## Research Article

# Product of Locally Primitive Graphs 

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Received 21 January 2014; Revised 2 April 2014; Accepted 16 April 2014; Published 4 May 2014
Academic Editor: Seppo Hassi
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Many large graphs can be constructed from existing smaller graphs by using graph operations, such as the product of two graphs. Many properties of such large graphs are closely related to those of the corresponding smaller ones. In this paper we consider the product of two locally primitive graphs and prove that only tensor product of them will also be locally primitive.

## 1. Introduction

Let $\Gamma=(V, E)$ be a simple graph, where $V$ is the set of vertices and $E$ is the set of edges of $\Gamma$. An edge joining the vertices $u$ and $v$ is denoted by $\{u, v\}$. The group of automorphisms of $\Gamma$ is denoted by $\operatorname{Aut}(\Gamma)$, which acts on vertices of $\Gamma$. $\Gamma$ is called vertex transitive if $\operatorname{Aut}(\Gamma)$ acts transitively on the set of vertices. $\Gamma$ is called $X$-locally transitive or $X$-locally primitive for $X \leqslant$ $\operatorname{Aut}(\Gamma)$ (or simply locally primitive or locally transitive when $X=\operatorname{Aut}(\Gamma))$ if $X_{v}$ acts transitively or primitively on $\Gamma(v)$, respectively, for each vertex $v \in V(\Gamma)$, where $\Gamma(v)$ is the set of vertices which are adjacent to $v$ and $X_{v}$ is the stabilizer of $v$ in $X$. It is known that 2-arc-transitive graphs form a proper subclass of vertex transitive locally primitive graphs.

Let $G$ be a group and let $S$ be a nonempty subset of $G$. The Cayley graph of $G$ with respect to $S$ is denoted by $\Gamma=$ $\operatorname{Cay}(G, S)$ is defined as a graph with vertex set $G$ and $\{x, y\}$ is an edge of $\Gamma$ if and only if for some $s \in S$ we have $y=s x$.

Many large graphs can be constructed by expanding of small graphs; thus it is important to know which properties of small graphs can be transferred to the expanded one; for example, Li et al. in [1] proved that the lexicographic product of vertex transitive graphs is also vertex transitive as well as the lexicographic product of edge transitive graphs, and Jaradat et al. in [2] found the basis number of the semicomposition product of two paths and a cycle with a path. Here we consider seven products of graphs as the expander graph which is described below, and hence when we talk about the product of graphs, we mean that the product is one of the following products.

Definition 1. Let $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$ be two graphs. $\Gamma=(V, E)$, the product of them, is a graph with vertex set $V=V_{1} \times V_{2}$, and the vertex $\left(u_{1}, u_{2}\right)$ is adjacent to $\left(v_{1}, v_{2}\right)$ in $\Gamma$ if one of the relevant conditions happens depending on the product.
(1) Cartesian product: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}=$ $v_{2}$ or $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.
(2) Tensor product: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.
(3) Strong product: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}=v_{2}$ or $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$ or $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.
(4) Lexicographic: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ or $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.
(5) Conormal product: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ or $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.
(6) Modular product: $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$ or $u_{1}$ is not adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}$ is not also adjacent to $v_{2}$ in $\Gamma_{2}$.
(7) Rooted product with root $h_{1} \in V_{2}: u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}=v_{2}=h_{1}$ or $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.

Locally primitive Cayley graphs and 2-arc-transitive graphs have been extensively studied; see, for example, [3-8] and references therein. These motivated the author to investigate if the product of two locally transitive or locally primitive graphs has the property as well.

## 2. Main Results

Theorem 2. Let $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$ be two simple graphs, let $X$ be a subgroup of $A u t\left(\Gamma_{1}\right)$, and let $Y$ be a subgroup of $\operatorname{Aut}\left(\Gamma_{2}\right)$. Then $X \times Y$ is also a subgroup of $\operatorname{Aut}(\Gamma)$, where $\Gamma$ is the product of $\Gamma_{1}$ and $\Gamma_{2}$.
Proof. Suppose $\alpha=(\sigma, \delta) \in X \times Y$. Thus $\sigma$ is an automorphism of $\Gamma_{1}$ and $\delta$ is an automorphism of $\Gamma_{2}, \sigma$ is a bijection of $V_{1}$ and $\delta$ is a bijection of $V_{2}$, which implies $\alpha$ is also a bijection of $V=V_{1} \times V_{2}$, the vertex set of $\Gamma$. Now assume $\left(u_{1}, u_{2}\right)$ and ( $v_{1}, v_{2}$ ) are two arbitrary adjoint vertices of $\Gamma$. We distinguish seven kinds of products as follows.
(1) Cartesian product: suppose $\Gamma$ is the Cartesian product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$. If $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjoint in the Cartesian product, by the definition of Cartesian product, we have $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$ or vice versa.

