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Research Article

Fibonacci Collocation Method for Solving High-Order Linear Fredholm Integro-Differential-Difference Equations

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A new collocation method based on the Fibonacci polynomials is introduced for the approximate solution of high order-linear Fredholm integro-differential-difference equations with the mixed conditions. The proposed method is analyzed to show the convergence of the method. Some further numerical experiments are carried out to demonstrate the method.

1. Introduction

The integro-differential-difference equations (IDDEs) have been developed very rapidly in recent years. This is an important branch of mathematics which has a lot of interest in many application fields such as engineering, mechanics, physics, astronomy, chemistry, biology, economics, and potential theory, electrostatics [1–14]. Since some IDDEs are hard to solve numerically, they are solved by using the approximated methods. Several numerical methods were used such as the successive approximations, Adomian decomposition, Haar Wavelet, and Tau and Walsh series methods [15–20]. Additionally the Monte Carlo method for linear Fredholm integro-differential-difference equation has been presented by Farnoosh and Ebrahimi [21] and the Direct method based on the Fourier and block-pulse method functions by Asady et al. [22].

Since the beginning of 1994, the Taylor and Chebyshev matrix methods have also been used by Sezer et al. to solve linear differential, Fredholm integral, and Fredholm integro-differential equations [23–35]. Lately, the Fibonacci collocation method has been used to find the approximate solutions of differential, integral, and integro-differential equations [36].

In this study, we consider the approximate solution of the mth-order Fredholm integro-differential-difference equations,

$$\sum_{k=0}^{m} P_k(x)\, y^{(k)}(x) + \sum_{r=0}^{s} Q_r(x)\, y^{(r)}(\mu_r x + \tau_r) = g(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt, \tag{1}$$

where $s \le m$, τ_r are the integer, $0 \le a \le x, t \le b$, under the mixed conditions

$$\sum_{k=0}^{m-1} \left[a_{jk} y^{(k)}(a) + b_{jk} y^{(k)}(b) \right] = \lambda_j, \quad j = 1, 2, 3, \ldots, m, \tag{2}$$

where $P_k(x)$, $Q_r(x)$, $g(x)$, and $K(x,t)$ are functions defined on $a \le x, t \le b$; a_{jk}, b_{jk}, λ, and λ_j are suitable constants.

Our aim is to obtain an approximate solution of (1) expressed in the truncated Fibonacci series form:

$$y(x) = \sum_{n=1}^{N} a_n F_n(x), \tag{3}$$

where a_n, $n = 1, 2, 3, \ldots, N$, are the unknown Fibonacci coefficients. Here N is positive integer such that $N \geq m$ and $F_n(x)$, $n = 1, 2, 3, \ldots, N$, are the Fibonacci polynomials defined by

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1},$$
$$\left[\frac{(n-1)}{2}\right] = \begin{cases} \dfrac{(n-2)}{2}, & n \text{ even}, \\ \dfrac{(n-1)}{2}, & n \text{ odd}. \end{cases} \tag{4}$$

2. Fundamental Matrix Relations

Firstly, we can write the Fibonacci polynomials $F_n(x)$ in the matrix form as follows:

$$\mathbf{F}^T(x) = \mathbf{C}\mathbf{X}^T(x) \Longleftrightarrow \mathbf{F}(x) = \mathbf{X}(x)\mathbf{C}^T, \tag{5}$$

where

$$\mathbf{F}(x) = \left[F_1(x) \;\; F_2(x) \;\; \cdots \;\; F_N(x)\right],$$
$$\mathbf{X}(x) = \left[1x \;\; \cdots \;\; x^{N-1}\right]. \tag{6}$$

If N is even,

$$\mathbf{C} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0 \\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{\frac{(n-2)}{2}}{\frac{(n-2)}{2}} & 0 & \binom{\frac{n}{2}}{\frac{(n-4)}{2}} & 0 & \cdots & 0 \\ 0 & \binom{\frac{n}{2}}{\frac{(n-2)}{2}} & 0 & \binom{\frac{(n+2)}{2}}{\frac{(n-4)}{2}} & \cdots & \binom{n-1}{0} \end{bmatrix}_{N\times N}; \tag{7}$$

if N is odd,

$$\mathbf{C} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0 \\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \binom{\frac{(n-1)}{2}}{\frac{(n-3)}{2}} & 0 & \binom{\frac{(n+1)}{2}}{\frac{(n-5)}{2}} & \cdots & 0 \\ \binom{\frac{(n-1)}{2}}{\frac{(n-1)}{2}} & 0 & \binom{\frac{(n+1)}{2}}{\frac{(n-3)}{2}} & \cdots & \cdots & \binom{n-1}{0} \end{bmatrix}_{N\times N}. \tag{8}$$

