup \\ \{ (n\,(l+m), n\,(l+m) + ij) : \\ ij \in \{1\,(l+m), 1\,(l+3m), 3\,(l+m), 3\,(l+3m)\} \} \end{array} \right\} \end{array} \right\}.$$

(10) $\square$

Theorem 12. *For all positive integers $m, n \geq 2$, there exists an ODC of $K_{2mn,2mn}$ by $K_{2,m} \cup K_{2,m(n-1)} \cup (3mn - 4)K_1$.*

Proof. Since $v(K_{1,m})$ and $v(K_{1,2} \cup K_{2,n-1})$ are symmetric starter vectors, then $v(K_{1,m}) \times v(K_{1,2} \cup K_{2,n-1})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(K_{2,m} \cup K_{2,m(n-1)}\right)$$

$$= \bigcup_{\alpha=0}^{m-1} \{(01, \alpha\beta), (00, \alpha 0), (0n, \alpha 0),$$

(11)

$$(0\,(n+1), \alpha\beta) : 2 \leq \beta \leq n\}.$$ $\square$

Lemma 13. *For any positive integer m, there exists an ODC of $K_{6m,6m}$ by $K_{2,3m} \cup (9m - 2)K_1$.*

Proof. Since $v(K_{1,m})$ and $v(K_{2,3})$ are symmetric starter vectors, then $v(K_{1,m}) \times v(K_{2,3})$ is a symmetric starter vector with respect to $\mathbb{Z}_m \times \mathbb{Z}_6$ (Theorem 4), and the resulting symmetric starter graph has the following edges set:

$$E\left(K_{2,3m}\right) = \bigcup_{\alpha=0}^{m-1} \{(00, \alpha\beta), (03, \alpha\beta) : \beta \in \{0, 1, 5\}\}.$$ (12) $\square$

4. Conclusion

In conclusion, the known symmetric starters are used as ingredients for the cartesian product construction to obtain new symmetric starters which are $mK_{2,2n} \cup 2m(3n - 1)K_1$, $mK_{4,4} \cup 24mK_1$, $2mK_{4,4} \cup 48mK_1$, $nmK_{4,4} \cup 24mnK_1$, $K_{1,2m} \cup K_{1,2m(n-1)} \cup 2(mn-1)K_1$, $K_{1,4} \cup K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)} \cup 4(mn-1)K_1$, $mK_{2,4} \cup mK_{2,4(n-1)} \cup 4m(3n-1)K_1$, $K_{2,m} \cup K_{2,m(n-1)} \cup (3mn-4)K_1$, and $K_{2,3m} \cup (9m-2)K_1$.

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