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Research Article

On Certain Classes of Convex Functions

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For real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathscr{K}(\alpha, \beta)$ the class of normalized analytic functions which satisfy the following two sided-inequality: $\alpha < \Re\{1 + (zf''(z)/f'(z))\} < \beta$ $(z \in \mathbb{U})$, where $\mathbb{U}$ denotes the open unit disk. We find some relationships involving functions in the class $\mathscr{K}(\alpha, \beta)$. And we estimate the bounds of coefficients and solve the Fekete-Szegö problem for functions in this class. Furthermore, we investigate the bounds of initial coefficients of inverse functions or biunivalent functions.

1. Introduction

Let $\mathscr{A}$ denote the class of analytic functions in the unit disc

$$\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\} \tag{1}$$

which is normalized by

$$f(0) = 0, \qquad f'(0) = 1. \tag{2}$$

Also let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ which is composed of functions which are univalent in $\mathbb{U}$. And, as usual, we denote by $\mathscr{K}$ the class of functions in $\mathscr{A}$ which are convex in $\mathbb{U}$.

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, if and only if $f(z) = F(w(z))$ for some Schwarz function $w(z)$ such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}). \tag{3}$$

If F is univalent in $\mathbb{U}$, then the subordination $f \prec F$ is equivalent to

$$f(0) = F(0), \qquad f(\mathbb{U}) \subset F(\mathbb{U}). \tag{4}$$

Definition 1. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. The function $f \in \mathscr{A}$ belongs to the class $\mathscr{K}(\alpha, \beta)$ if f satisfies the following inequality:

$$\alpha < \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \tag{5}$$

It is clear that $\mathscr{K}(\alpha, \beta) \subset \mathscr{K}$. And we remark that, for given real numbers α and β $(0 \leq \alpha < 1 < \beta)$, $f \in \mathscr{K}(\alpha, \beta)$ if and only if f satisfies each of the following two subordination relationships:

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &\prec \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (z \in \mathbb{U}), \\ 1 + \frac{zf''(z)}{f'(z)} &\prec \frac{1 + (1 - 2\beta) z}{1 - z} \quad (z \in \mathbb{U}). \end{aligned} \tag{6}$$

Now, we define an analytic function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} z}{1 - z}\right). \tag{7}$$

The above function p was introduced by Kuroki and Owa [1], and they proved that p maps $\mathbb{U}$ onto a convex domain

$$\Lambda = \{w : \alpha < \Re\{w\} < \beta\} \tag{8}$$

conformally. Using this fact and the definition of subordination, we can obtain the following lemma, directly.

Lemma 2. *Let $f(z) \in \mathscr{A}$ and $0 \leq \alpha < 1 < \beta$. Then $f \in \mathscr{K}(\alpha, \beta)$ if and only if*

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))} z}{1 - z}\right) \quad (z \in \mathbb{U}). \tag{9}$$

And we note that the function p, defined by (7), has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{10}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right) \quad (n \in \mathbb{N}). \tag{11}$$

For given real numbers α and β such that $0 \leq \alpha < 1 < \beta$, we denote by $\mathscr{K}_\sigma(\alpha, \beta)$ the class of biunivalent functions consisting of the functions in $\mathscr{A}$ such that

$$f \in \mathscr{K}(\alpha, \beta), \qquad f^{-1} \in \mathscr{K}(\alpha, \beta), \tag{12}$$

where f^{-1} is the inverse function of f.

In our present investigation, we first find some relationships for functions in bounded positive class $\mathscr{K}(\alpha, \beta)$. And we solve several coefficient problems including Fekete-Szegö problems for functions in the class. Furthermore, we estimate the bounds of initial coefficients of inverse functions and biunivalent functions. For the coefficient bounds of functions in special subclasses of $\mathscr{S}$, the readers may be referred to the works [2–4].

2. Relations Involving Bounds on the Real Parts

In this section, we will find some relations involving the functions in $\mathscr{K}(\alpha, \beta)$. And the following lemma will be needed in finding the relations.

