

Oscillation in pest population and its management: A mathematical study

Samit Bhattacharyya¹ and Suma Ghosh

supplementary materials

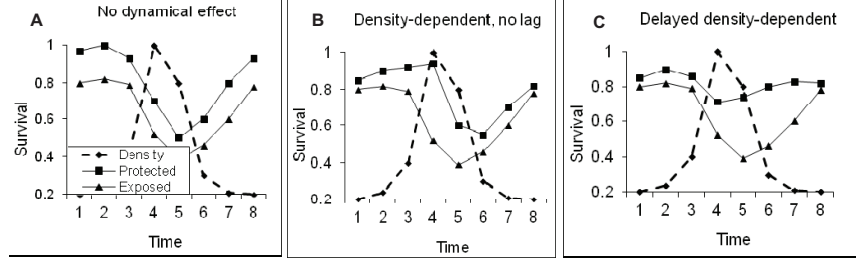


Figure 1: Possible dynamical effects of predation. In all graphs, the dotted line indicates SBP population density during the course of a single oscillation, peaking in year 4. The solid line indicates the survival rate that determines the course of the oscillation. The broken line indicates the survival rate when predators are excluded, and the separation between the solid and broken lines measures the predation impact. (A) The expected or mean predation impact does not vary with density. (B) Predation acting as a first-order process, with the greatest impact occurring during the peak year. (C) Predation acting as a second-order process, with the greatest impact occurring during the period of population collapse. If predation were the dynamical factor completely responsible for population change, then the broken line in (C) would be completely flat (Turchin et al., 1999).

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1 Local stability of system (4.1)-(4.2) around E^*

To investigate the nature of the system around E^* , we perturb system (4.1) – (4.2) around $E^* = (x^*, y^*)$. Neglecting the higher order terms, we obtain the linearized the system around E^* , which is given by

$$\frac{dx'}{dt} = A_1 x' + A_2 y', \quad (1)$$

$$\frac{dy'}{dt} = B_1 y' + B_2 x'(t - \tau), \quad (2)$$

where $A_1 = -\left(\frac{rx^*}{K} - \frac{\alpha x^* y^*}{(\gamma + x^*)^2}\right)$, $A_2 = -\frac{\alpha x^*}{\gamma + x^*}$, $B_1 = -\epsilon y^*$, $B_2 = \frac{\beta \gamma y^*}{(\gamma + x^*)^2}$.

It may be mentioned that the control u is involved in equations (4.3)-(4.4) via components of equilibrium, i.e., x^* , y^* .

Further, this linearized coupled system can be written as

$$\frac{d^2 x'}{dt^2} - (A_1 + B_1) \frac{dx'}{dt} + A_1 B_1 x' - A_2 B_2 x'(t - \tau) = 0. \quad (3)$$

Now consider the equation (4.5) and the space of all real valued continuous functions defined on $[-\tau, \infty)$ satisfying the initial conditions

$$x'(t) = 0 \text{ for } -\tau \leq t < 0, \quad x'(0+) = P_1 > 0 \quad \text{and} \quad \dot{x}'(0+) = P_2 > 0.$$

Laplace transformation of equation (3) gives then

$$s^2 C - sP_1 - P_2 - (A_1 + B_1)[sC - P_1] + A_1 B_1 C - A_2 B_2 C e^{-s\tau} = 0, \quad (4)$$

where $C := C(s)$ is the Laplace transform of x' .

Now, the inverse of Laplace transform of $C(s)$ will give terms that exponentially increases with time if $C(s)$ has any pole with positive real part.

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A necessary and sufficient condition for local stability is that all poles of $C(s)$ have negative real parts. The easiest way of determining whether any poles are located in the right half plane, is to the use of Nyquist-plot technique (Thingstad and Langland, 1974; Nyquist, 1932). The Nyquist criteria states that if we let s traverse a curve encircling the right half plane, the curve $C(s)$ will encircle the origin a number of times which is equal to the difference between number of poles and number of zeros of $C(s)$ in the right half plane. As zeros of

$$\Psi(s) = s^2 - (A_1 + B_1)s + A_1B_1 - A_2B_2e^{-s\tau} \quad (5)$$

are identical to the poles of $C(s)$, so this criteria leads to the following necessary and sufficient condition for local stability of E^* :

$$\text{Im}\Psi(i\omega_0) > 0, \quad (6)$$

$$\text{Re}\Psi(i\omega_0) = 0, \quad (7)$$

where ω_0 is the smallest positive value of ω for which the equation (7) holds. From (5) we have

$$\text{Im}\Psi(i\omega_0) = -(A_1 + B_1)\omega_0 + A_2B_2\sin(\omega_0\tau), \quad (8)$$

$$\text{Re}\Psi(i\omega_0) = -\omega_0^2 + A_1B_1 - A_2B_2\cos(\omega_0\tau). \quad (9)$$

Now, suppose $z = \omega^2 - A_1B_1$ and $z = -A_2B_2\cos(\omega\tau)$. So, existence of such an positive ω_0 satisfying (9) implies that these two curves must intersect and as ω_0 is the smallest one among such roots, we have from Fig (a)

$$\tau < \frac{\pi}{2\sqrt{A_1B_1}} \quad (10)$$

and ω_0 is restricted to

$$\sqrt{A_1B_1} < \omega_0 < \frac{\pi}{2\tau}. \quad (11)$$

Note that A_1B_1 is a positive quantity due to (3.7). Again, (8) implies that

$$\sin(\omega_0\tau) < \left(\frac{A_1 + B_1}{A_2B_2}\right)\omega_0 \quad (12)$$

for all ω_0 satisfies (11). From Fig (b), It is clearly true, if

$$\sin(\sqrt{A_1B_1}\tau) \leq \left(\frac{A_1 + B_1}{A_2B_2}\right)\sqrt{A_1B_1} \quad (13)$$

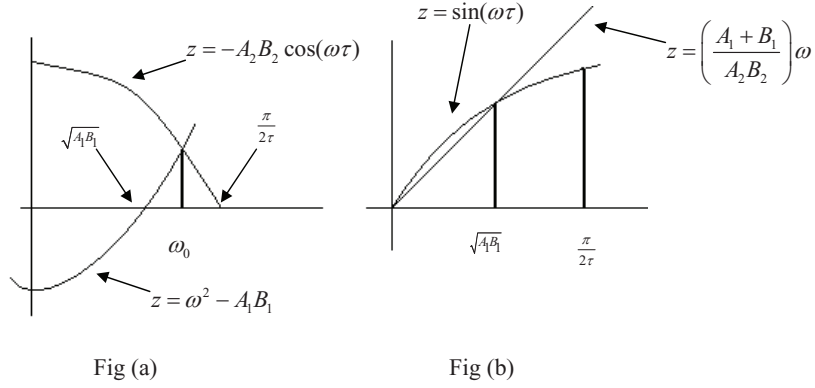


Figure 1: Graphical representations of equations (4.9) (Fig (a)) and (4.8) (Fig (b)) in $\omega - z$ plane.

holds and this must be true for all $\tau < \bar{\tau}$ for some $\bar{\tau}$, (13) holds. Thus, we conclude that in our case Nyquist criterion holds and the interior equilibrium E^* of the system (4.1) – (4.2) is locally asymptotically stable for all values of τ satisfying

$$0 < \tau < \tau_{\min} = \min \left(\frac{\pi}{2\sqrt{A_1 B_1}}, \bar{\tau} \right). \quad (14)$$

