Research Article

On the Products of k -Fibonacci Numbers and k -Lucas Numbers

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In this paper we investigate some products of k -Fibonacci and k -Lucas numbers. We also present some generalized identities on the products of k -Fibonacci and k -Lucas numbers to establish connection formulas between them with the help of Binet's formula.

1. Introduction

Fibonacci numbers possess wonderful and amazing properties; though some are simple and known, others find broad scope in research work. Fibonacci and Lucas numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy [1] and Vajda [2]. The Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, 8, \dots\}$ wherein each term is the sum of the two previous terms beginning with the initial values $F_0 = 0$ and $F_1 = 1$. Also the ratio of two consecutive Fibonacci numbers converges to the Golden mean, $\phi = (1 + \sqrt{5})/2$. The Fibonacci numbers and Golden mean find numerous applications in modern science and have been extensively used in number theory, applied mathematics, physics, computer science, and biology.

The well-known Fibonacci sequence is defined as

$$\begin{aligned} F_0 &= 0, & F_1 &= 1, \\ F_n &= F_{n-1} + F_{n-2} & \text{for } n \geq 2. \end{aligned} \quad (1)$$

In a similar way, Lucas sequence is defined as

$$\begin{aligned} L_0 &= 2, & L_1 &= 1, \\ L_n &= L_{n-1} + L_{n-2} & \text{for } n \geq 2. \end{aligned} \quad (2)$$

The second order Fibonacci sequence has been generalized in several ways. Some authors have preserved the recurrence relation and altered the first two terms of the sequence while others have preserved the first two terms of the sequence and altered the recurrence relation slightly.

The k -Fibonacci sequence introduced by Falcón and Plaza [3] depends only on one integer parameter k and is defined as follows:

$$\begin{aligned} F_{k,0} &= 0, & F_{k,1} &= 1, \\ F_{k,n+1} &= kF_{k,n} + F_{k,n-1}, & \text{where } n \geq 1, k \geq 1. \end{aligned} \quad (3)$$

The first few terms of this sequence are

$$\{0, 1, k, k^2 + 1, k^2 + 2, \dots\}. \quad (4)$$

The particular cases of the k -Fibonacci sequence are as follows.

If $k = 1$, the classical Fibonacci sequence is obtained:

$$\begin{aligned} F_0 &= 0, & F_1 &= 1, \\ F_{n+1} &= F_n + F_{n-1} & \text{for } n \geq 1, \\ \{F_n\}_{n \in \mathbb{N}} &= \{0, 1, 1, 2, 3, 5, 8, \dots\}. \end{aligned} \quad (5)$$

If $k = 2$, the Pell sequence is obtained:

$$\begin{aligned} P_0 &= 0, & P_1 &= 1, & P_{n+1} &= 2P_n + P_{n-1} & \text{for } n \geq 1, \\ \{P_n\}_{n \in \mathbb{N}} &= \{0, 1, 2, 5, 12, 29, 70, \dots\}. \end{aligned} \quad (6)$$

Motivated by the study of k -Fibonacci numbers in [4], the k -Lucas numbers have been defined in a similar fashion as

$$\begin{aligned} L_{k,0} &= 2, & L_{k,1} &= k, \\ L_{k,n+1} &= kL_{k,n} + L_{k,n-1}, & \text{where } n \geq 1, k \geq 1. \end{aligned} \quad (7)$$

The first few terms of this sequence are

$$\{2, k, k^2 + 2, k^3 + 3 \dots\}. \quad (8)$$

The particular cases of the k -Lucas sequence are as follows.

