Research Article

Coproximality in the Space of Bounded Functions

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As a counterpart to best approximation in Banach spaces, the best coapproximation was introduced by Franchetti and Furi (1972). In this paper, we shall consider the relation between coproximality of a nonempty subset M of a Banach space X and of $C(S, M)$ in $C(S, X)$.

1. Introduction

Let M be a nonempty subset of a Banach space X . We say that M is proximal in X if each $x \in X$ there corresponds at least to one point $m_0 \in M$ such that

$$\|x - m_0\| \leq \|x - m\|, \quad \forall m \in M. \quad (1)$$

Recently, another kind of approximation from M has been introduced by Franchetti and Furi [1] who have considered those elements (if any) $m_0 \in M$ satisfying

$$\|m - m_0\| \leq \|x - m\|, \quad \forall m \in M, \text{ and for any fixed } x \in X. \quad (2)$$

Such an element m_0 is called best coapproximation of x in M . The set of all such elements, satisfying above inequality, is denoted by $R_M(x)$. The subset M is called coproximal in X if $R_M(x)$ is nonempty for any $x \in X$. If $R_M(x)$ is singleton for any $x \in X$, then M is called coChebyshev; see [2–4].

It is clear that $R_M(x)$ is convex if M is convex and closed. The kernel of R_M is the set defined by

$$\ker R_M = \{x \in X : \|m\| \leq \|m - x\|, \forall m \in M\}. \quad (3)$$

Many results in the theory of best coapproximation have appeared since Franchetti and Furi's paper, 1972. These results are largely concerned with the question of existence and uniqueness of best coapproximation (see for example [5–8]). Let $\ell_\infty(S, X)$ and $C(S, X)$ denote the Banach spaces of all bounded (resp., continuous) functions from a topological

space S into a Banach space X and let M be a closed subset of X . It should be remarked that if S is a compact space, then $C(S, X)$ is a subspace of $\ell_\infty(S, X)$. In this paper, we discuss the coproximality of $\ell_\infty(S, M)$ and $C(S, M)$ in $\ell_\infty(S, X)$ and $C(S, X)$, respectively. For uniqueness and existence of best coapproximation in $C(S, X)$, see [7].

Definition 1. Let M be a coproximal subset of a Banach space X . A map $F_M : X \rightarrow M$ which associates with each element in X one of its best coapproximation in M (i.e., $F_M(x) \in R_M(x)$ for all $x \in X$) is called a coproximity map.

Remark 2. Let M be a coproximal subset of a Banach space X . We state some basic properties of a proximity map F_M .

(1) If M is coChebyshev, [5], then

- (a) $F_M(m) = m$, for any $m \in M$.
- (b) $F_M(\alpha x) = \alpha F_M(x)$, for any scalar α and $x \in X$. (i.e., F_M is homogeneous).
- (c) $F_M(m + x) = m + F_M(x)$, for any $x \in X$ and $m \in M$.

(2) If M is a subspace of X , then

- (a) $\|F_M(x) - F_M(y)\| \leq \|x - m\| + \|y - m\|$, for any $x, y \in X$ and $m \in M$.
- (b) $\|F_M(x)\| \leq \|x\|$, for all $x \in X$. (set $y = 0$ and $m = 0$ in (a) and take into account that $F_M(0) = 0$).

- (c) F_M is continuous at $x = 0$.
 (d) If F_M is linear, then F_M is continuous.

Theorem 3 (see [3]). *If M is coproximal hyperplane or 1-dimensional subspace of a Banach space X , then M has a continuous coproximity map.*

Lemma 4 (see [3]). *Let M be a coproximal subspace of X . Then the following are equivalent.*

- (1) M has linear coproximity map.
- (2) $\ker R_M$ contains a closed subspace W such that $X = W \oplus M$. Moreover, if (2) holds, then the linear coproximity map for M can be defined by $P(m + w) = m$, $m \in M$, and $w \in W$.

Definition 5. Let X be a Banach space. Consider the following.

- (a) A subspace M of X is called one complemented in X if there is a closed subspace W such that $X = W \oplus M$ and the projection $P : X \rightarrow M$ is contractive.
- (b) A linear projection P is called an M -projection if $\|x\| = \max\{\|P(x)\|, \|x - P(x)\|\}$ for all $x \in X$. A closed subspace J of X is called an M -summand if it is the range of an M -projection, [9].
- (c) A linear projection P is called an L -projection if $\|x\| = \|P(x)\| + \|x - P(x)\|$ for all $x \in X$. A closed subspace J of X is called an L -summand if it is the range of an L -projection.

Clearly every M -summand is one complemented.