2 Bifurcation of the solution

We consider the parameter time delay τ as bifurcation parameter of the system (4.1) – (4.2). Assuming $x' = e^{\lambda t}$ as a solution of the following characteristic equation

$$\Delta(\lambda, \tau) = \lambda^2 - (A_1 + B_1)\lambda + A_1 B_1 - A_2 B_2 e^{-\lambda \tau} = 0. \quad (15)$$

Clearly, $\lambda = 0$ is not a solution of the above equation. Assume that for some $\tau > 0$, $i\omega$ with $\omega > 0$ is a solution of (15). Substituting this in (15) and separating the real and imaginary parts, we have

$$-\omega^2 + A_1 B_1 - A_2 B_2 \cos(\omega\tau) = 0, \quad (16)$$

$$-(A_1 + B_1)\omega + A_2 B_2 \sin(\omega\tau) = 0. \quad (17)$$

Squaring and adding (16) & (17), we arrive at the result

$$\omega^4 + (A_1^2 + B_1^2)\omega^2 + [(A_1 B_1)^2 - (A_2 B_2)^2] = 0, \quad (18)$$

It is easy to see that under the condition

$$A_1 B_1 + A_2 B_2 < 0 \quad (19)$$

the equation (18) has only one positive real root, namely

$$\omega_+ = \sqrt{\frac{1}{2}[-(A_1^2 + B_1^2) + \sqrt{\{(A_1^2 - B_1^2)^2 + 4A_2^2 B_2^2\}}]}. \quad (20)$$

Again, using (16) & (17), we have

$$(A_2 B_2)^2 \cos^2(\omega\tau) - (A_2 B_2)(A_1 + B_1)^2 \cos(\omega\tau) + (A_1 B_1)(A_1 + B_1)^2 - (A_2 B_2)^2 = 0. \quad (21)$$

Set

$$f(z) = (A_2 B_2)^2 z^2 - (A_2 B_2)(A_1 + B_1)^2 z + (A_1 B_1)(A_1 + B_1)^2 - (A_2 B_2)^2 = 0. \quad (22)$$

Then it is easy to see that under condition (19)

$$\begin{aligned} f(1) &= (A_1 + B_1)^2 (A_1 B_1 - A_2 B_2) > 0, \\ f(-1) &= (A_1 + B_1)^2 (A_1 B_1 + A_2 B_2) < 0. \end{aligned}$$

This shows that $f(z)$ has real solution in $(-1, 1)$ of the form $\cos(\omega\tau) = k$, where $|k| < 1$. From equation (17), τ is given by

$$\tau_k = \frac{1}{\omega_+} \arcsin\left(\frac{A_1 + B_1}{A_2 B_2}\right) \omega_+ + \frac{2k\pi}{\omega_+} \quad k = 0, 1, 2, \dots \quad (23)$$

Again, from analysis of the system (2.1) – (2.2) (i.e., without delay), it reveals that real part of all the eigenvalues are negative, when $\tau = 0$. Thus

applying Rouché theorem (in the form of lemma in Cooke and Grossman, 1982), we obtain the conclusion that whenever $\tau = \tau_k$ equation (15) has a pair of conjugate purely imaginary roots $\pm i\omega_+$ and all other roots have negative real parts. Thus we proved the following lemma:

Lemma 1 *Suppose (3.7) holds and $A_1B_1 + A_2B_2 < 0$. Then at*

$$\tau_k = \frac{1}{\omega_+} \arcsin \left(\frac{A_1 + B_1}{A_2B_2} \right) \omega_+ + \frac{2k\pi}{\omega_+} \quad k = 0, 1, 2, \dots$$

equation (15) has a simple pair of conjugate purely imaginary roots $\pm i\omega_+$ and all other roots have negative real parts, where

$$\omega_+ = \sqrt{\frac{1}{2}[-(A_1^2 + B_1^2) + \sqrt{\{(A_1^2 - B_1^2)^2 + 4A_2^2B_2^2\}}]}.$$

Next we compute the transversality condition. Substituting $\lambda = \nu + i\omega$ in equation (15) and separating the real and imaginary parts we have the following transcendental equation

$$\nu^2 - \omega^2 + A_1B_1 - (A_1 + B_1)\nu - A_2B_2e^{-\nu\tau} \cos(\omega\tau) = 0, \quad (24)$$

$$2\nu\omega - (A_1 + B_1)\omega + A_2B_2 \sin(\omega\tau) = 0. \quad (25)$$

As the characteristic polynomial in (15) is an analytic function of λ and τ , then a root $\lambda(\tau)$ of it is a differentiable function of τ near $i\omega_+$. Differentiating (24) and (25) with respect to τ , we get

$$\begin{aligned} & \{2\nu - (A_1 + B_1) + A_2B_2e^{-\nu\tau} \cos(\omega\tau)\} \frac{d\nu}{d\tau} + \{-2\omega + A_2B_2e^{-\nu\tau} \sin(\omega\tau)\} \frac{d\omega}{d\tau} \\ &= -A_2B_2e^{-\nu\tau} (\nu \cos(\omega\tau) + \omega \sin(\omega\tau)) \end{aligned}$$

and

$$\begin{aligned} & \{2\omega - A_2B_2e^{-\nu\tau} \sin(\omega\tau)\} \frac{d\nu}{d\tau} + \{2\nu - (A_1 + B_1) + A_2B_2e^{-\nu\tau} \cos(\omega\tau)\} \frac{d\omega}{d\tau} \\ &= A_2B_2e^{-\nu\tau} (\nu \sin(\omega\tau) - \omega \cos(\omega\tau)), \end{aligned}$$

which gives under simplification

$$\begin{aligned} & [\{2\nu - (A_1 + B_1) + A_2B_2e^{-\nu\tau} \cos(\omega\tau)\}^2 + \{2\omega - A_2B_2e^{-\nu\tau} \sin(\omega\tau)\}^2] \frac{d\nu}{d\tau} \\ &= A_2B_2e^{-\tau\nu} [-(\nu \cos(\omega\tau) + \omega \sin(\omega\tau))\{2\nu - (A_1 + B_1) + A_2B_2e^{-\nu\tau} \cos(\omega\tau)\} - \\ & \quad \{2\omega - A_2B_2e^{-\nu\tau} \sin(\omega\tau)\}(\nu \sin(\omega\tau) - \omega \cos(\omega\tau))] \end{aligned}$$

$$(\nu \sin(\omega\tau) - \omega \cos(\omega\tau))\{-2\omega + A_2 B_2 e^{-\nu\tau} \sin(\omega\tau)\}].$$

At $\nu = 0$, $\omega = \omega_+$ and $\tau = \tau_0$, it reduces to

$$\begin{aligned} & [\{A_2 B_2 \tau_0 \cos(\omega_+ \tau_0) - (A_1 + B_1)\}^2 + \{2\omega_+ - A_2 B_2 \tau_0 \sin(\omega_+ \tau_0)\}^2] \frac{d\nu}{d\tau} \\ &= A_2 B_2 \omega_+ [(A_1 + B_1) \sin(\omega_+ \tau_0) - 2\omega_+ \cos(\omega_+ \tau_0)] \\ &= \omega_+^2 [(A_1^2 + B_1^2) + 2\omega_+^2] > 0, \text{ whenever } \omega_+ \neq 0. \end{aligned}$$

So the root must pass from the negative to the positive half plane as τ increases. On the other hand, $\omega_+ = 0$ corresponds to a zero root $\lambda = 0$ of (15), which is also impossible due to (3.7). Thus root can cross the imaginary axis only left to right as τ increases. If the stability is lost at some critical value of τ , it can never be regained.