Let us show (1) in the following form:

$$P(x) + Q(x) = g(x) + \lambda I(x), \tag{9}$$

where

$$P(x) = \sum_{k=0}^{m} P_k(x)\, y^{(k)}(x),$$
$$Q(x) = \sum_{r=0}^{s} Q_r(x)\, y^{(r)}(\mu_r x + \tau_r),$$
$$I(x) = \lambda \int_a^b K(x,t)\, y(t)\, dt. \tag{10}$$

2.1. Matrix Relations for the Differential Part $P(x)$. Firstly, we consider the solution $\mathbf{y}(x)$ and its kth derivate $\mathbf{y}^{(k)}(x)$ in the matrix form:

$$\mathbf{y}(x) = \mathbf{F}(x)\mathbf{A}, \quad \mathbf{A} = [a_1 \ a_2 \ \cdots \ a_N]^T, \tag{11}$$

$$\mathbf{y}^{(k)}(x) = \mathbf{F}^{(k)}(x)\mathbf{A}. \tag{12}$$

Then, from relations (5) and (11), we can obtain the following matrix form:

$$\mathbf{y}(x) = \mathbf{X}(x)\mathbf{C}^T\mathbf{A}. \tag{13}$$

Similar to (13), from relations (5), (11), and (12), we can find $\mathbf{y}^{(k)}(x)$ matrix form as

$$\mathbf{y}^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{C}^T\mathbf{A}. \tag{14}$$

To find the matrix $\mathbf{X}^{(k)}(x)$ in terms of the matrix $\mathbf{X}(x)$, we can use the following relation:

$$\begin{gathered}\mathbf{X}^{(1)}(x) = \mathbf{X}(x)\mathbf{T}^T, \\ \mathbf{X}^{(2)}(x) = \mathbf{X}^{(1)}(x)\mathbf{T}^T = \left(\mathbf{X}(x)\mathbf{T}^T\right)\mathbf{T}^T = \mathbf{X}(x)\left(\mathbf{T}^T\right)^2 \\ \vdots \\ \mathbf{X}^{(k)}(x) = \mathbf{X}^{(k-1)}(x)\mathbf{T}^T = \mathbf{X}(x)\left(\mathbf{T}^T\right)^k,\end{gathered} \tag{15}$$

where

$$\mathbf{T}^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & N-1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

Subsequently, by substituting the matrix form (15) into (14), we obtain the matrix relations

$$\mathbf{y}^{(k)}(x) = \mathbf{X}(x)\left(\mathbf{T}^T\right)^k\mathbf{C}^T\mathbf{A}. \tag{17}$$

2.2. Matrix Relations for the Difference Part $Q(x)$. If we put $x \to \mu_r x + \tau_r$ in the relation (11), we have the matrix form

$$\mathbf{y}(\mu_r + \tau_r) = \mathbf{F}(\mu_r x + \tau_r)\mathbf{A}. \tag{18}$$

It is seen that the relation between the matrices $\mathbf{X}(x)$ and $\mathbf{X}(\mu_r x + \tau_r)$ is

$$\mathbf{X}(\mu_r x + \tau_r) = \mathbf{X}(x)\boldsymbol{\beta}(\mu_r, \tau_r), \tag{19}$$

where

$$\boldsymbol{\beta}(\mu_r, \tau_r) = \begin{bmatrix} \binom{0}{0}\mu_r{}^0\tau_r{}^0 & \binom{1}{0}\mu_r{}^0\tau_r{}^1 & \cdots & \binom{N-1}{0}\mu_r{}^0\tau_r{}^{N-1} \\ 0 & \binom{1}{1}\mu_r{}^1\tau_r{}^0 & \cdots & \binom{N-1}{1}\mu_r{}^1\tau_r{}^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N-1}{N-1}\mu_r{}^{N-1}\tau_r{}^0 \end{bmatrix}. \tag{20}$$

By using the relations (15) and (19), we can get

$$\mathbf{X}^{(k)}(\mu_r x + \tau_r) = \mathbf{X}(x)\boldsymbol{\beta}(\mu_r, \tau_r)\left(\mathbf{T}^T\right)^k. \tag{21}$$

Thus from (14) and (21), we can find

$$\mathbf{y}^{(k)}(\mu_r + \tau_r) = \mathbf{X}(x)\boldsymbol{\beta}(\mu_r, \tau_r)\left(\mathbf{T}^T\right)^k\mathbf{C}^T\mathbf{A}. \tag{22}$$