Lemma 3 (see Miller and Mocanu [5, Theorem 2.3.i]). *Let Ξ be a set in the complex plane $\mathbb{C}$ and let b be a complex number such that $\Re\{b\} > 0$. Suppose that a function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfies the condition*

$$\psi(i\rho, \sigma; z) \notin \Xi \tag{13}$$

for all real $\rho, \sigma \leq -|b - i\rho|^2/(2\Re\{b\})$ and all $z \in \mathbb{U}$. If the function $p(z)$ defined by $p(z) = b + b_1 z + b_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and if

$$\psi\left(p(z), zp'(z); z\right) \in \Xi, \tag{14}$$

then $\Re\{p(z)\} > 0$ in $\mathbb{U}$.

Theorem 4. *Let $f \in \mathscr{A}$, $0 \leq \alpha < 1$ and*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}). \tag{15}$$

Then

$$\Re\left\{\sqrt{f'(z)}\right\} > \frac{1}{2 - \alpha} \quad (z \in \mathbb{U}). \tag{16}$$

Proof. First of all, we put $\gamma = 1/(2 - \alpha)$ and note that $1/2 \leq \gamma < 1$ for $0 \leq \alpha < 1$. Let

$$p(z) = \frac{1}{1 - \gamma}\left(\sqrt{f'(z)} - \gamma\right). \tag{17}$$

Differentiating (17), we can obtain

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2(1-\gamma) zp'(z)}{(1-\gamma) p(z) + \gamma} = \psi\left(p(z), zp'(z)\right), \tag{18}$$

where

$$\psi(r, s) = 1 + \frac{2(1-\gamma) s}{(1-\gamma) r + \gamma}. \tag{19}$$

Using (15), we have

$$\left\{\psi\left(p(z), zp'(z)\right) : z \in \mathbb{U}\right\} \subset \{w \in \mathbb{C} : \Re(w) > \alpha\} := \Omega_\alpha. \tag{20}$$

Now for all real $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$,

$$\begin{aligned} \Re\{\psi(i\rho, \sigma)\} &= 1 + \frac{2\gamma(1-\gamma)\sigma}{(1-\gamma)^2 \rho^2 + \gamma^2} \\ &\leq 1 - \frac{\gamma(1-\gamma)\left(1 + \rho^2\right)}{(1-\gamma)^2 \rho^2 + \gamma^2}. \end{aligned} \tag{21}$$

Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(\rho) = \frac{1 + \rho^2}{(1-\gamma)^2 \rho^2 + \gamma^2}. \tag{22}$$

Then g is a continuous even function and

$$g'(\rho) = \frac{2(2\gamma - 1)\rho}{\left((1-\gamma)^2 \rho^2 + \gamma^2\right)^2}. \tag{23}$$

Hence $g'(0) = 0$ and g is increasing on $(0, \infty)$, since $1/2 \leq \gamma < 1$. Hence g satisfies that

$$\frac{1}{\gamma^2} \leq g(\rho) < \frac{1}{(1-\gamma)^2}, \tag{24}$$

for all $\rho \in \mathbb{R}$. Therefore, by combining (21) and (24), we can get

$$\Re\{\psi(i\rho, \sigma)\} \leq 1 - \gamma(1-\gamma) g(\rho) \leq 2 - \frac{1}{\gamma} = \alpha. \tag{25}$$

And this shows that $\Re\{\psi(i\rho, \sigma)\} \notin \Omega_\alpha$ for all $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$. By Lemma 3, we get $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (16) holds and the proof of Theorem 4 is completed. □

Theorem 5. *Let $f \in \mathcal{A}$, $1 < \beta < 2$ and*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \tag{26}$$

Then

$$\Re\left\{\sqrt{f'(z)}\right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \tag{27}$$

Proof. We put $\delta = 1/(2-\beta)$ and note that $\delta > 1$ for $1 < \beta < 2$. And let

$$\begin{aligned} p(z) &= \frac{1}{1-\delta}\left(\sqrt{f'(z)} - \delta\right), \\ \psi(r,s) &= 1 + \frac{2(1-\delta)s}{(1-\delta)r + \delta}. \end{aligned} \tag{28}$$