For the case $u_{1}=v_{1}, u_{2}$ and $v_{2}$ are adjoint in $\Gamma_{2} . \sigma$ is a map; thus $\sigma\left(u_{1}\right)=\sigma\left(v_{1}\right) . \delta$ is an automorphism of $\Gamma_{2}$; hence $\delta\left(u_{2}\right)$ is adjacent to $\delta\left(v_{2}\right)$ in $\Gamma_{2}$. Thus ( $\left.\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right), \delta\left(v_{2}\right)\right)$, which says $\alpha\left(\left(u_{1}, u_{2}\right)\right)$ is also adjacent to $\alpha\left(\left(v_{1}, v_{2}\right)\right)$ in $\Gamma$.

Similar argument can be done for the case $u_{1}$ being adjacent to $v_{1}$ in $\Gamma_{1}$ and $u_{2}=v_{2}$ as well, which says $\alpha$ preserves the edges of the graph $\Gamma$.
(2) Tensor product: suppose $\Gamma$ is tensor product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$. If $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are adjoint in the graph $\Gamma$, by the definition of tensor product, we imply that $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are edges of $\Gamma_{1}$ and $\Gamma_{2}$, respectively.
$\sigma$ is an automorphism of the graph $\Gamma_{1}$, implying $\left(\sigma\left(u_{1}\right)\right.$, $\left.\sigma\left(v_{1}\right)\right)$ is an edge of the graph $\Gamma_{1}$. Being $\delta$ an automorphism of the graph $\Gamma_{2}$ yields $\left(\delta\left(u_{2}\right), \delta\left(v_{2}\right)\right)$ will be an edge of $\Gamma_{2}$; that is, $\alpha\left(\left(u_{1}, u_{2}\right)\right)$ and $\alpha\left(\left(v_{1}, v_{2}\right)\right)$ are also adjacent in the graph $\Gamma$; that is, $\alpha$ preserves the edges of the graph $\Gamma$.
(3) Strong product: from the case 1 and case 2 we can deduce that in this case we can also say $\alpha$ preserves the edges of the graph $\Gamma$.
(4) Lexicographic product: if $\Gamma$ is the lexicographic product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$ and $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are joint by an edge in the graph $\Gamma$, by the definition of lexicographic product, $\left(u_{1}, v_{1}\right)$ is an edge of $\Gamma_{1}$, or $u_{1}=v_{1}$ and $u_{2}$ is adjacent to $v_{2}$ in $\Gamma_{2}$.

If $u_{1}$ is adjacent to $v_{1}$ in $\Gamma_{1}$, then $\sigma\left(u_{1}\right)$ is also adjacent to $\sigma\left(v_{1}\right)$ in $\Gamma_{1}$, since $\sigma$ is an automorphism of the graph $\Gamma_{1}$, and hence by the definition of the lexicographic product, $\left(\sigma\left(u_{1}\right)\right.$, $\left.\delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right), \delta\left(v_{2}\right)\right)$ in the graph $\Gamma$; that is, $\alpha\left(\left(u_{1}, u_{2}\right)\right)$ and $\alpha\left(\left(v_{1}, v_{2}\right)\right)$ are joined by an edge in the graph $\Gamma$.

For the case $u_{1}=v_{1}$ and $u_{2}$ being adjacent to $v_{2}$ in $\Gamma_{2}$ the same argument in case 1 can be done to show that $\alpha\left(\left(u_{1}, u_{2}\right)\right)$ is also adjacent to $\alpha\left(\left(v_{1}, v_{2}\right)\right)$ in the graph $\Gamma$.
(5) Conormal product: if $\Gamma$ is the conormal product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$ and $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are joint by an edge in the graph $\Gamma$, by the definition of conormal product we have either $\left(u_{1}, v_{1}\right) \in E_{1}$ or $\left(u_{2}, v_{2}\right) \in E_{2}$, where $E_{1}$ and $E_{2}$ are the edge set of the graphs $\Gamma_{1}$ and $\Gamma_{2}$, respectively.