By using the expressions (14) and (22), we obtain the matrix form

$$\begin{gathered}P(x) = \sum_{k=0}^{m} p_k(x)\mathbf{X}(x)\left(\mathbf{T}^T\right)^k\mathbf{C}^T\mathbf{A}, \\ Q(x) = \sum_{r=0}^{s} Q_r(x)\mathbf{X}(x)\boldsymbol{\beta}(\mu_r, \tau_r)\left(\mathbf{T}^T\right)^r\mathbf{C}^T\mathbf{A}.\end{gathered} \tag{23}$$

2.3. Matrix Relations for the Integral Part. Let us find the matrix relation for the Fredholm integral part $I(x)$ in (9). The kernel function $K(x,t)$ can be shown by the truncated Fibonacci series,

$$K(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}^f F_m(x) F_n(t), \tag{24}$$

and the truncated Taylor series,

$$K(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn}^t x^m t^n, \tag{25}$$

where

$$k_{mn}^t = \frac{1}{m!n!}\frac{\partial^{m+n}K(0,0)}{\partial x^m \partial x^n}; \quad m,n = 0,1,\ldots,N. \tag{26}$$

The expressions (24) and (25) can be put in matrix forms as

$$K(x,t) = \mathbf{F}(x)\mathbf{K}_F\mathbf{F}^T(t), \quad \mathbf{K}_F = \left[k_{mn}^F\right], \quad m,n = 0,1,\ldots,N, \tag{27}$$

$$K(x,t) = \mathbf{X}(x)\mathbf{K}_t\mathbf{X}^T(t), \quad \mathbf{K}_t = \left[k_{mn}^t\right], \quad m,n = 0,1,\ldots,N. \tag{28}$$

From (11), (27), and (28) we can obtain

$$\mathbf{X}(x)\mathbf{K}_t\mathbf{X}^T(t) = \mathbf{F}(x)\mathbf{K}_F\mathbf{F}^T(t) \Longrightarrow \mathbf{X}(x)\mathbf{K}_t\mathbf{X}^T(t) = \mathbf{X}(x)\mathbf{C}^T\mathbf{K}_F\mathbf{C}\mathbf{X}^T(t). \tag{29}$$

Thus

$$\mathbf{K}_t = \mathbf{C}^T\mathbf{K}_F\mathbf{C} \Longrightarrow \mathbf{K}_F = \left(\mathbf{C}^T\right)^{-1}\mathbf{K}_t\mathbf{C}^{-1}. \tag{30}$$

By substituting the matrix forms (22) and (27) into the integral part $I(x)$ in (9), we can have the matrix relation as follows:

$$\begin{aligned}[I(x)] &= \int_a^b \mathbf{F}(x)\mathbf{K}_F\mathbf{F}^T(t)\mathbf{X}(t)\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^k\mathbf{C}^T\mathbf{A}\,dt \\ &= \mathbf{F}(x)\mathbf{K}_F\mathbf{Q}\mathbf{A}\end{aligned} \tag{31}$$

so that

$$\mathbf{Q} = \int_a^b \mathbf{F}^T(t)\mathbf{X}(t)\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^k\mathbf{C}^T dt. \tag{32}$$

From (5) and (32), we have

$$\begin{aligned}\mathbf{Q} &= \int_a^b \mathbf{C}\mathbf{X}^T(t)\mathbf{X}(t)\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^k\mathbf{C}^T dt \\ &= \mathbf{C}\mathbf{H}\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^k\mathbf{C}^T,\end{aligned} \tag{33}$$

where

$$\begin{aligned}\mathbf{H} &= \int_a^b \mathbf{X}^T(t)\mathbf{X}(t)\,dt, \quad \mathbf{H} = \left[h_{ij}\right], \\ h_{ij} &= \frac{b^{i+j+1}-a^{i+j+1}}{i+j+1}i, \quad j = 1,2,\ldots,N.\end{aligned} \tag{34}$$

If we substitute the matrix relation (5) into (31), we have the matrix form

$$[\mathbf{I}(x)] = \mathbf{X}(x)\mathbf{C}^T\mathbf{K}_F\mathbf{Q}\mathbf{A}. \tag{35}$$

2.4. Matrix Relations for the Conditions. The corresponding matrix form for the conditions (2) can be shown, by means of (17), as

$$\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{X}(a) + b_{jk}\mathbf{X}(b)\right]\left(\mathbf{T}^T\right)^k\mathbf{C}^T\mathbf{A} = \lambda_j, \quad j = 1,2,\ldots,m. \tag{36}$$