As in the proof of Theorem 4, we can get

$$\left\{\psi\left(p(z), zp'(z)\right) : z \in \mathbb{U}\right\} \subset \left\{w \in \mathbb{C} : \Re(w) < \beta\right\} := \Omega_\beta, \tag{29}$$

by (26). And for all real ρ, σ with $\sigma \le -(1+\rho^2)/2$,

$$\begin{aligned} \Re\{\psi(i\rho,\sigma)\} &= 1 + \frac{2\delta(1-\delta)\sigma}{(1-\delta)^2\rho^2 + \delta^2} \\ &\ge 1 - \frac{\delta(1-\delta)\left(1+\rho^2\right)}{(1-\delta)^2\rho^2 + \delta^2} \\ &= 1 - \delta(1-\delta)g(\rho), \end{aligned} \tag{30}$$

where $g(\rho)$ is given by

$$g(\rho) = \frac{1+\rho^2}{(1-\delta)^2\rho^2 + \delta^2}. \tag{31}$$

Since $\delta > 1$, g satisfies the inequality

$$\frac{1}{\delta^2} \le g(\rho) < \frac{1}{(1-\delta)^2}, \tag{32}$$

for all $\rho \in \mathbb{R}$. Therefore,

$$\Re\{\psi(i\rho,\sigma)\} \ge 1 - \delta(1-\delta)g(\rho) \ge 2 - \frac{1}{\delta} = \beta. \tag{33}$$

And this shows that $\Re\{\psi(i\rho,\sigma)\} \notin \Omega_\beta$ for all $\rho,\sigma \in \mathbb{R}$ with $\sigma \le -(1+\rho^2)/2$. By Lemma 3, we get $\Re\{p(z)\} > 0$ for all $z \in \mathbb{U}$, and this shows that the inequality (27) holds and the proof of Theorem 5 is completed. □

By combining Theorems 4 and 5, we can obtain the following result.

Theorem 6. *Let $f \in \mathcal{A}$, $0 \le \alpha < 1 < \beta < 2$ and*

$$\alpha < \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta \quad (z \in \mathbb{U}). \tag{34}$$

Then

$$\frac{1}{2-\alpha} < \Re\left\{\sqrt{f'(z)}\right\} < \frac{1}{2-\beta} \quad (z \in \mathbb{U}). \tag{35}$$

3. Coefficient Problems Involving Functions in $\mathcal{K}(\alpha,\beta)$

In the present section, we will solve some coefficient problems involving functions in the class $\mathcal{K}(\alpha,\beta)$. And our first result on the coefficient estimates involves the function class $\mathcal{K}(\alpha,\beta)$ and the following lemma will be needed.

Lemma 7 (see Rogosinski [6, Theorem 10]). *Let $p(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in $\mathbb{U}$ and suppose that $p(z)$ maps $\mathbb{U}$ onto a convex domain. If $q(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies the following subordination:*

$$q(z) \prec p(z) \quad (z \in \mathbb{U}), \tag{36}$$

then

$$|A_n| \le |C_1| \quad (n = 1, 2, \ldots). \tag{37}$$

Theorem 8. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$. If the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}(\alpha,\beta)$, then*

$$|a_n| \le \begin{cases} \dfrac{1}{2}|B_1|, & n = 2, \\ \dfrac{|B_1|}{n(n-1)}\displaystyle\prod_{k=1}^{n-2}\left(1 + \frac{|B_1|}{k}\right), & n = 3, 4, 5, \ldots, \end{cases} \tag{38}$$

where $|B_1|$ is given by

$$|B_1| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{39}$$

Proof. Let us define

$$q(z) = 1 + \frac{zf''(z)}{f'(z)}, \tag{40}$$

$$p(z) = 1 + \frac{\beta-\alpha}{\pi}i\log\left(\frac{1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1-z}\right). \tag{41}$$