Hence, using the above transversality condition and lemma in (Cooke and Grossman, 1982), we obtain the following lemma:

Lemma 2 *Suppose (3.7) holds and $A_1 B_1 + A_2 B_2 < 0$. If $\tau > \tau_0$, equation (15) has at least one root with strictly positive real part.*

Summarizing the above discussions and using the standard Hopf-bifurcation theorem for retarded DDEs (Hale, 1993), we conclude theorem 1.

3 Stability of the bifurcation

In this section, we determine the formula that establishes the stability of bifurcating periodic orbits using the approach adapted in Hassard et al. (1981). We assume the case where the Hopf bifurcation occurs (at $\tau = \tau_0$ and $\omega = \omega_+$). Normalizing the delay τ in the system (4.1) – (4.2) by the time scaling $t \rightarrow t/\tau$, we have

$$\frac{dx}{dt} = \tau r x \left(1 - \frac{x}{K}\right) - \frac{\tau \alpha x y}{\gamma + x} - q_1 \tau u x, \quad (26)$$

$$\frac{dy}{dt} = \tau y \left(-d - \epsilon y + \frac{\beta x(t-1)}{\gamma + x(t-1)}\right) - q_2 \tau u y. \quad (27)$$

Expanding the system (26)–(27) around $E^* = (x^*, y^*)$ up to 3 order terms and using the same variables, we obtain

$$\frac{dx}{dt} = a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{111} x^3 + a_{112} x^2 y, \quad (28)$$

$$\frac{dy}{dt} = b_1 y + b_2 x(t-1) + b_{22} y^2 + b_{12} x(t-1)y + b_{111} x^3(t-1) + b_{112} x^2(t-1)y, \quad (29)$$

where $a_1 = \tau A_1$, $a_2 = \tau A_2$, $a_{11} = -\left(\frac{\tau r}{K} - \frac{\tau \alpha \gamma y^*}{(\gamma + x^*)^3}\right)$, $a_{12} = -\frac{\tau \alpha \gamma}{(\gamma + x^*)^2}$, $a_{111} = -\frac{\tau \alpha \gamma y^*}{(\gamma + x^*)^4}$, $a_{112} = \frac{\tau \alpha \gamma}{(\gamma + x^*)^3}$, $b_1 = \tau B_1$, $b_2 = \tau B_2$, $b_{22} = -\epsilon \tau$, $b_{12} = \frac{\tau \beta \gamma}{(\gamma + x^*)^2}$, $b_{111} = \frac{\tau \beta \gamma y^*}{(\gamma + x^*)^4}$ and $b_{112} = -\frac{\tau \beta \gamma}{(\gamma + x^*)^3}$.

We will see later that other higher terms will not contribute in the computation. Using the standard notation of Hassard et al (1981), we transform the equations (28)-(29) into the operator differential equation

$$\dot{x}_t = Ax_t + Rx_t, \quad (30)$$

where $x = \begin{pmatrix} x \\ y \end{pmatrix}$, $x_t(\theta) = x(t + \theta)$, $\theta \in (-1, 0]$. The linear operator A and R are defined as follows:

$$A\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta} & -1 \leq \theta < 0 \\ \int_{-1}^0 d\eta(s, \tau)\phi(s) & \theta = 0 \end{cases} \quad (31)$$

and

$$R\phi(\theta) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -1 \leq \theta < 0 \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} & \theta = 0, \end{cases} \quad (32)$$

where

$$\begin{aligned} f_1(\phi(\theta)) &= a_{11}\phi_1^2(\theta) + a_{12}\phi_1(\theta)\phi_2(\theta) + a_{111}\phi_1^3(\theta) + a_{112}\phi_1^2(\theta)\phi_2(\theta), \\ f_2(\phi(\theta)) &= b_{22}\phi_2^2(\theta) + b_{12}\phi_1(\theta-1)\phi_2(\theta) + b_{111}\phi_1^3(\theta-1) + b_{112}\phi_1^2(\theta-1)\phi_2(\theta-1), \end{aligned} \quad (33)$$

and

$$d\eta(\theta, \tau) = \begin{pmatrix} a_1\delta(\theta) & a_2\delta(\theta) \\ b_1\delta(\theta) & b_2\delta(\theta+1) \end{pmatrix} d\theta, \quad (35)$$

where $\delta(\theta)$ is dirac delta function.

Note that $d\eta$ depends upon the bifurcation parameter τ . Obviously $A = A(\tau)$, but as we said earlier, in most of the following computations, unless explicitly specified, it is assumed that $A = A(\tau_0)$. Note that A and its adjoint operator A^* can have complex eigenvectors. It is therefore suitable to allow for ϕ functions $\phi : (-1, 0] \rightarrow C^2$ instead of R^2 .

In order to determine the Poincare normal form of operator A , we need to compute the eigenvector q of operator A belonging to the eigenvalue $i\omega_+$ and eigenvector q^* of the adjoint operator A^* belonging to the eigenvalue $-i\omega_+$. We obtain

$$q(\theta) = \begin{pmatrix} D \\ 1 \end{pmatrix} e^{i\omega_+\theta} \quad -1 \leq \theta \leq 0, \quad (36)$$

where

$$D = \frac{a_2(a_1 + i\omega_+)}{a_1^2 + \omega_+^2}.$$

To determine the eigenvector q^* , we use the relations

$$\langle q, q^* \rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q} \rangle = 0, \quad (37)$$

where the scalar product $\langle \cdot, \cdot \rangle$ is defined as follows:

$$\langle \psi, \phi \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi. \quad (38)$$

Here ‘ $\bar{\cdot}$ ’ denotes the complex conjugate, and $\phi : (-1, 0] \rightarrow C^2$ and $\psi : [0, 1) \rightarrow C^2$ are continuous and bounded functions. Moreover, $\bar{\psi}(0) \cdot \phi(0) = \sum_{j=1}^2 \bar{\psi}_j(0) \phi_j(0)$.

Suppose $q^*(\theta) = (\nu_1, \nu_2) e^{-i\omega_+\theta}$, where ν_1 and ν_2 have to be determined from the relations in (37).

Now according to (38),

$$\begin{aligned} \langle q^*, q \rangle &= D\bar{\nu}_1 + \bar{\nu}_2 - \int_{\theta=-1}^0 \left[\int_{\xi=0}^{\theta} e^{i\omega_+\theta} (\bar{\nu}_1, \bar{\nu}_2) \begin{pmatrix} a_1\delta(\theta) & a_2\delta(\theta) \\ b_1\delta(\theta) & b_2\delta(\theta+1) \end{pmatrix} \begin{pmatrix} D \\ 1 \end{pmatrix} d\xi \right] d\theta \\ &= D\bar{\nu}_1 + \bar{\nu}_2 - \int_{\theta=-1}^0 \left[\int_{\xi=0}^{\theta} e^{i\omega_+\theta} \{\varphi_1\delta(\theta) + \varphi_2\delta(\theta+1)\} d\xi \right] d\theta \end{aligned}$$

where, $\varphi_1 = \bar{\nu}_1(a_1 D + a_2) + \bar{\nu}_2 b_1 D$ and $\varphi_2 = \bar{\nu}_2 b_2$.