3. Method of Solution

We can construct the fundamental matrix equation corresponding for (1). For this aim, we substitute the matrix relations (23) and (35) into (9). So we obtain the matrix equation

$$\begin{aligned}&\sum_{k=0}^{m}P_k(x)\mathbf{X}(x)\mathbf{C}^T\left(\mathbf{T}^T\right)^k\mathbf{A} \\ &\quad + \sum_{r=0}^{s}Q_r(x)\mathbf{X}(x)\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^r\mathbf{C}^T\mathbf{A} \\ &\quad = \mathrm{g}(x) + \lambda\mathbf{X}(x)\mathbf{C}^T\mathbf{K}_F\mathbf{Q}\mathbf{A}.\end{aligned} \tag{37}$$

By using in (37) the collocation points x_i defined by

$$x_i = a + \left(\frac{b-a}{N-1}\right)(i-1), \quad i = 1,2,\ldots,N, \tag{38}$$

the system of the matrix equations is obtained

$$\begin{aligned}&\sum_{k=0}^{m}P_k(x_i)\mathbf{X}(x_i)\mathbf{C}^T\left(\mathbf{T}^T\right)^k\mathbf{A} \\ &\quad + \sum_{r=0}^{s}Q_r(x_i)\mathbf{X}(x_i)\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^r\mathbf{C}^T\mathbf{A} \\ &\quad = \mathrm{g}(x_i) + \lambda\mathbf{X}(x_i)\mathbf{C}^T\mathbf{K}_F\mathbf{Q}\mathbf{A}\end{aligned} \tag{39}$$

or shortly the fundamental matrix equation becomes

$$\begin{aligned}&\left\{\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}\left(\mathbf{T}^T\right)^k\mathbf{C}^T + \sum_{r=0}^{s}\mathbf{Q}_r\mathbf{X}\boldsymbol{\beta}(\mu_r,\tau_r)\left(\mathbf{T}^T\right)^r\mathbf{C}^T\right. \\ &\quad \left. -\lambda\mathbf{X}\mathbf{C}^T\mathbf{K}_F\mathbf{Q}\right\}\mathbf{A} = \mathbf{G},\end{aligned} \tag{40}$$

where

$$\begin{aligned}\mathbf{P}_k &= \begin{bmatrix} p_k(x_1) & 0 & \cdots & 0 \\ 0 & p_k(x_2) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & p_k(x_N)\end{bmatrix}, \\ \mathbf{Q}_r &= \begin{bmatrix} Q_r(x_1) & 0 & \cdots & 0 \\ 0 & Q_r(x_2) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & Q_r(x_N)\end{bmatrix}, \\ \mathbf{X} &= \begin{bmatrix} X(x_1) \\ X(x_2) \\ \vdots \\ X(x_N)\end{bmatrix} = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{N-1} \\ 1 & x_2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^{N-1}\end{bmatrix}, \\ \mathbf{G} &= \begin{bmatrix} \mathrm{g}(x_1) \\ \mathrm{g}(x_2) \\ \vdots \\ \mathrm{g}(x_N)\end{bmatrix}.\end{aligned} \tag{41}$$

Therefore, the fundamental matrix equation (40) corresponding for (1) can be written as

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \text{or} \quad [\mathbf{W};\mathbf{G}], \tag{42}$$

where

$$\mathbf{W}=\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}\left(\mathbf{T}^T\right)^k\mathbf{C}^T + \sum_{r=0}^{s}\mathbf{Q}_r\mathbf{X}\boldsymbol{\beta}\left(\mu_r,\tau_r\right)\left(\mathbf{T}^T\right)^r\mathbf{C}^T-\lambda\mathbf{X}\mathbf{C}^T\mathbf{K}_F\mathbf{Q}. \tag{43}$$

Equation (42) corresponds to a system of N linear algebraic equations with unknown Fibonacci coefficients $a_1, a_2, \ldots, a_N$. Further, we can express the matrix form (36) conditions

$$\mathbf{U}_jA=\left[\lambda_j\right] \quad \text{or} \quad \left[\mathbf{U}_j;\lambda_j\right], \quad j=1,2,\ldots,m, \tag{44}$$

where

$$\mathbf{U}_j=\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{X}(a)+b_{jk}\mathbf{X}(b)\right]\left(\mathbf{T}^T\right)^k\mathbf{C}^T = \left[u_{j1}\ \ u_{j2}\ \ u_{j3}\ \ \ldots\ \ u_{jN}\right]. \tag{45}$$