Then, the subordination (9) can be written as follows:

$$q(z) \prec p(z) \quad (z \in \mathbb{U}). \tag{42}$$

Note that the function $p(z)$ defined by (41) is convex in $\mathbb{U}$ and has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{43}$$

where

$$B_n = \frac{\beta-\alpha}{n\pi}i\left(1 - e^{2n\pi i((1-\alpha)/(\beta-\alpha))}\right). \tag{44}$$

If we let

$$q(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \tag{45}$$

then by Lemma 7, we see that the subordination (42) implies that

$$|A_n| \leq |B_1| \quad (n = 1, 2, \ldots), \tag{46}$$

where

$$|B_1| = \frac{2(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha}. \tag{47}$$

Now, equality (40) implies that

$$f'(z) + zf''(z) = q(z) f'(z) \quad (z \in \mathbb{U}). \tag{48}$$

Then, the coefficients of z^{n-1} in both sides lead to

$$a_n = \frac{1}{n(n-1)} \left(A_{n-1} + 2a_2 A_{n-2} + 3a_3 A_{n-3} + \cdots + (n-1) a_{n-1} A_1\right). \tag{49}$$

A simple calculation combined with the inequality (46) yields that

$$\begin{aligned} |a_n| &= \frac{1}{n(n-1)} \\ &\quad \times \left|A_{n-1} + 2a_2 A_{n-2} + 3a_3 A_{n-3} + \cdots + (n-1) a_{n-1} A_1\right| \\ &\leq \frac{1}{n(n-1)} \\ &\quad \times \left(|A_{n-1}| + 2|a_2||A_{n-2}| + 3|a_3||A_{n-3}| + \cdots + (n-1)|a_{n-1}||A_1|\right) \\ &\leq \frac{|B_1|}{n(n-1)} \sum_{k=2}^{n} (k-1)|a_{k-1}|, \end{aligned} \tag{50}$$

where $|B_1|$ is given by (47) and $|a_1| = 1$. Hence, we have $|a_2| \leq |B_1|/2$. To prove the assertion of the theorem, we need to show that

$$\begin{aligned} |a_n| &\leq \frac{|B_1|}{n(n-1)} \sum_{k=2}^{n} (k-1)|a_{k-1}| \\ &\leq \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k}\right) \quad (n = 3, 4, 5, \ldots). \end{aligned} \tag{51}$$

We now use the mathematical induction for the proof of the theorem. For the case $n = 3$, it is clear. We assume that the inequality (51) holds for $n = m$. Then, some calculation gives us that

$$\begin{aligned} |a_{m+1}| &\leq \frac{|B_1|}{(m+1)m} \sum_{k=2}^{m+1} (k-1)|a_{k-1}| \\ &\leq \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k}\right) \\ &\quad + \frac{|B_1|^2}{m(m+1)(m-1)} \prod_{k=1}^{m-2} \left(1 + \frac{|B_1|}{k}\right) \\ &= \frac{|B_1|}{(m+1)m} \prod_{k=1}^{m-1} \left(1 + \frac{|B_1|}{k}\right), \end{aligned} \tag{52}$$

which implies that the inequality (51) is true for $n = m + 1$. Hence, by the mathematical induction, we prove that

$$|a_n| \leq \frac{|B_1|}{n(n-1)} \prod_{k=1}^{n-2} \left(1 + \frac{|B_1|}{k}\right) \quad (n = 3, 4, 5, \ldots), \tag{53}$$

where $|B_1|$ is given by (47). This completes the proof of Theorem 8. □

And now, we will solve the Fekete-Szegö problem for $f \in \mathcal{K}(\alpha, \beta)$, and we will need the following lemma.

Lemma 9 (see Keogh and Merkes [7]). *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be a function with positive real part in $\mathbb{U}$. Then, for any complex number ν,*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |1 - 2\nu|\}. \tag{54}$$

Now, the following result holds for the coefficient of $f \in \mathcal{K}(\alpha, \beta)$.