After computation the double integration using the properties of Dirac Delta function, we obtain

$$\langle q^*, q \rangle = D\bar{\nu}_1 + \bar{\nu}_2 e_1, \quad (39)$$

where $e_1 = 1 + \frac{1}{2}b_2(\cos \omega_+ - i \sin \omega_+)$. In a similar manner, we compute that

$$\langle q^*, \bar{q} \rangle = \bar{D}\bar{\nu}_1 + \bar{\nu}_2 e_2, \quad (40)$$

where $e_2 = 1 + \frac{b_2}{4\omega_+} \sin \omega_+$.

Using the relations in (37); (38) and (40) finally give

$$\bar{\nu}_1 = \frac{e_2}{De_2 - \bar{D}e_1} \quad \text{and} \quad \bar{\nu}_2 = \frac{-\bar{D}}{De_2 - \bar{D}e_1} \quad (41)$$

Thus we obtain the eigenvector q^* using (41).

Now center manifold \mathcal{C}_0 near the origin $x = 0$ for equation (30), is an attracting, locally invariant, 2-dimensional manifold in C . If we define

$$z = \langle q^*, x_t \rangle \quad \text{and} \quad w = x_t - zq - \bar{z}\bar{q}, \quad (42)$$

where x_t is a solution of the original equation (30). Then on \mathcal{C}_0

$$w(t, \theta) = w(z(t), \bar{z}(t), \theta) = \frac{1}{2}w_{20}(\theta)z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2 + \dots \quad (43)$$

In fact, z and \bar{z} are local coordinates for \mathcal{C}_0 in the directions of q^* and \bar{q}^* . The existence of center manifold \mathcal{C}_0 enables us to reduce the equation (30) to an ordinary differential equation of a single complex variable on \mathcal{C}_0 . In the variable z and w , (30) becomes:

$$\dot{z} = i\omega_0 z + \langle q^*(\theta), R(w + 2\text{Re}\{zq(\theta)\}) \rangle, \quad (44)$$

$$\dot{w} = Aw - 2\text{Re}\{\langle q^*(\theta), R(w + 2\text{Re}\{zq(\theta)\}) \rangle q(\theta)\} + R(w + 2\text{Re}\{zq(\theta)\}) \quad (45)$$

By the definition of scalar product in (38) and from definition of R in (32), we observe that

$$\langle q^*(\theta), R(w + 2\text{Re}\{zq(\theta)\}) \rangle = \bar{q}^*(0).R(w(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\}). \quad (46)$$

Then according to the Hassard et al.(1981), we define the functions

$$g(z, \bar{z}) := \bar{q}^*(0).R(w(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\}), \quad (47)$$

$$H(z, \bar{z}, \theta) := R(w(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\}) - 2\text{Re}\{g(z, \bar{z})q(\theta)\} \quad (48)$$

In this notation, we rewrite the equations (44) and (45) as:

$$\dot{z} = i\omega_0 z + g(z, \bar{z}), \quad (49)$$

$$\dot{w} = Aw + H(z, \bar{z}, \theta) \quad (50)$$

The objective is to expand on \mathcal{C}_0 the function $g(z, \bar{z})$ in powers of z and \bar{z} :

$$g(z, \bar{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \dots \quad (51)$$

and to determine the coefficients of the expansion of (51). This can be done by comparison of (51) with (47) when substituting for w its expansion (43). Now to compute the coefficients $w_{ij}(\theta)$ of (43) on \mathcal{C}_0 we expand the function $H(z, \bar{z}, \theta)$ in powers of z and \bar{z} :

$$H(z, \bar{z}, \theta) = \frac{1}{2}H_{20}(\theta)z^2 + H_{11}(\theta)z\bar{z} + \frac{1}{2}H_{02}(\theta)\bar{z}^2 + \dots \quad (52)$$

The coefficients of the expansion (52) can be computed from (48) as

$$H_{20} = \left[\frac{\partial^2}{\partial z^2} H \right]_{z=\bar{z}=0}, \quad H_{11} = \left[\frac{\partial^2}{\partial z \partial \bar{z}} H \right]_{z=\bar{z}=0}.$$

Thus we obtain (Details are given in APPENDIX A)

$$H_{20}(\theta) = -(\bar{E}_1 q(\theta) + E_1 \bar{q}(\theta)) + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -1 \leq \theta < 0 \\ \begin{pmatrix} a_{11}D^2 + a_{12}D \\ b_{22} + b_{12}De^{-i\omega_+} \end{pmatrix} & \theta = 0, \end{cases} \quad (53)$$

where $E_1 = \nu_1(a_{11}\bar{D}^2 + a_{12}\bar{D}) + \nu_2(b_{22} + b_{12}\bar{D}e^{i\omega_+})$,

and

$$H_{11}(\theta) = -(\bar{E}_2 q(\theta) + E_2 \bar{q}(\theta)) + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -1 \leq \theta < 0 \\ \begin{pmatrix} 2a_{11} |D|^2 + 2a_{12}\text{Re}\{D\} \\ 2b_{22} + 2b_{12}\text{Re}\{De^{-i\omega_+}\} \end{pmatrix} & \theta = 0, \end{cases} \quad (54)$$

where $E_2 = \nu_1(2a_{11} |D|^2 + 2a_{12}\text{Re}\{D\}) + \nu_2(2b_{22} + 2b_{12}\text{Re}\{De^{-i\omega_+}\})$.

On the other hand, near the origin, we write $w(z, \bar{z})$ as

$$\dot{w}(z, \bar{z}) = w_z \dot{z} + w_{\bar{z}} \dot{\bar{z}}. \quad (55)$$

By using (43) to replace the derivatives w_z and $w_{\bar{z}}$ and (49) to compute \dot{z} and $\dot{\bar{z}}$, we get another expression for \dot{w} . Equating the two expressions of \dot{w} from (50) and (52), we have

$$(2i\omega_+ I - A)w_{20}(\theta) = H_{20}(\theta), \quad (56)$$

$$-Aw_{11}(\theta) = H_{11}(\theta), \quad (57)$$

where $w_{02}(\theta) = \bar{w}_{20}(\theta)$.

Solving these two operator equations (56) and (57), we obtain the expression of $w_{20}(\theta)$ and $w_{11}(\theta)$ for $-1 \leq \theta \leq 0$. For details see the APPENDIX B.