To obtain the solution of (1) under the conditions (2), by replacing the row matrices (44) by the last m rows of the matrices (42), we have the new augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{G}}. \tag{46}$$

If the last m rows of the (30) are replaced, the augmented matrix of the above system is obtained as

$$\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right] = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} & ; & g(t_1) \\ w_{21} & w_{22} & \cdots & w_{2N} & ; & g(t_2) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(t_{N-m}) \\ u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{21} & u_{22} & \cdots & u_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mN} & ; & \lambda_m \end{bmatrix}. \tag{47}$$

If rank $\widetilde{\mathbf{W}}$ = rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = N$, then we can write

$$\mathbf{A}=\left(\widetilde{\mathbf{W}}\right)^{-1}\widetilde{\mathbf{G}}. \tag{48}$$

And so, the matrix $\mathbf{A}$ (thereby the coefficients $a_1, a_2, \ldots, a_N$) is uniquely determined.

4. Accuracy of Solution

We can check the accuracy of the method. The truncated Fibonacci series in (3) have to be approximately satisfying (1). For each $x = x_i \in [a,b]$, $i = 1,2,3,\ldots$,

$$E(x_i)=\left|P(x_i)-Q(x_i)-g(x_i)-\lambda I(x_i)\right| \cong 0 \tag{49}$$

or

$$E(x_i)\le 10^{-k_i} \quad (k_i \text{ is any positive integer}). \tag{50}$$

If $\max(10^{-k_i}) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

5. Numerical Examples

In this section, several examples are given to illustrate the applicability of the method and all of them are performed on the computer MATLAB. Also, the absolute errors in tables are the values of $|y(x) - y_N(x)|$ at selected points.

Example 1. Let us consider the linear Fredholm integro-differential-difference equation given by

$$y''(x)-y''(x-1)+2xy'(x-2) = 4x^2-15x+4+12\int_0^1 xty(t)\,dt, \quad 0\le x,t\le 1, \tag{51}$$

with the boundary conditions

$$y(0)=1, \quad y(1)=1 \tag{52}$$

and the approximate solution $y(x)$ by the truncated Fibonacci series

$$y(x)=\sum_{n=1}^{3}a_nF_n(x), \tag{53}$$

where $N = 3$, $P_2(x) = 1$, $Q_1(x) = 2x$, $Q_2(x) = 1$, $\mu_1 = 1$, $\tau_1 = -2$, $\mu_2 = 1$, and $\tau_2 = -1$,

$$\lambda = 12, \qquad K(x,t)=xt, \qquad g(x)=4x^2-15x+4. \tag{54}$$

From (38), the collocation points for $N = 3$, are computed

$$\left\{x_1=0, x_2=\frac{1}{2}, x_3=1\right\} \tag{55}$$

and from (40), the fundamental matrix equation of the problem is

$$\left\{\mathbf{P}_2\mathbf{X}\left(\mathbf{T}^T\right)^2\mathbf{C}^T+\mathbf{Q}_0\mathbf{X}\boldsymbol{\beta}(1,-2)\left(\mathbf{T}^T\right)^0\mathbf{C}^T + \mathbf{Q}_1\mathbf{X}\boldsymbol{\beta}(1,-1)\left(\mathbf{T}^T\right)^2\mathbf{C}^T-\boldsymbol{\lambda}\mathbf{X}\mathbf{C}^T\mathbf{K}_f\mathbf{Q}\right\}\mathbf{A}=\mathbf{G}, \tag{56}$$

where

$$\mathbf{P}_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad \mathbf{Q}_1=\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}, \quad \mathbf{Q}_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$\mathbf{T}^T=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}, \quad \mathbf{C}^T=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}, \quad \mathbf{G}=\begin{bmatrix}4\\-\frac{5}{2}\\-7\end{bmatrix},$$

$$\boldsymbol{\beta}(1,-2)=\begin{bmatrix}1&-2&4\\0&1&-4\\0&0&1\end{bmatrix}, \quad \boldsymbol{\beta}(1,-1)=\begin{bmatrix}1&-1&1\\0&1&-2\\0&0&0\end{bmatrix},$$

$$\mathbf{K}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \quad \mathbf{Q}=\begin{bmatrix}1&\frac{1}{2}&\frac{4}{3}\\\frac{1}{2}&\frac{1}{3}&\frac{3}{4}\\\frac{4}{3}&\frac{3}{4}&\frac{28}{15}\end{bmatrix}. \tag{57}$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\mathbf{W};\mathbf{G}] = \begin{bmatrix} 0 & 0 & 4 & ; & 4 \\ -3 & -1 & -\frac{7}{2} & ; & -\frac{5}{2} \\ -6 & -2 & -9 & ; & -7 \end{bmatrix}. \tag{58}$$