Theorem 10. *Let $0 \leq \alpha < 1 < \beta$ and let the function $f(z)$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{K}(\alpha, \beta)$. Then, for a complex number μ,*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ &\quad \leq \frac{\beta - \alpha}{3\pi} \sin\left(\frac{1-\alpha}{\beta - \alpha}\pi\right) \\ &\quad \cdot \max\left\{1; \left|\frac{1}{2} + \left(1 - \frac{3}{2}\mu\right)\frac{\beta - \alpha}{\pi} i \right.\right. \\ &\quad \left.\left. + \left(\frac{1}{2} - \left(1 - \frac{3}{2}\mu\right)\frac{\beta - \alpha}{\pi} i\right) \times e^{2\pi i((1-\alpha)/(\beta-\alpha))}\right|\right\}. \end{aligned} \tag{55}$$

The result is sharp.

Proof. Let us consider a function $q(z)$ given by $q(z) = 1 + zf''(z)/f'(z)$. Then, since $f \in \mathcal{K}(\alpha,\beta)$, we have $q(z) \prec p(z) (z \in \mathbb{U})$, where

$$p(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\alpha)/(\beta-\alpha))}z}{1-z}\right) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{56}$$

where B_n is given by (11). Let

$$h(z) = \frac{1+p^{-1}(q(z))}{1-p^{-1}(q(z))} = 1 + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{57}$$

Then h is analytic and has positive real part in the open unit disk $\mathbb{U}$. We also have

$$q(z) = p\left(\frac{h(z)-1}{h(z)+1}\right) \quad (z \in \mathbb{U}). \tag{58}$$

We find from (58) that

$$a_2 = \frac{1}{4} B_1 h_1, \qquad a_3 = \frac{1}{12} B_1 h_2 - \frac{1}{24} B_1 h_1^2 + \frac{1}{24} B_2 h_1^2 + \frac{1}{24} B_1^2 h_1^2, \tag{59}$$

which imply that

$$a_3 - \mu a_2^2 = \frac{1}{12} B_1 \left(h_2 - \nu h_1^2\right), \tag{60}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - B_1 + \frac{3}{2}\mu B_1\right). \tag{61}$$

Applying Lemma 9, we can obtain

$$\left|a_3 - \mu a_2^2\right| = \frac{1}{12}\left|B_1\right|\left|h_2 - \nu h_1^2\right| \leq \frac{1}{6}\left|B_1\right| \cdot \max\{1; |1-2\nu|\}. \tag{62}$$

And substituting

$$B_1 = \frac{\beta-\alpha}{\pi} i \left(1 - e^{2\pi i((1-\alpha)/(\beta-\alpha))}\right), \tag{63}$$

$$B_2 = \frac{\beta-\alpha}{2\pi} i \left(1 - e^{4\pi i((1-\alpha)/(\beta-\alpha))}\right) \tag{64}$$

in (62), we can obtain the result as asserted. The estimate is sharp for the function $f : \mathbb{U} \to \mathbb{C}$ defined by

$$f(z) = \int_0^z \left\{\exp\left(\int_0^\zeta \frac{p(\xi)-1}{\xi} d\xi\right)\right\} d\zeta, \tag{65}$$

where the function p is given by (7). Hence the proof of Theorem 10 is completed. □

Using Theorem 10, we can get the following result.

Corollary 11. *Let $0 \leq \alpha < 1 < \beta$ and let the function f, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}(\alpha,\beta)$. Also let the function f^{-1}, defined by*

$$f^{-1}(f(z)) = z = f\left(f^{-1}(z)\right), \tag{66}$$

be the inverse of f. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad \left(|w| < r_0;\ r_0 > \frac{1}{4}\right), \tag{67}$$

then

$$\left|b_2\right| \leq \frac{\beta-\alpha}{\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right),$$
$$\left|b_3\right| \leq \frac{\beta-\alpha}{3\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right) \cdot \max\left\{1; \left|\frac{1}{2} - \frac{2(\beta-\alpha)}{\pi} i + \left(\frac{1}{2} + \frac{2(\beta-\alpha)}{\pi} i\right) e^{2\pi i((1-\alpha)/(\beta-\alpha))}\right|\right\}. \tag{68}$$

Proof. The relations (66) and (67) give

$$b_2 = -a_2, \qquad b_3 = 2a_2^2 - a_3. \tag{69}$$

Thus, we can get the estimate for $|b_2|$ by

$$\left|b_2\right| = \left|a_2\right| \leq \frac{1}{2}\left|B_1\right| = \frac{\beta-\alpha}{\pi} \sin\left(\frac{1-\alpha}{\beta-\alpha}\pi\right), \tag{70}$$

immediately. Furthermore, an application of Theorem 10 (with $\mu = 2$) gives the estimates for $|b_3|$; hence, the proof of Corollary 11 is completed. □

Finally, we will estimate some initial coefficients for the bi-univalent functions $f \in \mathcal{K}_\sigma(\alpha,\beta)$.