Theorem 6. *Let M be a subspace of a Banach space X . Then the following are equivalent.*

- (1) M is one complemented in X .
- (2) M is coproximal in X and has linear coproximity map.

Proof. (1) $\rightarrow$ (2): Let $X = M \oplus W$, $P : X \rightarrow M$ a contractive projection, and $x \in X$. Then $x = m_1 + w_1$,

$$\begin{aligned} \|P(x - m)\| &\leq \|x - m\|, \quad \forall m \in M, \\ \|m - m_1\| &\leq \|x - m\|, \quad \forall m \in M. \end{aligned} \quad (4)$$

Hence M is coproximal in X . Now let $w \in W$. Then for any $m \in M$, $\|P(w - m)\| = \|m\| \leq \|w - m\|$. Thus $W \subset \ker R_M$ and M has linear coproximity map by Lemma 4.

(2) $\rightarrow$ (1): If M has a linear coproximity map, then $\ker R_M$ contains a closed subspace W such that $X = M \oplus W$. Moreover, $P(m + w) = m$, $m + w \in M \oplus W$ is a linear coproximity map for M and so P is the projection of M along W by Lemma 4. To show that P is contractive, if $x \in X$, then $\|m - P(x)\| \leq \|m - x\|$ for all $m \in M$, and so $\|p(x)\| \leq \|x\|$ which means that P is contractive. $\square$

Corollary 7. *If M is L -summand, then M is coproximal and has continuous coproximity map.*

For more information about coproximal sets, optimal sets, and contractive sets, the reader is referred to [10–18].

2. Coproximality in $\ell_\infty(S, X)$ and $C(S, X)$

Let S be a compact Hausdorff space and X be a Banach space. We denote $\ell_\infty(S, X)$ and $C(S, X)$ to be the Banach space of all bounded (resp., continuous) functions from S into X .

Lemma 8 (see [19]). *Let A be a continuous map of a Banach space X into a closed subset Y of X . If S is compact Hausdorff space, then the map defined by $\Phi(z) = A \circ z$ is continuous from $C(S, X)$ to $C(S, Y)$.*

Lemma 9. *Let M be a closed subset of a Banach space X . If there is a continuous coproximity map of X onto M , then $C(S, M)$ is coproximal in $C(S, X)$ and in fact it has a continuous coproximity map.*

Proof. Let $\phi : X \rightarrow M$ be a continuous coproximity map for M . Define

$$\Phi : C(S, X) \longrightarrow C(S, M) \quad (5)$$

by $\Phi(f) = \phi \circ f$, then Φ is continuous by Lemma 8. Let $h \in C(S, M)$, then

$$\|h(s) - \phi(f(s))\| \leq \|f(s) - h(s)\| \quad (6)$$

for all $s \in S$,

$$\|h - \phi \circ f\| \leq \|f - h\|. \quad (7)$$

Hence

$$\|h - \Phi(f)\| \leq \|f - h\| \quad (8)$$

for all $h \in C(S, M)$ and Φ is a continuous coproximity map from $C(S, X)$ onto $C(S, M)$. $\square$

Lemma 10. *Let M be a closed subset of a Banach space X . If $C(S, M)$ is coproximal in $C(S, X)$, then M is coproximal in X . Moreover, if $C(S, M)$ has a continuous coproximity map, then M has a continuous coproximity map.*

Proof. For $x \in X$, define $f_x : S \rightarrow X$ by $f_x(s) = x$ for all $s \in S$. The map $f_x \in C(S, X)$. By assumption, there exists $g_0 \in C(S, M)$ such that

$$\|g_0 - h\| \leq \|f_x - h\| \quad (9)$$

for all $h \in C(S, M)$. In particular,

$$\|g_0 - f_m\| \leq \|f_x - f_m\| \quad (10)$$

for all $m \in M$. For any fixed $s_0 \in S$,

$$\|g_0(s_0) - m\| \leq \|x - m\| \quad (11)$$

for all $m \in M$ and so $g_0(s_0)$ is a best coapproximation of x . Now let $\Phi : C(S, X) \rightarrow C(S, M)$ be a continuous coproximity map. For any fixed $s_0 \in S$, define $\phi : X \rightarrow M$

by $\phi(x) = \Phi f_x(s_0)$. Then ϕ is a proximity map. Suppose that $x_n \rightarrow x$ in X . Then $\Phi(f_{x_n}) \rightarrow \Phi(f_x)$. and

$$\begin{aligned} \|\phi(x_n) - \phi(x)\| &= \|\Phi(f_{x_n})(s_0) - \Phi(f_x)(s_0)\| \\ &\leq \|\Phi(f_{x_n}) - \Phi(f_x)\| \rightarrow 0. \end{aligned} \quad (12)$$

Hence ϕ is continuous. $\square$

Theorem 11. Let M be a closed subset of a Banach space X . Then the following are equivalent.