From (37), the matrix forms for the boundary conditions are

$$\begin{aligned} [\mathbf{U}_0;\lambda_0] &= [1 \;\; 0 \;\; 1 \;\; ; \;\; 1], \\ [\mathbf{U}_1;\lambda_1] &= [0 \;\; 1 \;\; 2 \;\; ; \;\; 1]. \end{aligned} \tag{59}$$

The new augmented matrix based on conditions can be written as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} 0 & 0 & 4 & ; & 4 \\ 1 & 0 & 1 & ; & 1 \\ 0 & 1 & 2 & ; & 1 \end{bmatrix}. \tag{60}$$

Solving this system, the unknown Fibonacci coefficients are obtained as

$$A = [0 \;\; -1 \;\; 1]^T. \tag{61}$$

Hence, by substituting the Fibonacci coefficients matrix into (11), we have the approximate solution $y(x) = x^2 - x + 1$, which is the exact solution

$$\begin{aligned} y(x) &= \sum_{n=1}^{3} a_n F_n(x) = a_1 F_1(x) + a_2 F_2(x) + a_3 F_3(x) \\ &= 0.1 + (-1)\cdot x + 1\cdot\left(x^2+1\right) = x^2 - x + 1. \end{aligned} \tag{62}$$

Example 2 (see [26]). Consider the following linear Fredholm integro-differential-difference equation with variable coefficients

$$\begin{aligned} &y'''(x) - xy'(x) + y''(x-1) - xy(x-1) \\ &\quad = -(x+1)(\sin(x-1) + \cos(x)) \\ &\qquad - \cos(2) + 1 + \int_{-1}^{1} y(t-1)\,dt \qquad -1 \le x, t \le 1 \end{aligned} \tag{63}$$

with the conditions $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 0$ and the exact solution $y(x) = \sin(x)$.

So $P_1(x) = -x$, $P_3(x) = 1$, $Q_1(x) = 2x$, $Q_2(x) = 1$, $\mu_1 = 1$, $\tau_1 = -1$, $\mu_0 = 1$, and $\tau_0 = -1$,

$$\begin{aligned} &\lambda = 1, \quad K(x,t) = t - 1, \\ &g(x) = -(x+1)(\sin(x-1) + \cos(x)) - \cos(2) + 1. \end{aligned} \tag{64}$$

The fundamental matrix equation of the problem from (40) becomes

$$\begin{aligned} &\left\{\mathbf{P}_1\mathbf{X}\left(\mathbf{T}^T\right)^1\mathbf{C}^T + \mathbf{P}_3\mathbf{X}\left(\mathbf{T}^T\right)^3\mathbf{C}^T + \mathbf{Q}_0\mathbf{X}\boldsymbol{\beta}(1,-1)\left(\mathbf{T}^T\right)^0\mathbf{C}^T\right. \\ &\quad \left. + \mathbf{Q}_1\mathbf{X}\boldsymbol{\beta}(1,-1)\left(\mathbf{T}^T\right)^2\mathbf{C}^T - \boldsymbol{\lambda}\mathbf{X}\mathbf{C}^T\mathbf{K}_f\mathbf{Q}\right\}\mathbf{A} = \mathbf{G}. \end{aligned} \tag{65}$$

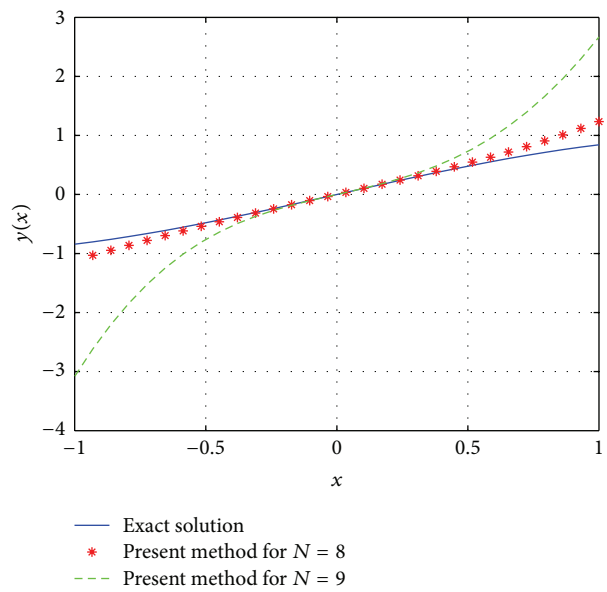

Figure 1: Numerical and exact solutions of Example 2 for $N = 8, 9$.