Theorem 12. *For given α and β such that $0 \leq \alpha < 1 < \beta$, let f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{K}_\sigma(\alpha,\beta)$. Then*

$$\left|a_2\right| \leq \frac{\left|B_1\right|\sqrt{\left|B_1\right|}}{\sqrt{2\left|B_1^2 - 2B_1 + 2B_2\right|}}, \tag{71}$$

$$\left|a_3\right| \leq \frac{1}{2}\left(\left|B_1\right| + \left|B_2 - B_1\right|\right), \tag{72}$$

where B_1 and B_2 are given by (63) and (64).

Proof. If $f \in \mathcal{K}_\sigma(\alpha,\beta)$, then $f \in \mathcal{K}(\alpha,\beta)$ and $g \in \mathcal{K}(\alpha,\beta)$, where $g = f^{-1}$. Hence

$$Q(z) := 1 + \frac{zf''(z)}{f'(z)} \prec p(z) \quad (z \in \mathbb{U}), \qquad L(z) := 1 + \frac{zg''(z)}{g'(z)} \prec p(z) \quad (z \in \mathbb{U}), \tag{73}$$

where $p(z)$ is given by (7). Let

$$h(z) = \frac{1+p^{-1}(Q(z))}{1-p^{-1}(Q(z))} = 1 + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U}),$$
$$k(z) = \frac{1+p^{-1}(L(z))}{1-p^{-1}(L(z))} = 1 + k_1 z + k_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{74}$$

Then h and k are analytic and have positive real part in $\mathbb{U}$. Also, we have

$$Q(z) = p\left(\frac{h(z)-1}{h(z)+1}\right) \quad (z \in \mathbb{U}),$$
$$L(z) = p\left(\frac{k(z)-1}{k(z)+1}\right) \quad (z \in \mathbb{U}). \tag{75}$$

By suitably comparing coefficients, we get

$$a_2 = \frac{1}{4} B_1 h_1, \tag{76}$$

$$3a_3 - 2a_2^2 = \frac{1}{4} B_1 h_2 - \frac{1}{8} B_1 h_1^2 + \frac{1}{8} B_2 h_1^2, \tag{77}$$

$$b_2 = \frac{1}{4} B_1 k_1, \tag{78}$$

$$3b_3 - 2b_2^2 = \frac{1}{4} B_1 k_2 - \frac{1}{8} B_1 k_1^2 + \frac{1}{8} B_2 k_1^2, \tag{79}$$

where B_1 and B_2 are given by (63) and (64), respectively. Now, considering (76) and (78), we get

$$h_1 = -k_1. \tag{80}$$

Also, from (77), (78), (79), and (80), we find that

$$a_2^2 = \frac{B_1^3 (h_2 + k_2)}{8(B_1^2 - 2B_1 + 2B_2)}. \tag{81}$$

Therefore, we have

$$\left|a_2^2\right| \le \frac{|B_1|^3}{8\left|B_1^2 - 2B_1 + 2B_2\right|}\left(|h_2| + |k_2|\right) \le \frac{|B_1|^3}{2\left|B_1^2 - 2B_1 + 2B_2\right|}. \tag{82}$$

This gives the bound on $|a_2|$ as asserted in (71). Now, further computations from (77), (79), (80), and (81) lead to

$$a_3 = \frac{1}{12} B_1 (2h_2 + k_2) + \frac{1}{8} h_1^2 (B_2 - B_1). \tag{83}$$