Now in (47), let us consider the expression $R(w(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\})$. According to the definition of operator R in (32), when $\theta = 0$, if we substitute for w its expression (43), then we obtain the components of $R(w(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\})$ defined as:

$$\begin{aligned} f_1^0 &:= a_{11} \left[zD + z\bar{D} + \frac{1}{2}w_{20}^{(1)}(0)z^2 + w_{11}^{(1)}(0)z\bar{z} + \frac{1}{2}w_{02}^{(1)}(0)\bar{z}^2 \right]^2 + a_{12} \left[z + \bar{z} + \frac{1}{2}w_{20}^{(2)}(0)z^2 + w_{11}^{(2)}(0)z\bar{z} + \frac{1}{2}w_{02}^{(2)}(0)\bar{z}^2 \right] \times \left[zD + z\bar{D} + \frac{1}{2}w_{20}^{(1)}(0)z^2 + w_{11}^{(1)}(0)z\bar{z} + \frac{1}{2}w_{02}^{(1)}(0)\bar{z}^2 \right] \\ &+ a_{111} \left[zD + z\bar{D} + O\left(|z|^2\right) \right]^3 + a_{112} \left[zD + z\bar{D} + O\left(|z|^2\right) \right]^2 \times \left[z + \bar{z} + O\left(|z|^2\right) \right] \\ &= z^2 \left[a_{11}D^2 + a_{12}D \right] + \bar{z}^2 \left[a_{11}\bar{D}^2 + a_{12}\bar{D} \right] + z\bar{z} \left[2a_{11}|D|^2 + 2a_{12}\text{Re}(D) \right] \\ &+ z^2\bar{z} \left[a_{11} \left\{ \bar{D}w_{20}^{(1)}(0) + 2Dw_{11}^{(1)}(0) \right\} + a_{12} \left\{ \bar{D}w_{20}^{(2)}(0) + w_{20}^{(1)}(0) + Dw_{11}^{(2)}(0) + w_{11}^{(1)}(0) \right\} \right] + \\ &3a_{111}|D|^2 D + a_{112} \left\{ D^2 + 2|D|^2 \right\} + O\left(|z|^3\right) \\ &= A_{20}z^2 + A_{02}\bar{z}^2 + A_{11}z\bar{z} + A_{21}z^2\bar{z} + O\left(|z|^3\right) \text{ (say).} \\ f_2^0 &:= b_{22} \left[z + \bar{z} + \frac{1}{2}w_{20}^{(2)}(0)z^2 + w_{11}^{(2)}(0)z\bar{z} + \frac{1}{2}w_{02}^{(2)}(0)\bar{z}^2 \right]^2 + b_{12} \left[zD + z\bar{D} + \frac{1}{2}w_{20}^{(1)}(-1)z^2 + w_{11}^{(1)}(-1)z\bar{z} + \frac{1}{2}w_{02}^{(1)}(-1)\bar{z}^2 \right] \times \left[z + \bar{z} + \frac{1}{2}w_{20}^{(2)}(0)z^2 + w_{11}^{(2)}(0)z\bar{z} + \frac{1}{2}w_{02}^{(2)}(0)\bar{z}^2 \right] \end{aligned}$$

$$\begin{aligned}
& \left. \frac{1}{2} w_{02}^{(2)}(0) \bar{z}^2 \right] + b_{111} \left[zD + z\bar{D} + O\left(|z|^2 \right) \right]^3 + b_{112} \left[zD + z\bar{D} + O\left(|z|^2 \right) \right]^2 \times \left[z + \bar{z} + O\left(|z|^2 \right) \right] \\
& = z^2 \left[b_{22} + b_{12} \bar{D} \right] + \bar{z}^2 \left[b_{22} + b_{12} D \right] + z\bar{z} \left[2b_{22} b_{12} \operatorname{Re}(D) \right] + z^2 \bar{z} \left[b_{22} w_{20}^{(2)}(0) + \right. \\
& 2b_{22} w_{11}^{(2)}(0) + b_{12} \left\{ \frac{1}{2} w_{20}^{(1)}(-1) + \frac{D}{2} w_{20}^{(2)}(0) + \bar{D} w_{11}^{(2)}(0) + w_{11}^{(1)}(-1) \right\} + 3b_{111} | \\
& D|^2 D + b_{112} \left\{ \bar{D}^2 + 2 |D|^2 \right\} \left. \right] + O\left(|z|^3 \right) \\
& = B_{20} z^2 + B_{02} \bar{z}^2 + B_{11} z\bar{z} + B_{21} z^2 \bar{z} + O\left(|z|^3 \right) \text{ (say)}.
\end{aligned}$$

It should be noted that all the coefficients in the above expressions can be explicitly computed. Hence according to the definition of $g(z, \bar{z})$ in (47), we finally obtain:

$$g(z, \bar{z}) = \bar{\nu}_1 f_1^0 + \bar{\nu}_2 f_2^0 \quad (58)$$

Equating the right hand side of (58) with that of (51), we obtain:

$$g_{20} = 2[\bar{\nu}_1 A_{20} + \bar{\nu}_2 B_{20}], \quad (59)$$

$$g_{11} = \bar{\nu}_1 A_{02} + \bar{\nu}_2 B_{02}, \quad (60)$$

$$g_{02} = 2[\bar{\nu}_1 A_{11} + \bar{\nu}_2 B_{11}], \quad (61)$$

$$g_{21} = 2[\bar{\nu}_1 A_{21} + \bar{\nu}_2 B_{21}]. \quad (62)$$

Finally, from (59)-(62), we can deduce the complex number (using the notation of Hassard et al. (1981)),

$$C_1(0) = \frac{i}{2\omega_+} (g_{20} \cdot g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2) + \frac{1}{2} g_{21}, \quad (63)$$

at the bifurcation value $\tau = \tau_0$. Then we have (e.g.: Chap. 2: A Recipe-Summery in Hassard et al. 1981):

$$\mu_2 = -\frac{\operatorname{Re} C_1(0)}{\alpha'(0)}, \quad (64)$$

$$\tau_2 = -\frac{\operatorname{Im} C_1(0) + \mu_2 \omega'(0)}{\omega_+}, \quad (65)$$

$$\beta_2 = 2\operatorname{Re} C_1(0), \quad (66)$$

where

$$\alpha'(0) = \frac{d\nu}{d\tau} \big|_{\tau=\tau_0}, \quad \omega'(0) = \frac{d\omega}{d\tau} \big|_{\tau=\tau_0}, \quad (67)$$

provided that $\mu_2 \neq 0$. Hence the bifurcation is supercritical if $\mu_2 > 0$ and subcritical if $\mu_2 < 0$. Further, if $\tau_2 > 0$, the period of the solution increases with increase of τ .

