Research Article

Faber Polynomial Coefficient Estimates for Meromorphic Bi-Starlike Functions

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We consider meromorphic starlike univalent functions that are also bi-starlike and find Faber polynomial coefficient estimates for these types of functions. A function is said to be bi-starlike if both the function and its inverse are starlike univalent.

Consider the function

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \quad (1)$$

where the coefficients $(b_0, b_1, b_2, \dots, b_n, \dots)$ are in the submanifold M on C^N such that $g(z)$ is univalent in $\Delta := \{z : 1 < |z| < \infty\}$. Therefore

$$\frac{zg'(z)}{g(z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, b_2, \dots, b_n) \frac{1}{z^{n+1}}, \quad (2)$$

where $F_{n+1}(b_0, b_1, b_2, \dots, b_n)$ is a Faber polynomial of degree $n+1$. (Also see [1, 2].) We note that

$$F_1 = -b_0,$$

$$F_2 = b_0^2 - 2b_1,$$

$$F_3 = -b_0^3 + 3b_1b_0 - 3b_2,$$

$$F_4 = b_0^4 - 4b_0^2b_1 + 4b_0b_2 + 2b_1^2 - 4b_3,$$

$$F_5 = -b_0^5 + 5b_0^3b_1 - 5b_0^2b_2 - 5b_0b_1^2 + 5b_1b_2 + 5b_0b_3 - 5b_4,$$

$$\begin{aligned} F_6 = & b_0^6 + 3b_2^2 + 6b_0^3b_2 - 12b_0b_1b_2 - 6b_0^4b_1 - 2b_1^3 \\ & + 9b_0^2b_1^2 + 6b_0b_4 + 6b_1b_3 - 6b_0^2b_3 - 6b_5. \end{aligned} \quad (3)$$

In general (also see Bouali [3, page 52])

$$\begin{aligned} F_{n+1}(b_0, b_1, \dots, b_n) \\ = \sum_{i_1+2i_2+\dots+(n+1)i_{n+1}=n+1} A(i_1, i_2, \dots, i_{n+1}) b_0^{i_1} b_1^{i_2} \dots b_n^{i_{n+1}}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} A(i_1, i_2, \dots, i_{n+1}) := & (-1)^{(n+1)+2i_1+\dots+(n+2)i_{n+1}} \\ & \times \frac{(i_1 + i_2 + \dots + i_{n+1} - 1)!(n+1)}{(i_1!)(i_2!) \dots (i_{n+1}!)}. \end{aligned} \quad (5)$$

The coefficients of g^{-1} , the inverse map of g are given by

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n \geq 1} \frac{1}{n} K_{n+1} \frac{1}{w^n}, \quad (6)$$

where

$$\begin{aligned} K_{n+1}^n &= nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 \\ &+ \frac{1}{2}n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2) \\ &+ \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) \\ &+ \sum_{j \geq 5} b_0^{n-j}V_j \end{aligned} \quad (7)$$

and V_j with $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \dots, b_n$. (Also see [1, page 349].)

Similarly

$$\frac{wh'(w)}{h(w)} = 1 + \sum_{n=1}^{\infty} F_n(B_0, B_1, B_2, \dots, B_n) \frac{1}{w^n}, \quad (8)$$

where $F_n(B_0, B_1, B_2, \dots, B_n)$ is a Faber polynomial of degree n and

$$\begin{aligned} F_n &= \frac{-n(n-(n+1))!}{n!(n-2n)!} B_0^n - \frac{n(n-(n+1))!}{(n-2)!(n-(2n-1))!} B_0^{n-2} B_1 \\ &- \frac{n(n-(n+1))!}{(n-3)!(n-(2n-2))!} B_0^{n-3} B_2 \\ &- \frac{n(n-(n+1))!}{(n-4)!(n-(2n-3))!} B_0^{n-4} \left(B_3 + \frac{n-(2n-3)}{2} B_1^2 \right) \\ &- \sum_{j \geq 5} B_0^{n-j} K_j, \end{aligned} \quad (9)$$

where K_j for $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $B_1, B_2, \dots, B_{n-1}$.

The Faber polynomials introduced by Faber [4] play an important role in various areas of mathematical sciences, especially in geometric function theory (e.g., see Gong [5] and Schiffer [6]). The recent interest in the calculus of the Faber polynomials, especially when it involves the function $h = g^{-1}$, the inverse map of g (see [2, page 186]) beautifully fits the case for the meromorphic bi-univalent functions.