This equation, together with the well-known estimates

$$|h_1| \le 2, \qquad |h_2| \le 2, \qquad |k_2| \le 2, \tag{84}$$

leads us to the inequality (72). Therefore, the proof of Theorem 12 is completed. □

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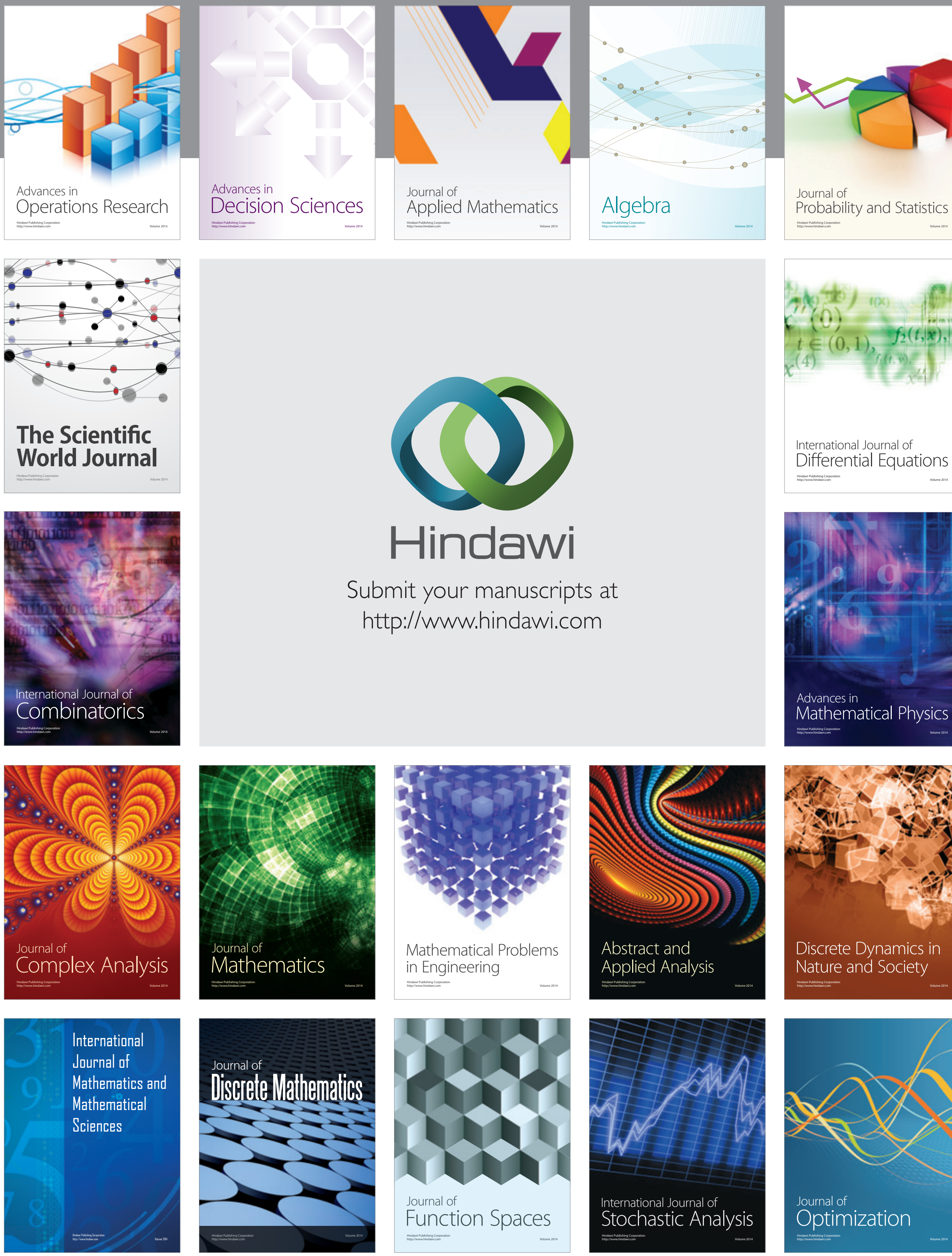
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