4 Derivation of the optimal biomass and optimal value of the control parameter

The Hamiltonian as in (6.3) is given by

$$H = (p_1 q_1 x - p_2 q_2 y - c)u(t) + \lambda_1 G_1 + \lambda_2 G_2, \quad (68)$$

where $G_1 = rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{\gamma + x} - q_1 ux$ and $G_2 = y \left(-d - \epsilon y + \frac{\beta x_\tau}{\gamma + x_\tau}\right) - q_2 uy$ and $\lambda_i(t)$ are costate variables, for $i = 1, 2$. For steady state solution, we have

$$r \left(1 - \frac{x}{K}\right) - \frac{\alpha y}{\gamma + x} - q_1 u = 0, \quad (69)$$

$$\left(-d - \epsilon y + \frac{\beta x_\tau}{\gamma + x_\tau}\right) - q_2 u = 0. \quad (70)$$

Hence applying the necessary conditions (69) and (70) for steady state solution, we have

$$\begin{aligned} \lambda_i(T) &= 0, \quad i = 1, 2, \\ \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial x_\tau}, \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial y}, \end{aligned}$$

which on application of (75) and (76) gives,

$$\dot{\lambda}_1(t) = p_1 q_1 u + \lambda_1 \left[-\frac{rx}{K} + \frac{\alpha xy}{(\gamma + x)^2} \right] - \lambda_2 \frac{\beta \gamma y}{(\gamma + x_\tau)^2}, \quad (71)$$

$$\dot{\lambda}_2(t) = -p_2 q_2 u - \lambda_1 \frac{\alpha x}{\gamma + x} + \lambda_2 \left[\frac{\beta \gamma (x_\tau - x)}{(\gamma + x)(\gamma + x_\tau)} - \epsilon y \right] \quad (72)$$

Again $x_\tau = x - \tau G_1$, for small delay τ . Putting it in (77) and (78), we get,

$$\dot{\lambda}_1(t) = p_1 q_1 u + \lambda_1 A'_1 + \lambda_2 C'_1, \quad (73)$$

$$\dot{\lambda}_2(t) = -p_2 q_2 u + \lambda_1 B'_1 - \lambda_2 C'_2, \quad (74)$$

where $A'_1 = -\frac{rx}{K} + \frac{\alpha xy}{(\gamma+x)^2}$, $B'_1 = -\frac{\alpha x}{\gamma+x}$, $C'_1 = \frac{\beta \gamma y}{(\gamma+x-\tau G_1)^2}$, $C'_2 = \frac{\tau G_1}{(\gamma+x)(\gamma+x-\tau G_1)} + \epsilon y$.

Now the dynamical systems (79)-(80) with $(\lambda_1(T), \lambda_2(T)) = (0, 0)$ has unique solution. We get the particular solution as,

$$\lambda_1(t) = \frac{(C'_1 p_2 q_2 - C'_2 p_1 q_1)u}{C'_1 B'_1 + C'_2 A'_1}, \quad (75)$$

$$\lambda_2(t) = \frac{(B'_1 p_1 q_1 - A'_1 p_2 q_2)u}{C'_1 B'_1 + C'_2 A'_1}, \quad (76)$$

Now, if H is optimum at $u = u^*$ (say), $0 < u^* < u_{\max}$, then by (71) we have

$$\frac{\partial H}{\partial u} = \tau \frac{\partial H}{\partial x_\tau} \frac{\partial f}{\partial u}, \quad \text{at} \quad u = u^*,$$

i.e.,

$$(p_1 q_1 x - p_2 q_2 y - c) - \lambda_1 q_1 x - \lambda_2 \left[q_2 y - \tau \frac{q_1 \beta \gamma x y}{(\gamma + x - \tau G_1)^2} \right] = 0, \quad (77)$$

where $\lambda_i(t)$ $i = 1, 2$ correspond to $u = u^*$.

Again as steady state optimal solution (x^*, y^*) is desired, so u^* is given by

$$u^* = \frac{1}{q_1} \left[r \left(1 - \frac{x}{K} \right) - \frac{\alpha y}{\gamma + x} \right] = \frac{1}{q_2} \left[-d - \epsilon y + \frac{\beta x}{\gamma + x} \right]. \quad (78)$$

Putting the value of u^* from (84) into $\lambda_i(t)$ $i = 1, 2$, we obtain the value of $\lambda_i(t)$ in terms of x and y . Finally, by substituting these modified value of λ_i into (83), we obtain the equation of optimal path. This optimal path on intersection with locus of dynamic equilibria

$$y = \frac{\beta x - (\gamma + x) \{ q_1 d K + r(K - x) \}}{K \{ q_1 \epsilon (\gamma + x) - q_2 \alpha \}} \quad (79)$$

gives the value of optimal biomass x^* and y^* .

5 APPENDIX A: Computation of the coefficients $H_{20}(\theta)$ and $H_{11}(\theta)$

From its definition (48) of the function $H(z, \bar{z}, \theta)$ is given by

$$H(z, \bar{z}, \theta) = R(w(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\}) - 2\text{Re}\{g(z, \bar{z})q(\theta)\} \quad (80)$$

with

$$g(z, \bar{z}) = \bar{q}^*(0).R(w(z, \bar{z}, 0) + 2\text{Re}\{zq(0)\}). \quad (81)$$

Let us compute first the argument of R :

$$w + zq(\theta) + z\bar{q}(\theta) = \begin{pmatrix} w^{(1)}(\theta) + zDe^{i\omega_+ \theta} + z\bar{D}e^{-i\omega_+ \theta} \\ w^{(2)}(\theta) + ze^{i\omega_+ \theta} + \bar{z}e^{-i\omega_+ \theta} \end{pmatrix}, \quad (82)$$

where (36) has been used. According to the definition (32) of operator R ,

$$R(w + 2\text{Re}\{zq(\theta)\}) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -1 \leq \theta < 0 \\ \begin{pmatrix} f'_1 \\ f'_2 \end{pmatrix} & \theta = 0, \end{cases} \quad (83)$$

where

$$\begin{aligned} f'_1 &= a_{11}W_1^2(0) + a_{12}W_1(0)W_2(0) + a_{111}W_1^3(0) + a_{112}W_1^2(0)W_2(0), \\ f'_2 &= b_{22}W_2^2(0) + b_{12}W_1(-1)W_2(0) + b_{111}W_1^3(-1) + b_{112}W_1^2(-1)W_2(0) \end{aligned} \quad (84)$$

with

$$W_1(0) = w^{(1)}(0) + zD + z\bar{D}, \quad (86)$$

$$W_2(0) = w^{(2)}(0) + z + \bar{z}, \quad (87)$$

$$W_1(-1) = w^{(1)}(-1) + zDe^{-i\omega_+} + z\bar{D}e^{i\omega_+}, \quad (88)$$

$$W_2(-1) = w^{(2)}(-1) + ze^{-i\omega_+} + \bar{z}e^{i\omega_+}. \quad (89)$$

We may observe that because of (43) and the expression of f'_1 and f'_2 ; w cannot contribute to H_{20} and H_{11} . In fact, it introduces only terms of higher order than z^2 , \bar{z}^2 or $z\bar{z}$. Again, (47) gives

$$g(z, \bar{z}) = \bar{\nu}_1 f'_1 + \bar{\nu}_2 f'_2. \quad (90)$$

Thus finally we have

$$H(z, \bar{z}, \theta) = -2\text{Re}\{(\bar{\nu}_1 f'_1 + \bar{\nu}_2 f'_2)q(\theta)\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -1 \leq \theta < 0 \\ \begin{pmatrix} f'_1(z, \bar{z}) \\ f'_2(z, \bar{z}) \end{pmatrix} & \theta = 0. \end{cases} \quad (91)$$

Now, by (52) we have

$$H_{20}(\theta) = \left[\frac{\partial^2}{\partial z^2} H(z, \bar{z}, \theta) \right]_{z=\bar{z}=0}, \quad H_{11}(\theta) = \left[\frac{\partial^2}{\partial z \partial \bar{z}} H(z, \bar{z}, \theta) \right]_{z=\bar{z}=0}.$$

Let us first consider H_{20} . By inspection,

$$\left[\frac{\partial^2}{\partial z^2} f'_1(z, \bar{z}) \right]_{z=\bar{z}=0} = a_{11} D^2 + a_{12} D \quad \text{and} \quad \left[\frac{\partial^2}{\partial z^2} f'_2(z, \bar{z}) \right]_{z=\bar{z}=0} = b_{22} + b_{12} D e^{-i\omega_+}.$$

Hence from above expression and using (37), we obtain the expression of $H_{20}(\theta)$ given by (53). Following the same procedure, we have

$$\left[\frac{\partial^2}{\partial z \partial \bar{z}} f'_1(z, \bar{z}) \right]_{z=\bar{z}=0} = 2[a_{11} |D|^2 + a_{12} \text{Re}D]$$

and

$$\left[\frac{\partial^2}{\partial z \partial \bar{z}} f'_2(z, \bar{z}) \right]_{z=\bar{z}=0} = 2[b_{22} b_{12} \text{Re}(D e^{-i\omega_+})].$$

Thus we finally obtain the expression of $H_{11}(\theta)$ given by (5.29).