For $N = 8, 9$ the solutions of the problem are found as, respectively,

$$\begin{aligned} y(x) &= (0.003823057225)\,x^7 + (0.01008064682)\,x^6 \\ &\quad - (0.02660425406)\,x^5 + (0.05046207175)\,x^4 \\ &\quad + (0.1974487774)\,x^3 - (2.925448512)\,x^2 \\ &\quad + x + (5.629719781e-17), \\ y(x) &= -(0.003384338243)\,x^8 - (0.001420180534)\,x^7 \\ &\quad + (0.06727516657)\,x^6 - (0.1256152219)\,x^5 \\ &\quad - (0.2693617976)\,x^4 + (2.000085149)\,x^3 \\ &\quad - (6.513886652e-16)\,x^2 \\ &\quad + x - (1.153157431e-15). \end{aligned} \tag{66}$$

We compare the solutions found by the present method for $N = 8$, $N = 9$ and the absolute errors in Figure 1 and Table 1. It is seen that when we increase integer N, the errors decrease.

Example 3 (see [30]). Now consider the linear Fredholm integro-differential-difference equation given by

$$\begin{aligned} &y'''(x) - (x-1)\,y'' + (x-1)\,y'(x) - y(x) + y'(x-1) \\ &\quad = e^{x-1} + x\left(ex - \frac{1}{e}x - 2\frac{1}{e}\right) \\ &\qquad + \int_{-1}^{1}\left(xt - x^2\right)y(t)\,dt \qquad -1 \le x, t \le 1 \end{aligned} \tag{67}$$

Table 1: Comparison of the absolute errors of Example 2.

x_i	Exact solution	Present method			
		$N = 8$		$N = 9$	
		$y(x_i)$	Absolute errors	$y(x_i)$	Absolute errors
−1	−0.8414711	−1.114125	0.2726539	−3.078521	2.23705
−0.8	−0.7173561	−0.869866	0.1525099	−1.875847	1.158491
−0.6	−0.5646425	−0.633677	0.06903453	−1.054038	0.4893957
−0.4	−0.3894183	−0.4110374	0.0216191	−0.5333391	0.1439208
−0.2	−0.1986693	−0.2014897	0.00282041	−0.2163871	0.01771782
0	0	$5.62972e-17$	$5.62972e-17$	$-1.153157e-15$	$1.153157e-15$
0.2	0.1986693	0.2016525	0.00298318	0.2155338	0.01686445
0.4	0.3894183	0.4137037	0.02428533	0.5200945	0.1306762
0.6	0.5646425	0.6476974	0.08305494	0.9903835	0.4257410
0.8	0.7173561	0.9164897	0.1991336	1.689322	0.9719655
1	0.841471	1.235210	0.3937393	2.667579	1.826108

Table 2: Comparison of the absolute errors of Example 5.3.

x_i	Exact solution	Present method			
		$N = 6$		$N = 9$	
		$y(x_i)$	Absolute errors	$y(x_i)$	Absolute errors
−1	0.3678794	0.368349	$0.4695169e-3$	0.3678795	$9.388396e-8$
−0.8	0.449329	0.4496545	$0.3255685e-3$	0.4493289	$4.540547e-8$
−0.6	0.5488116	0.5490096	$0.1980008e-3$	0.5488116	$3.612436e-8$
−0.4	0.67032	0.6704069	$0.8684809e-4$	0.67032	$8.495309e-9$
−0.2	0.8187308	0.8187468	0.1605797	0.8187308	$7.845555e-10$
0	1.0	1.0	0	1.0	0
0.2	1.221403	1.221369	$0.3335666e-4$	1.221403	$3.813474e-9$
0.4	1.491825	1.491453	$0.3721909e-3$	1.491825	$2.903571e-8$
0.6	1.822119	1.820404	$0.171522e-2$	1.822119	$0.2499077e-6$
0.8	2.225541	2.220096	$0.5445047e-2$	2.225537	$0.3627591e-5$
1	2.718282	2.704284	0.01399794	2.71826	$0.2222775e-4$