- (1) M is coproximal in X and has a continuous coproximity map.
- (2) $C(S, M)$ is coproximal in $C(S, X)$ and has a continuous coproximity map.

Proof. The proof follows from Lemmas 9 and 10. $\square$

Theorem 12. Let M be a closed subset of a Banach space X and let S be a topological space. Then the following are equivalent.

- (1) M is coproximal in X .
- (2) $\ell_\infty(S, M)$ is coproximal in $\ell_\infty(S, X)$.

Proof. (1) $\rightarrow$ (2). Let $f \in \ell_\infty(S, X)$. Since M is coproximal in X , then $R_M(f(s))$ is nonempty for any $s \in S$. By axiom of choice, we may define a function $h : S \rightarrow M$ such that $h(s) \in R_M(f(s))$ for all $s \in S$. Since $\|h(s)\| \leq \|f(s)\|$ for all $s \in S$, $h \in \ell_\infty(S, M)$ and h is a best coapproximation of f in $\ell_\infty(S, X)$.

(2) $\rightarrow$ (1). For $x \in X$, define $f_x : S \rightarrow X$ by $f_x(s) = x$ for all $s \in S$. Since $\ell_\infty(S, M)$ is coproximal in $\ell_\infty(S, X)$, the map $f_x \in \ell_\infty(S, X)$. By assumption, there exists $g_0 \in \ell_\infty(S, M)$ such that

$$\|g_0 - h\| \leq \|f_x - h\| \quad (13)$$

for all $h \in \ell_\infty(S, M)$. In particular,

$$\|g_0 - f_m\| \leq \|f_x - f_m\| \quad (14)$$

for all $m \in M$. For any fixed $s_0 \in S$,

$$\|g_0(s_0) - m\| \leq \|x - m\| \quad (15)$$

for all $m \in M$ and $g_0(s_0)$ is a best coapproximation of x .

We recall that. $\square$

Definition 13. Let Φ be a set valued mapping, taking points of a topological space S into the family of all subsets of a topological space T . The mapping Φ is said to be lower semicontinuous if, for each open set O in T , the set $\Phi^- = \{s \in S : \Phi(s) \cap O \neq \emptyset\}$ is open.

Theorem 14 (see [19, Michael Selection Theorem]). Let Φ be a lower semicontinuous of a paracompact space S into the family of nonvoid closed convex subsets of a Banach space X . Then Φ has a continuous selection; that is, there exists a continuous map $\varphi : S \rightarrow X$ such that $\varphi(s) \in \Phi(s)$ for all $s \in S$.

Definition 15. A sequence of sets $\{A_n\}$ in a topological space (X, T) is convergent and has the limit A if and only if $\limsup\{A_n\} = A = \liminf\{A_n\}$ where $\limsup\{A_n\}$ is the set of those elements which are limits of points in A_n for infinitely many n and $\liminf\{A_n\}$ is the set of those elements which are limits of points in A_n for cofinitely many (for all but finitely many) n .

Lemma 16. Let S be a compact Hausdorff space, X a Banach space, and G a compact subset of X . For $f \in C(S, X)$, we define $\pi_f : S \rightarrow 2^G$ by

$$\pi_f(s) = \{g_0 \in G : \|g_0 - g\| \leq \|g - f(s)\|, \forall g \in G\}. \quad (16)$$

Then π_f is lower semicontinuous.

Proof. Let O be an open set in G . We need to prove that

$$\pi_f^-(O) = \{s \in S : \pi_f(s) \cap O \neq \emptyset\} \quad (17)$$

is open. For this purpose, let $\{x_n\}$ be a sequence in $S/\pi_f^-(O)$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, so $x_n \notin \pi_f^-(O)$ for all n which means that

$$\pi_f(x_n) \cap O = \emptyset. \quad (18)$$

Thus $\pi_f(x_n) \subset G/O$ for all n , and $\{\pi_f(x_n)\}$ is a sequence of subsets of G . We shall show that $\pi_f(x_n) \rightarrow \pi_f(x)$ as $n \rightarrow \infty$. First, we show that

$$\limsup \pi_f(x_n) \subseteq \pi_f(x). \quad (19)$$

Let $z \in \limsup \pi_f(x_n)$. Then there is a sequence of points $\{z_k\}$ and a subsequence $\{\pi_f(x_{n_k})\}$ of $\{\pi_f(x_n)\}$ such that $z_k \in \pi_f(x_{n_k})$ and $z_k \rightarrow z$ as $k \rightarrow \infty$. By the definition of π_f , we get