For the first case, since $\sigma$ is an automorphism of $\Gamma_{1}$, thus $\left(\sigma\left(u_{1}\right), \sigma\left(v_{1}\right)\right) \in E_{1}$, and by the definition of conormal product, $\left(\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right), \delta\left(v_{2}\right)\right)$ in the graph $\Gamma$.

And for the case $\left(u_{2}, v_{2}\right) \in E_{2}$ we will have $\left(\delta\left(u_{2}\right), \delta\left(v_{2}\right)\right) \in$ $E_{2}$, since $\delta$ is an automorphism of the graph $\Gamma_{2}$. And hence $\left(\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right), \delta\left(v_{2}\right)\right)$ in the graph $\Gamma$.

Therefore in both cases we have that $\alpha\left(\left(u_{1}, u_{2}\right)\right)$ and $\alpha\left(\left(v_{1}, v_{2}\right)\right)$ are joint by an edge in the graph $\Gamma$; that is $\alpha$ preserves the edges.
(6) Modular product: we have $\left(u_{i}, v_{i}\right) \in E_{i}$ for $i=1,2$ if and only if $\left(\sigma\left(u_{1}\right), \sigma\left(v_{1}\right)\right) \in E_{1}$ and $\left(\delta\left(u_{2}\right), \delta\left(v_{2}\right)\right) \in E_{2}$, because $\sigma$ and $\delta$ are automorphisms of the graphs $\Gamma_{1}$ and $\Gamma_{2}$ respectively. Therefore $\left(u_{i}, v_{i}\right) \notin E_{i}$ for $i=1,2$ if and only if $\left(\sigma\left(u_{1}\right), \sigma\left(v_{1}\right)\right) \notin E_{1}$ and $\left(\delta\left(u_{2}\right), \delta\left(v_{2}\right)\right) \notin E_{2}$.

Now if $\Gamma$ is the modular product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$ and $\left(u_{1}, u_{2}\right)$ and $\left(v_{1}, v_{2}\right)$ are joint by an edge in the graph $\Gamma$, by the definition of modular product, we have $\left(u_{i}, v_{i}\right) \in E_{i}$, or $\left(u_{i}, v_{i}\right) \notin E_{i}$ for $i=1,2$, where $E_{i}$ is the set of edges of $\Gamma_{i}$.

For the first case we have $\left(\sigma\left(u_{1}\right), \sigma\left(v_{1}\right)\right) \in E_{1}$ and $\left(\delta\left(u_{2}\right)\right.$, $\left.\delta\left(v_{2}\right)\right) \in E_{2}$, implying $\left(\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right)\right.$, $\left.\delta\left(v_{2}\right)\right)$ in the graph $\Gamma$.

For the latter case we have $\left(\sigma\left(u_{1}\right), \sigma\left(v_{1}\right)\right) \notin E_{1}$ and $\left(\delta\left(u_{2}\right)\right.$, $\left.\delta\left(v_{2}\right)\right) \notin E_{2}$; that is $\left(\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)$ is adjacent to $\left(\sigma\left(v_{1}\right), \delta\left(v_{2}\right)\right)$ in the graph $\Gamma$.

And we conclude that $\Gamma$ preserve the edges.
(7) Rooted product: similar argument of case (1) can be done to prove that $\Gamma$, the rooted product of graphs $\Gamma_{1}$ and $\Gamma_{2}$, also preserve the edges, but we have considered that if $\Gamma$ is rooted product with root $h_{1}$, then the image of it with respect to $\alpha$ will be a rooted graph with root $\delta\left(h_{1}\right)$.

Thus if $\Gamma$ is one of the 7 kinds of products of $\Gamma_{1}$ and $\Gamma_{2}$, then $X \times Y$ will be a bijection on $V=V_{1} \times V_{2}$ which preserves the edges of the graph $\Gamma$, implying $X \times Y$ is an automorphism of the graph $\Gamma$.