The function g is said to be meromorphic bi-univalent in Δ if both g and its inverse $h = g^{-1}$ are meromorphic univalent in Δ . By the same token, the function g is said to be meromorphic bi-starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ if both g and its inverse map $h = g^{-1}$ are meromorphic starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ , that is,

$$\begin{aligned} \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) &> \alpha \quad (z \in \Delta), \\ \operatorname{Re} \left(\frac{wh'(w)}{h(w)} \right) &> \alpha \quad (w \in \Delta). \end{aligned} \quad (10)$$

Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer [6] obtained the estimate $|b_2| \leq 2/3$ for

meromorphic univalent functions g with $b_0 = 0$ and Duren ([7] or [8, Theorem 4.9, page 139]) proved that if $b_1 = b_2 = \dots = b_k = 0$ for $1 \leq k < (1/2)n$ then $|b_n| \leq 2/(n+1)$. He then proved that this bound also holds for meromorphic starlike univalent functions g of order zero (Duren [8, Theorem 4.8, page 137]). So far, the latest known results are given by the following two articles. Kapoor and Mishra [9] found sharp bounds for the coefficients of starlike univalent functions of order α ; $0 \leq \alpha < 1$ in Δ and for its inverse functions they obtained the bound $2(1-\alpha)/(n+1)$ when $((n-1)/n) \leq \alpha < 1$. More recently, Srivastava et al. [10] found sharp bounds for the coefficients of starlike univalent functions of order α , $0 \leq \alpha < 1$, having m -fold gaps in their series representation in Δ and also for their inverse functions. The above two articles settled the coefficient bounds for starlike functions and their inverses but they have not considered the bi-starlike case. The problem arises when the bi-univalence condition is imposed on the meromorphic functions g . The bi-univalence requirement makes the task of finding bounds for the coefficients of g and its inverse map $h = g^{-1}$ more involved. In this paper, for the first time, we use the Faber polynomial expansions to study the coefficients of meromorphic bi-starlike functions. As a result, we are able to prove.

Theorem 1. Let $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n(1/z^n)$ be meromorphic bi-starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ . If $b_1 = b_2 = \dots = b_{n-1} = 0$ for n being odd or if $b_0 = b_1 = \dots = b_{n-1} = 0$ for n being even, then

$$|b_n| \leq \frac{2(1-\alpha)}{n+1}; \quad 0 \leq \alpha < 1, \quad n \in \mathbb{N}. \quad (11)$$

Proof. Suppose that the function g is a meromorphic bi-starlike function of order $\alpha : (0 \leq \alpha < 1)$ in Δ . Then both g and its inverse $h = g^{-1}$ are starlike of order $\alpha : (0 \leq \alpha < 1)$ in Δ . Therefore, by definition, there exist two functions p and q with positive real parts in Δ of the form

$$\begin{aligned} p(z) &= 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \dots \quad (z \in \Delta), \\ q(w) &= 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \dots \quad (w \in \Delta), \end{aligned} \quad (12)$$

so that

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= \alpha + (1-\alpha)p(z) \\ &= 1 + \frac{(1-\alpha)c_1}{z} + \frac{(1-\alpha)c_2}{z^2} + \frac{(1-\alpha)c_3}{z^3} + \dots, \\ \frac{wh'(w)}{h(w)} &= \alpha + (1-\alpha)q(w) \\ &= 1 + \frac{(1-\alpha)d_1}{w} + \frac{(1-\alpha)d_2}{w^2} + \frac{(1-\alpha)d_3}{w^3} + \dots. \end{aligned} \quad (13)$$

Note that, according to the Caratheodory lemma (see Duren [8, page 41]), $|c_n| \leq 2$ and $|d_n| \leq 2$ for $n = 1, 2, 3, \dots$. On

the other hand, comparing the corresponding coefficients of the functions g and $h = g^{-1}$, we obtain

$$B_0 = -b_0, \quad B_n = -\frac{1}{n}K_{n+1}^n. \quad (14)$$

Now, from $F_{n+1}(b_0, b_1, \dots, b_n)$ and $F_{n+1}(B_0, B_1, B_2, \dots, B_n)$, upon noting that there are just two choices of $i_1 = n$ and $i_2 = i_3 = \dots = i_n = 0$ or $i_1 = i_2 = \dots = i_{n-1} = 0$ and $i_n = 1$, we obtain

$$F_{n+1} = \begin{cases} b_0^{n+1} - (n+1)b_n & n = \text{odd} \\ -b_0^{n+1} - (n+1)b_n & n = \text{even}, \end{cases} \quad (15)$$

$$F_{n+1} = \begin{cases} B_0^{n+1} - (n+1)B_n & n = \text{odd} \\ -B_0^{n+1} - (n+1)B_n & n = \text{even}. \end{cases}$$