6 APPENDIX B: Solutions of (56) and (57)

Let us first consider the operator equation (5.31) where the operator A is defined in (31) and $H_{20}(\theta)$ given in (53). Explicitly writing (56) we obtain:

$$\begin{pmatrix} 2i\omega_+ - d/d\theta & 0 \\ 0 & 2i\omega_+ - d/d\theta \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \end{pmatrix} = - \begin{pmatrix} \bar{E}_1 D e^{i\omega_+ \theta} + E_1 \bar{D} e^{-i\omega_+ \theta} \\ \bar{E}_1 e^{i\omega_+ \theta} + E_1 e^{-i\omega_+ \theta} \end{pmatrix} \quad (92)$$

where $-\infty < \theta < 0$, and

$$\begin{pmatrix} 2i\omega_+ & 0 \\ 0 & 2i\omega_+ \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \end{pmatrix} - \int_{-1}^0 d\eta(s) \begin{pmatrix} w_{20}^{(1)}(s) \\ w_{20}^{(1)}(s) \end{pmatrix} = \begin{pmatrix} H_{20}^{(1)}(0) \\ H_{20}^{(2)}(0) \end{pmatrix} \quad (93)$$

whenever $\theta = 0$. Before solving both of the equations (92) and (93), we may observe that from second of (42), $x_t = w(\theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$, for all $-1 \leq \theta \leq 0$ and $t \in [0, +\infty)$. By the definition of $q(\theta)$ and $\bar{q}(\theta)$ are continuous functions of $\theta \in [-1, 0]$. So, to avoid the jump discontinuity for the solution $x_t(\theta) = x(t + \theta)$ at the actual time t obtained by putting $\theta = 0$, we actually need $w(\theta)$ is continuous for $\theta \in [-1, 0]$. Accordingly, we supplement the nonhomogeneous linear differential equations (92) by the boundary condition

$$\lim_{\theta \rightarrow 0^-} w_{20}(\theta) = w_{20}(0). \quad (94)$$

The same kind of boundary condition is required for the other operator equation (93).

The general solution of (92) is given by

$$w_{20}^{(1)}(\theta) = \sigma_1 e^{i\omega_+ \theta} + \sigma_2 e^{-i\omega_+ \theta} + \sigma_h e^{2i\omega_+ \theta}, \quad (95)$$

$$w_{20}^{(2)}(\theta) = \rho_1 e^{i\omega_+ \theta} + \rho_2 e^{-i\omega_+ \theta} + \rho_h e^{2i\omega_+ \theta}, \quad (96)$$

where $-1 \leq \theta < 0$. $\sigma_h e^{2i\omega_+ \theta}$ and $\mu_h e^{2i\omega_+ \theta}$ are the solutions of the homogeneous part of (92). By direct substitution of the particular solutions of (95), (96) into (92), we easily obtain the following:

$$\sigma_1 = \frac{i\bar{E}_1 D}{\omega_+}, \quad \sigma_2 = \frac{iE_1 \bar{D}}{3\omega_+}. \quad (97)$$

and

$$\rho_1 = \frac{i\bar{E}_1}{\omega_+}, \quad \rho_2 = \frac{iE_1}{3\omega_+}. \quad (98)$$

The free constant σ_h and ρ_h can be deduced from the following relations:

$$\sigma_h = w_{20}^{(1)}(0) - (\sigma_1 + \sigma_2), \quad (99)$$

$$\rho_h = w_{20}^{(2)}(0) - (\rho_1 + \rho_2) \quad (100)$$

Now, by substitution of (95) and (96) into the integral part of (93) the following system of algebraic equations are obtained:

$$(4i\omega_+ - a_1)w_{20}^{(1)}(0) - a_2w_{20}^{(2)}(0) = 2H_{20}^{(1)}(0), \quad (101)$$

$$-b_1w_{20}^{(1)}(0) + 4i\omega_+w_{20}^{(2)}(0) - b_2w_{20}^{(2)}(-1) = 2H_{20}^{(2)}(0) \quad (102)$$

Again, for $\theta = -1$, (96) reduces to

$$w_{20}^{(2)}(-1) = \rho_1 e^{-i\omega_+} + \rho_2 e^{i\omega_+} + \{w_{20}^{(2)}(0) - (\rho_1 + \rho_2)\}e^{-2i\omega_+}. \quad (103)$$

Consequently, the equations (101) and (102) are modified as follows:

$$(4i\omega_+ - a_1)w_{20}^{(1)}(0) - a_2w_{20}^{(2)}(0) = 2H_{20}^{(1)}(0), \quad (104)$$

$$-b_1w_{20}^{(1)}(0) + (4i\omega_+ - b_2e^{-2i\omega_+})w_{20}^{(2)}(0) = C_{20}^{(2)}(0) \quad (105)$$

where $C_{20}^{(2)}(0) = 2H_{20}^{(2)}(0) - b_2\{\rho_1 e^{-i\omega_+} + \rho_2 e^{i\omega_+} - (\rho_1 + \rho_2)e^{-2i\omega_+}\}$. Solving this system of algebraic equations, we obtain:

$$w_{20}^{(1)}(0) = \frac{2(4i\omega_+ - b_2e^{-2i\omega_+})H_{20}^{(1)}(0) + a_2C_{20}^{(2)}(0)}{\Lambda}, \quad (106)$$

$$w_{20}^{(2)}(0) = \frac{2b_1H_{20}^{(1)}(0) + (4i\omega_+ - a_1)C_{20}^{(2)}(0)}{\Lambda} \quad (107)$$

where $\Lambda = (4i\omega_+ - a_1)(4i\omega_+ - b_2e^{-2i\omega_+}) - a_2b_1$.

Putting the value of $w_{20}^{(2)}(0)$ in (103), we get the value of $w_{20}^{(2)}(-1)$.