Next lemma is simple to prove but useful in the literature. Thus we mention it without proof.

Lemma 3. For two simple graphs $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=$ $\left(V_{2}, E_{2}\right)$ and $\left(u_{1}, u_{2}\right) \in V=V_{1} \times V_{2}$, If $\Gamma$ is the product of graphs $\Gamma_{1}$ and $\Gamma_{2}$, then the neighbourhood of vertex $\left(u_{1}, u_{2}\right)$ of the vertex set of $\Gamma$ is as follows.
(1) Cartesian product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times\left\{u_{2}\right\}\right) \cup\left(\left\{u_{1}\right\} \times\right.$ $\left.\Gamma_{2}\left(u_{2}\right)\right)$.
(2) Tensor product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)$.
(3) Strong product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times\left\{u_{2}\right\}\right) \cup\left(\left\{u_{1}\right\} \times\right.$ $\left.\Gamma_{2}\left(u_{2}\right)\right) \cup\left(\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)\right)$.
(4) Lexicographic product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times V_{2}\right) \cup$ $\left(\left\{u_{1}\right\} \times \Gamma_{2}\left(u_{2}\right)\right)$.
(5) Conormal product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times V_{2}\right) \cup\left(V_{1} \times\right.$ $\left.\Gamma_{2}\left(u_{2}\right)\right)$.
(6) Modular product: $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)\right) \cup$ $\left(\Gamma_{1}\left(u_{1}\right)^{c} \times \Gamma_{2}\left(u_{2}\right)^{c}\right)$, where $\Gamma_{i}\left(u_{i}\right)^{c}$ is the complement set of $\Gamma_{i}\left(u_{i}\right)$ in the vertex set $V_{i}$ for $i=1,2$.
(7) Rooted product with root $\left\{h_{1}\right\}: \Gamma\left(\left(u_{1}, h_{1}\right)\right)=\left(\Gamma_{1}\left(u_{1}\right) \times\right.$ $\left.\left\{h_{1}\right\}\right) \cup\left(\left\{u_{1}\right\} \times \Gamma_{2}\left(h_{1}\right)\right)$ and $\Gamma\left(\left(u_{1}, u_{2}\right)\right)=\left\{u_{1}\right\} \times \Gamma_{2}\left(u_{2}\right)$, for $u_{2} \neq h_{1}$.

Now we focus on simple graphs, by which we mean an undirected graph with no loops.

Theorem 4. Let $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$ be $X$ and $Y$-locally transitive nonempty (nonedgeless) simple graphs, respectively, then $\Gamma$, the product of $\Gamma_{1}$ and $\Gamma_{2}$, is also $X \times Y$-locally transitive graph if and only if $\Gamma$ is the tensor product of them.

Proof. By the definition of locally transitive graph, we have to determine if $\Gamma$ is one of the seven kinds of graph product of $\Gamma_{1}$ and $\Gamma_{2} ;(X \times Y)_{w}$ acts transitively on the set $\Gamma(w)$, where $w=$ $\left(u_{1}, u_{2}\right)$ is any arbitrary element of the vertex set of $\Gamma$ and $(X \times$ $Y)_{w}$ is the stabilizer of $w$ in $X \times Y$.

If $\alpha=(\sigma, \delta)$ is in the stabilizer of $w=\left(u_{1}, u_{2}\right)$ in $X \times Y$, then $\alpha(w)=w$ and we have

$$
\begin{equation*}
\alpha(w)=(\sigma, \delta)\left(u_{1}, u_{2}\right)=\left(\sigma\left(u_{1}\right), \delta\left(u_{2}\right)\right)=\left(u_{1}, u_{2}\right)=w \tag{1}
\end{equation*}
$$

Thus $\sigma\left(u_{1}\right)=u_{1}$ and $\delta\left(u_{2}\right)=u_{2}$, implying $\sigma \in X_{u_{1}}$ and $\delta \in$ $Y_{u_{2}}$; that is, $\alpha=(\sigma, \delta) \in X_{u_{1}} \times Y_{u_{2}}$. The converse is also true which says

$$
\begin{equation*}
(X \times Y)_{w}=X_{u_{1}} \times Y_{u_{2}} \tag{2}
\end{equation*}
$$