Since $B_n = -b_n$ for the second system of equation we can write

$$F_{n+1} = \begin{cases} b_0^{n+1} + (n+1)b_n & n = \text{odd} \\ b_0^{n+1} + (n+1)b_n & n = \text{even}. \end{cases} \quad (16)$$

Therefore, for odd n , we obtain the system of equations

$$\begin{aligned} b_0^{n+1} - (n+1)b_n &= (1-\alpha)c_{n+1}, \\ b_0^{n+1} + (n+1)b_n &= (1-\alpha)d_{n+1}. \end{aligned} \quad (17)$$

Hence

$$2(n+1)b_n = (1-\alpha)(d_{n+1} - c_{n+1}). \quad (18)$$

Applying the Caratheodory Lemma yields

$$|b_n| \leq \frac{4(1-\alpha)}{2(n+1)} = \frac{2(1-\alpha)}{n+1}. \quad (19)$$

Similarly, for even n with $(b_0 = b_1 = \dots = b_{n-1} = 0)$, we obtain

$$\begin{aligned} (n+1)b_n &= -(1-\alpha)c_{n+1}, \\ (n+1)b_n &= +(1-\alpha)d_{n+1}. \end{aligned} \quad (20)$$

Hence

$$2(n+1)b_n = (1-\alpha)(d_{n+1} - c_{n+1}), \quad (21)$$

which upon applying the Caratheodory Lemma, we obtain

$$|b_n| \leq \frac{2(1-\alpha)}{n+1}. \quad (22)$$

Relaxing the coefficient restrictions imposed on Theorem 1, we can prove the following. $\square$

Theorem 2. Let $g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n(1/z^n)$ be meromorphic bi-starlike of order α : $(0 \leq \alpha < 1)$ in Δ . Then

- (i) $|b_0| \leq \sqrt[3]{2(1-\alpha)}$,
- (ii) $|b_1| \leq (1-\alpha)$.

Proof. Comparing the corresponding coefficients of

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= 1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} - \dots, \\ \frac{zg'(z)}{g(z)} &= \alpha + (1-\alpha)p(z) \\ &= 1 + \frac{(1-\alpha)c_1}{z} + \frac{(1-\alpha)c_2}{z^2} + \dots, \end{aligned} \quad (23)$$

we obtain

$$\begin{aligned} -b_0 &= (1-\alpha)c_1, \\ b_0^2 - 2b_1 &= (1-\alpha)c_2. \end{aligned} \quad (24)$$

Similarly, comparing the corresponding coefficients of

$$\begin{aligned} \frac{wh'(w)}{h(w)} &= 1 + \frac{b_0}{w} + \frac{b_0^2 + 2b_1}{w^2} + \dots, \\ \frac{wh'(w)}{h(w)} &= \alpha + (1-\alpha)q(w) \\ &= 1 + \frac{(1-\alpha)d_1}{w} + \frac{(1-\alpha)d_2}{w^2} + \dots, \end{aligned} \quad (25)$$

we obtain

$$\begin{aligned} b_0 &= (1-\alpha)d_1, \\ b_0^2 + 2b_1 &= (1-\alpha)d_2. \end{aligned} \quad (26)$$

Adding $b_0^2 - 2b_1 = (1-\alpha)c_2$ and $b_0^2 + 2b_1 = (1-\alpha)d_2$, we obtain

$$2b_0^2 = (1-\alpha)(c_2 + d_2) \quad (27)$$

which, upon applying the Caratheodory Lemma, yields

$$|b_0| \leq \sqrt[3]{2(1-\alpha)}. \quad (28)$$

On the other hand, subtracting $b_0^2 - 2b_1 = (1-\alpha)c_2$ from $b_0^2 + 2b_1 = (1-\alpha)d_2$, we obtain $4b_1 = (1-\alpha)(d_2 - c_2)$ which upon, applying the Caratheodory Lemma, yields

$$|b_1| \leq 1-\alpha. \quad (29)$$

$\square$

Remark 3. For the estimates of the first two coefficients of certain subclasses of analytic and bi-univalent functions, also see recent publications by Srivastava et al. [11] and Frasin and Aouf [12].

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