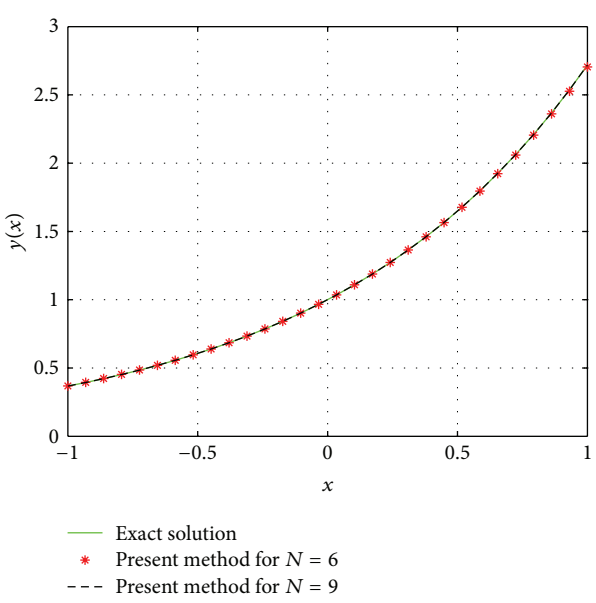

Figure 2: Numerical and exact solutions of Example 3 for $N = 6, 9$.

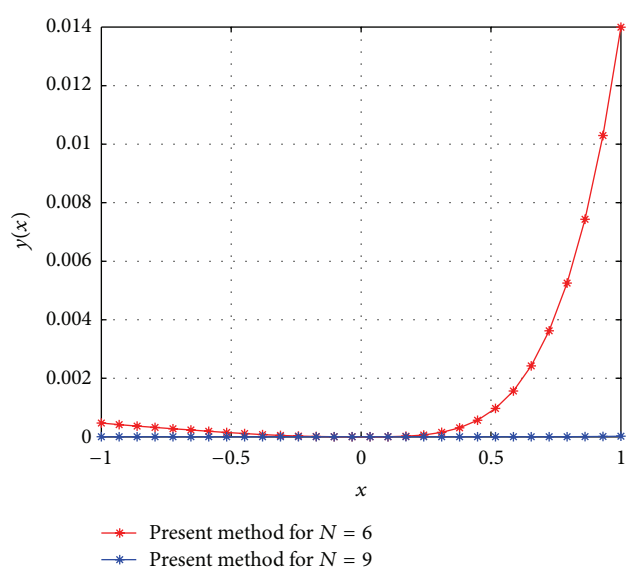

Figure 3: Comparison of the absolute errors of Example 3 for $N = 6, 9$.

with the initial conditions $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 0$ and the exact solution $y(x) = e^x$.

Here

$$\begin{gathered} P_0(x) = -1, \\ P_1(x) = x - 1, \\ P_2(x) = -(x-1), \qquad P_3(x) = 1, \\ Q_1(x) = 1, \qquad \mu_1 = 1, \tau_1 = -1, \\ \lambda = 1, \qquad K(x,t) = xt - x^2, \\ g(x) = e^{x-1} + x\left(ex - \frac{1}{e}x - 2\frac{1}{e}\right). \end{gathered} \tag{68}$$

From (40), the fundamental matrix equation of the problem becomes

$$\begin{aligned} &\left\{\mathbf{P}_0\mathbf{X}\left(\mathbf{T}^T\right)^0\mathbf{C}^T + \mathbf{P}_1\mathbf{X}\left(\mathbf{T}^T\right)^1\mathbf{C}^T + \mathbf{P}_2\mathbf{X}\left(\mathbf{T}^T\right)^2\mathbf{C}^T\right. \\ &+ \mathbf{P}_3\mathbf{X}\left(\mathbf{T}^T\right)^3\mathbf{C}^T + \mathbf{Q}_1\mathbf{X}\boldsymbol{\beta}(1,-1)\left(\mathbf{T}^T\right)^1\mathbf{C}^T \\ &\left.-\boldsymbol{\lambda}\mathbf{X}\mathbf{C}^T\mathbf{K}_f\mathbf{Q}\right\}\mathbf{A} = \mathbf{G}. \end{aligned} \tag{69}$$

The solutions obtained for $N = 6, 9$ are compared with the exact solution is e^x which are given in Figure 2 and we compare the absolute errors found by present method for $N = 6, 9$ in Figure 3. Also, the numerical solution and absolute errors are compared for $N = 6, 9$ in Table 2.

6. Conclusion

The Fibonacci collocation method is used to solve the linear Fredholm integro-differential-difference equations numerically. The obtained numerical results show that the accuracy improves when N is increased. Tables and figures indicate that as N increases, the errors decrease more rapidly. A considerable advantage of the method is that the Fibonacci polynomial coefficients of the solution are found very easily by using computer programs. This method can also be extended to the system of linear integro-differential-difference equation with variable coefficients, but some modifications are required.

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