Let $\Gamma$ be Cartesian product of $\Gamma_{1}$ and $\Gamma_{2}$. Lemma 3 implies that $\Gamma(w)=\left(\Gamma_{1}\left(u_{1}\right) \times\left\{u_{2}\right\}\right) \cup\left(\left\{u_{1}\right\} \times \Gamma_{2}\left(u_{2}\right)\right)$. Now for every $\alpha=$ $(\sigma, \delta) \in(X \times Y)_{w}$ we have $\sigma\left(u_{1}\right)=u_{1}$, since $\sigma$ is in the stabilizer of $u_{1}$ in $x$. Similar argument shows that $\delta\left(u_{2}\right)=u_{2} . \Gamma_{1}$ and $\Gamma_{2}$ both are edgeless and hence the sets $\Gamma_{1}\left(u_{1}\right)$ and $\Gamma_{2}\left(u_{2}\right)$ are nonempty for some $u_{1} \in V_{1}$ and $u_{2} \in V_{2}$ which are disjoint from the sets $\left\{u_{1}\right\}$ and $\left\{u_{2}\right\}$, respectively. Thus there is not any $\alpha \in(X \times Y)_{w}$ which sends $\left(u_{1}, u_{2}^{\prime}\right)$ to $\left(u_{1}^{\prime}, u_{2}\right)$ for some $u_{1}^{\prime} \in \Gamma_{1}\left(u_{1}\right)$ and $u_{2}^{\prime} \in \Gamma_{2}\left(u_{2}\right)$, implying $\Gamma$ is not locally transitive.

Similar argument shows that if $\Gamma$ is strong product, lexicographic product, conormal product, or rooted product (consider $\Gamma\left(u_{1}, h_{1}\right)$ for some $u_{1}$ in an edge of $\left.\Gamma_{1}\right)$; then it is not locally transitive graph.

For $\sigma \in X_{u_{1}}$ and $v \in \Gamma_{1}\left(u_{1}\right)$ we have $\sigma(v) \in \Gamma_{1}(v)$, which implies $\sigma(v)$ can not be in $\Gamma_{1}\left(u_{1}\right)^{c}$; that is, modular product of them can not be locally transitive.

If $\Gamma$ is the tensor product of $\Gamma_{1}$ and $\Gamma_{2}$, then by Lemma 3, $\Gamma(w)=\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)$. Now if $\Gamma_{1}$ and $\Gamma_{2}$ are $X$ - and $Y$-locally transitive, respectively, then $X_{u_{1}}$ and $Y_{u_{2}}$ act transitively on the set of $\Gamma_{1}\left(u_{1}\right)$ and $\Gamma_{2}\left(u_{2}\right)$, respectively, for every $u_{1} \in V_{1}$ and $u_{2} \in V_{2}$, and we conclude that $(X \times Y)_{w}$ acts transitively on the $\operatorname{set} \Gamma(w)$ for every $w \in V_{1} \times V_{2}$; that is, $\Gamma$ is locally transitive.

Lemma 5. Let $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$ be two simple graphs which are $X$ - and $Y$-locally primitive, respectively; then $\Gamma$ is $X \times Y$-locally primitive graph, where $\Gamma$ is the tensor product of $\Gamma_{1}$ and $\Gamma_{2}$.

Proof. By Theorem 2, $X \times Y$ is a subgroup of the automorphism group of graph $\Gamma$.

Suppose $\left(u_{1}, u_{2}\right) \in V_{1} \times V_{2}$ be an arbitrary vertex of the graph $\Gamma$. By Lemma 3 we have $\Gamma\left(u_{1}, u_{2}\right)=\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)$.
$\alpha=(\sigma, \delta)$ is in stabilizer of $\left(u_{1}, u_{2}\right)$ in $X \times Y$ if and only if $(\sigma, \delta)\left(u_{1}, u_{2}\right)=\left(u_{1}, u_{2}\right)$ if and only if $\sigma\left(u_{1}\right)=u_{1}$ and $\delta\left(u_{2}\right)=$ $u_{2}$; that is, $(X \times Y)_{\left(u_{1}, u_{2}\right)}=X_{u_{1}} \times Y_{u_{2}}$.