$$\|z_k - g\| \leq \|f(x_{n_k}) - g\| \quad \forall g \in G. \quad (20)$$

Taking $k \rightarrow \infty$ and $n_k \rightarrow \infty$, and since f is continuous, we have

$$\|z - g\| \leq \|f(x) - g\| \quad \forall g \in G, \quad (21)$$

so $z \in \pi_f(x)$. This proves that $\limsup \pi_f(x_n) \subseteq \pi_f(x)$. Now we shall show that $\pi_f(x) \subseteq \liminf \pi_f(x_n)$. Let $z \notin \liminf \pi_f(x_n)$. In this case, if $z_k \rightarrow z$ as $k \rightarrow \infty$ then $z_k \notin \pi_f(x_k)$ for infinitely many k , and, so,

$$\|z_k - g_k\| > \|f(x_k) - g_k\| \quad \text{for some } g_k \in G. \quad (22)$$

We may assume that, without loss of generality, $g_k \rightarrow g$ as $k \rightarrow \infty$, for some $g \in G$, since G is compact. Taking $k \rightarrow \infty$, we have that

$$\|z - g\| > \|f(x) - g\|, \quad (23)$$

so $z \notin \pi_f(x)$ and we have as a conclusion the following result:

$$\begin{aligned} \liminf \pi_f(x_n) &\subseteq \limsup \pi_f(x_n) \subseteq \pi_f(x) \\ &\subseteq \liminf \pi_f(x_n). \end{aligned} \quad (24)$$

Thus, $\lim \pi_f(x_n) = \pi_f(x)$. Finally, we know that G/O is a closed set and the members of the sequence of sets $\{\pi_f(x_n)\}$ are all subsets of G/O for all n . Now, to complete the proof, we still need to show that

$$\lim \pi_f(x_n) \subseteq \frac{G}{O}. \quad (25)$$

To do so, let $z \in \limsup \pi_f(x_n)$. By the definition of $\limsup \pi_f(x_n)$, there is a sequence of points $\{z_k\}$ and a subsequence $\{\pi_f(x_{n_k})\}$ of the sequence $\{\pi_f(x_n)\}$ such that

$$\begin{aligned} z_k \in \pi_f(x_{n_k}) &\subseteq \frac{G}{O} \quad \forall k, \\ z_k &\longrightarrow z \quad \text{as } k \longrightarrow \infty. \end{aligned} \quad (26)$$

Since G/O is closed, we get $z \in G/O$. This proves that $\pi_f(x) \subseteq G/O$. This means that $\pi_f(x) \cap O = \emptyset$, and so $x \notin \pi_f^-(O)$. Thus $\pi_f^-(O)$ is open set and so π_f is lower semicontinuous. $\square$

Theorem 17. *Let M be a compact convex subset of a Banach space X and S be a compact Hausdorff space. Then the following are equivalent.*

- (1) M is coproximal in X .
- (2) $C(S, M)$ is coproximal in $C(S, X)$.

Proof. (2) $\rightarrow$ (1). The proof is similar to that given in Lemma 10.

(1) $\rightarrow$ (2). Let $f \in C(S, X)$ and $\pi_f : S \rightarrow 2^M$ as defined in Lemma 16. Then π_f is lower semicontinuous by Lemma 16. Since M is closed, nonvoid, and convex, then $\pi_f(s)$ is closed, nonvoid, and convex. By Michael Selection Theorem, π_f has a continuous selection $g : S \rightarrow M$. Then $g \in C(S, M)$ and

$$\begin{aligned} \|g(s) - h(s)\| &\leq \|f(s) - h(s)\|, \quad \forall s \in S, h \in C(S, M), \\ \|g - h\| &\leq \|f - h\|, \quad \forall h \in C(S, M). \end{aligned} \quad (27)$$

Hence g is a best coapproximation of f in $C(S, M)$. $\square$

Corollary 18. *Let S be a compact Hausdorff space and let X be a reflexive Banach space. Then $B_1(X)$ (the closed unit ball of X) is coproximal in X if and only if $C(S, B_1(X))$ is coproximal in $C(S, X)$.*

Proof. Since X is reflexive, then $B_1(X)$ is compact by Alaoglu Theorem. As $B_1(X)$ is closed and convex subset of X , the result follows from Theorem 17. $\square$

Problem 19. If S is a compact Hausdorff space and X is a Banach space, when $C(S, X)$ is coproximal in $\ell_\infty(S, X)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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