Let us consider now the operator equation (57). We obtain:

$$\begin{pmatrix} -d/d\theta & 0 \\ 0 & -d/d\theta \end{pmatrix} \begin{pmatrix} w_{11}^{(1)}(\theta) \\ w_{11}^{(2)}(\theta) \end{pmatrix} = - \begin{pmatrix} \bar{E}_2 D e^{i\omega_+\theta} + E_2 \bar{D} e^{-i\omega_+\theta} \\ \bar{E}_2 e^{i\omega_+\theta} + E_2 e^{-i\omega_+\theta} \end{pmatrix} \quad (108)$$

where $-1 \leq \theta < 0$, and

$$- \int_{-1}^0 d\eta(s) \begin{pmatrix} w_{11}^{(1)}(s) \\ w_{11}^{(2)}(s) \end{pmatrix} = \begin{pmatrix} H_{11}^{(1)}(0) \\ H_{11}^{(2)}(0) \end{pmatrix} \quad (109)$$

whenever $\theta = 0$. We supplement (110) with the boundary condition $\lim_{\theta \rightarrow 0^-} = w_{11}(0)$. The general solution of (110) is

$$w_{11}^{(1)}(\theta) = \chi_1 e^{i\omega_+ \theta} + \chi_2 e^{-i\omega_+ \theta} + \chi_h, \quad (110)$$

$$w_{11}^{(2)}(\theta) = \varrho_1 e^{i\omega_+ \theta} + \varrho_2 e^{-i\omega_+ \theta} + \varrho_h, \quad (111)$$

where $-1 \leq \theta < 0$. χ_h and ϱ_h is the constant solution of the homogeneous part of (110). By direct substitution of the particular solution into (110), we get the values of χ_i and ϱ_i , $i = 1, 2$ given by

$$\chi_1 = -\frac{i\bar{E}_2 D}{\omega_+}, \quad \chi_2 = \frac{iE_2 \bar{D}}{\omega_+} \quad (112)$$

and

$$\varrho_1 = -\frac{i\bar{E}_2}{\omega_+}, \quad \varrho_2 = \frac{iE_2}{\omega_+}. \quad (113)$$

The constants χ_h and ϱ_h could be obtained by the following relations:

$$\chi_h = w_{11}^{(1)}(0) - (\chi_1 + \chi_2), \quad (114)$$

$$\varrho_h = w_{11}^{(2)}(0) - (\varrho_1 + \varrho_2) \quad (115)$$

Now, by substituting (110) and (111) into the integral part of (109), we obtain the system of algebraic equations:

$$a_1 w_{11}^{(1)}(0) + a_2 w_{11}^{(2)}(0) = -2H_{11}^{(1)}(0), \quad (116)$$

$$b_1 w_{11}^{(1)}(0) + b_2 w_{11}^{(2)}(-1) = -2H_{11}^{(2)}(0) \quad (117)$$

From (111), for $\theta = -1$ we obtain

$$w_{11}^{(2)}(-1) = \varrho_1 e^{-i\omega_+} + \varrho_2 e^{i\omega_+} + w_{20}^{(2)}(0) - (\varrho_1 + \varrho_2). \quad (118)$$

Putting this in (117), we finally have

$$a_1 w_{11}^{(1)}(0) + a_2 w_{11}^{(2)}(0) = -2H_{11}^{(1)}(0), \quad (119)$$

$$b_1 w_{11}^{(1)}(0) + b_2 w_{11}^{(2)}(-1) = -C_{11}^{(2)}(0), \quad (120)$$

where $C_{11}^{(2)}(0) = b_2 \{\varrho_1 e^{-i\omega_+} + \varrho_2 e^{i\omega_+} - (\varrho_1 + \varrho_2)\} + 2H_{11}^{(2)}(0)$. Solving this system of algebraic equations, we get

$$w_{11}^{(1)}(0) = \frac{2b_2 H_{11}^{(1)}(0) + a_2 C_{11}^{(2)}(0)}{a_1 b_2 - a_2 b_1} \quad (121)$$

$$w_{11}^{(2)}(0) = \frac{2b_1 H_{11}^{(1)}(0) + a_1 C_{11}^{(2)}(0)}{a_1 b_2 - a_2 b_1}. \quad (122)$$

Putting the value of $w_{11}^{(2)}(0)$ in (118), we get the value of $w_{11}^{(2)}(-1)$.

7 APPENDIX C: Computation of necessary conditions in optimization theory

proof: Let us construct the augmented functional J_a from given objective function J by introducing costate variables $\lambda(t) \in R^n$ given by

$$J_a(u) = g(T, x(T)) + \int_0^T [\pi(x, x_\tau, u, u_\tau) + \lambda^{tr}(f(x, x_\tau, u, u_\tau) - \dot{x})] dt. \quad (123)$$

Applying rule of integration of parts, we get

$$J_a(u) = g(T, x(T)) - \lambda^{tr} x \big|_0^T + \int_0^T [H(x, x_\tau, u, u_\tau) + \dot{\lambda}^{tr} x] dt, \quad (124)$$

where $H(x, x_\tau, u, u_\tau)$ denotes the Hamiltonian, given by

$$H(x, x_\tau, u, u_\tau) = \pi(x, x_\tau, u, u_\tau) + \lambda^{tr} f(x, x_\tau, u, u_\tau). \quad (125)$$

Note that this is similar to expression of (6.3). Thus the variation of the functional $j_a(u)$ is given by

$$\delta J_a = \left(\frac{\partial g}{\partial x} - \lambda^{tr} \right) \delta x \big|_{t=T} + \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^{tr} \right) \delta x + \frac{\partial H}{\partial x_\tau} \delta x_\tau + \frac{\partial H}{\partial u} \delta u + \frac{\partial H}{\partial u_\tau} \delta u_\tau \right] dt. \quad (126)$$

Now we see that $x_\tau = x(t - \tau) = x(t) - \tau \dot{x}(t)$, as τ is very small. Thus $x_\tau = x(t) - \tau f$.

This implies $\delta x_\tau = \delta x - \tau \delta f$.

Similarly, as $u(t)$ is linear in time t , we have $\delta u_\tau = \delta u$. Thus by substituting in (126) we have

$$\delta J_a = \left(\frac{\partial g}{\partial x} - \lambda^{tr} \right) \delta x \big|_{t=T} + \int_0^T \left[\left(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_\tau} \right) \delta u + \left(\frac{\partial H}{\partial x} + \frac{\partial H}{\partial x_\tau} + \dot{\lambda}^{tr} \right) \delta x - \tau \frac{\partial H}{\partial x_\tau} \delta f \right] dt \quad (127)$$

We now choose $\lambda(t)$ such that

$$\lambda^{tr} = \frac{\partial g}{\partial x}, \quad \text{at } t = T$$

and

$$\dot{\lambda}^{tr} = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial x_\tau}.$$

Hence under these choice of $\lambda(t)$, δj_a becomes

$$\delta J_a = \int_0^T \left[\left(\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_\tau} \right) \delta u - \tau \frac{\partial H}{\partial x_\tau} \delta f \right] dt. \quad (128)$$

Assuming that J_a is optimum corresponding to $u = u^*$, we have $\delta J_a = 0$ at $u = u^*$. We then have,

$$\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_\tau} = \tau \frac{\partial H}{\partial x_\tau} \frac{\delta f}{\delta u} \quad \text{at } u = u^*.$$

Taking the limit as $\delta u \rightarrow 0$, we have

$$\frac{\partial H}{\partial u} + \frac{\partial H}{\partial u_\tau} = \tau \frac{\partial H}{\partial x_\tau} \frac{\partial f}{\partial u} \quad \text{at } u = u^*.$$

Hence the theorem is proved.