By assumption $X_{u_{1}}$ acts on $\Gamma_{1}\left(u_{1}\right)$ primitively as well as $Y_{u_{2}}$ on $\Gamma_{2}\left(u_{2}\right)$. Thus it is enough to show that $X_{u_{1}} \times Y_{u_{2}}$ acts primitively on $\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)$. But transitivity of it arises from Theorem 4, and hence we should prove that it does not have any nontrivial block.

Suppose not, and take some nontrivial block $B$ of $X_{u_{1}} \times Y_{u_{2}}$ in $\Gamma_{1}\left(u_{1}\right) \times \Gamma_{2}\left(u_{2}\right)$. Set $A=\pi_{1}^{-1}(B)$ and $C=\pi_{2}^{-1}(B)$, where $\pi_{i}$ is the projective map of $i$ th coordinate.

For $\sigma \in X_{u_{1}}$, if $A^{\sigma} \cap A \neq \emptyset$, that is, $s^{\prime} \in A$ exists such that for some $s \in A$ we have $s^{\sigma}=s^{\prime}$, then for some $t, t^{\prime} \in \Gamma_{2}\left(u_{2}\right)$, both of $(s, t)$ and $\left(s^{\prime}, t^{\prime}\right)$ belong to $B$.

We also know $Y_{u_{2}}$ acts on $\Gamma_{2}\left(u_{2}\right)$ primitively and so transitively; thus we can say, for some $\delta \in Y_{u_{2}},(s, t)^{\alpha}=\left(s^{\prime}, t^{\prime}\right)$; that is $\left(s^{\prime}, t^{\prime}\right) \in B \cap B^{\alpha}$ for some $\alpha \in X_{u_{1}} \times Y_{u_{2}}$, and by permittivity condition $B$ should be the same as $B^{\alpha}$; thus, for every $s \in A$, some $t \in \Gamma_{2}\left(u_{2}\right)$ exists such that $(s, t) \in B$ implies $(s, t)^{\alpha}=$ $\left(s^{\sigma}, t^{\alpha}\right) \in B^{\alpha}=B$; therefore $s^{\sigma}$ should belong to $A$, and thus $A^{\sigma} \subseteq A$. But $\sigma$ is bijection and so $A^{\sigma}=A$; that is, $A$ is a block for $\Gamma_{1}\left(u_{1}\right)$.

Similarly, $C$ is also a block for $\Gamma_{2}\left(u_{2}\right)$. Thus if $B$ is a nontrivial block, then either $A$ or $C$ should also be a nontrivial block which is a contradiction to the assumption.

By the definition of locally permittivity and Lemma 5 and Theorem 4 we conclude the following theorem.

Theorem 6. Let $\Gamma_{1}=\left(V_{1}, E_{1}\right)$ and $\Gamma_{2}=\left(V_{2}, E_{2}\right)$ be two nonempty simple graphs which are $X$ - and $Y$-locally primitive, respectively, then $\Gamma$, the product of $\Gamma_{1}$ and $\Gamma_{2}$, is $X \times Y$-locally primitive if and only if $\Gamma$ is the tensor product.

In [9], the authors proved the following lemma for semigroups and hence we can conclude it is valid for groups.

Lemma 7. Let $H$ and $K$ be two groups and let $S$ and $T$ be inverse closed subsets of them, respectively, which does not contain the identity element. If $\Gamma_{1}=\operatorname{Cay}(H, S)$ and $\Gamma_{2}=$ $\operatorname{Cay}(K, T)$, then $\Gamma$ the tensor product of the graphs $\Gamma_{1}$ and $\Gamma_{2}$ is also a Cayley graph of the group $H \times K$ with respect to the subset $S \times T$.

Now by the Theorem 6 and Lemma 7 we can conclude the following theorem.

Theorem 8. Let $H$ and $K$ be two groups, let $S$ be an inverse closed subset of $H$, and let $T$ be an inverse closed subset of $K$ and none of them have the identity. If $\Gamma_{1}=\operatorname{Cay}(H, S)$ is X-locally primitive and $\Gamma_{2}=\operatorname{Cay}(K, T)$ is $Y$-locally primitive Cayley graph, then $\Gamma=\operatorname{Cay}(H \times K, S \times T)$ is $X \times Y$ locally primitive Cayley graph.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

The author would like to thank the anonymous referee for his useful comments and suggestions.

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