If $k = 1$, the classical Lucas sequence is obtained:

$$\{2, 1, 3, 4, 7, 11, 18 \dots\}. \quad (9)$$

If $k = 2$, the Pell-Lucas sequence is obtained:

$$\{2, 2, 6, 14, 34, 82 \dots\}. \quad (10)$$

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [2]. The same idea has been used to develop Binet formulas for other recursive sequences as well. The well-known Binet's formulas for k -Fibonacci numbers and k -Lucas numbers, see [3–5], are given by

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \quad (11)$$

$$L_{k,n} = r_1^n + r_2^n,$$

where r_1, r_2 are roots of characteristic equation

$$r^2 - kr - 1 = 0, \quad (12)$$

which are given by

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2}, \quad r_2 = \frac{k - \sqrt{k^2 + 4}}{2}. \quad (13)$$

We also note that

$$\begin{aligned} r_1 + r_2 &= k, \\ r_1 r_2 &= -1, \\ r_1 - r_2 &= \sqrt{k^2 + 4}. \end{aligned} \quad (14)$$

There are a huge number of simple as well as generalized identities available in the Fibonacci related literature in various forms. Some properties for common factors of Fibonacci and Lucas numbers are studied by Thongmoon [6, 7]. The k -Fibonacci numbers which are of recent origin were found by studying the recursive application of two geometrical transformations used in the well-known four-triangle longest-edge partition [3], serving as an example between geometry and numbers. Also in [8], authors established some new properties of k -Fibonacci numbers and k -Lucas numbers in terms of binomial sums. Falcón and Plaza [9] studied 3-dimensional k -Fibonacci spirals considering geometric point of view. Some identities for k -Lucas numbers may be found in [9]. In [10] many properties of k -Fibonacci numbers are obtained by easy arguments and related with so-called Pascal triangle. The aim of the present paper is to establish connection formulas between k -Fibonacci and k -Lucas numbers, thereby deriving some results out of them. In the following section we investigate some products of k -Fibonacci numbers and k -Lucas numbers. Though the results can be established by induction method as well, Binet's formula is mainly used to prove all of them.

2. On the Products of k -Fibonacci and k -Lucas Numbers

Theorem 1. $F_{k,2n}L_{k,2n} = F_{k,4n}$, where $n \geq 1$.

Proof.

$$\begin{aligned} F_{k,2n}L_{k,2n} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right] [r_1^{2n} + r_2^{2n}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n} + (r_1 r_2)^{2n} - (r_1 r_2)^{2n} - r_2^{4n}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n} - r_2^{4n}] \\ &= F_{k,4n}. \end{aligned} \quad (15)$$

□

Theorem 2. $F_{k,2n}L_{k,2n+1} = F_{k,4n+1} - 1$, where $n \geq 1$.

Proof.

$$\begin{aligned} F_{k,2n}L_{k,2n+1} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right] [r_1^{2n+1} + r_2^{2n+1}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n+1} + r_1^{2n} r_2^{2n+1} - r_1^{2n+1} r_2^{2n} - r_2^{4n+1}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n+1} - r_2^{4n+1}] + \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)} (r_2 - r_1) \\ &= F_{k,4n+1} - (-1)^{2n} \\ &= F_{k,4n+1} - 1. \end{aligned} \quad (16)$$

□

Theorem 3. $F_{k,2n}L_{k,2n+2} = F_{k,4n+2} - k$, where $n \geq 1$.

Proof.

$$\begin{aligned} F_{k,2n}L_{k,2n+2} &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right] [r_1^{2n+2} + r_2^{2n+2}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n+2} + r_1^{2n} r_2^{2n+2} - r_1^{2n+2} r_2^{2n} - r_2^{4n+2}] \\ &= \frac{1}{r_1 - r_2} [r_1^{4n+2} - r_2^{4n+2}] - \frac{(r_1 r_2)^{2n}}{(r_1 - r_2)} [r_1^2 - r_2^2] \\ &= F_{k,4n+2} - (r_1 r_2)^{2n} (r_1 + r_2) \\ &= F_{k,4n+2} - (-1)^{2n} k \\ &= F_{k,4n+2} - k. \end{aligned} \quad (17)$$

□

Theorem 4. $F_{k,2n}L_{k,2n+3} = F_{k,4n+3} - (k^2 + 1)$, where $n \geq 1$.

Proof.

$$\begin{aligned}
 & F_{k,2n}L_{k,2n+3} \\
 &= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right] [r_1^{2n+3} + r_2^{2n+3}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n+3} + r_1^{2n}r_2^{2n+3} - r_1^{2n+3}r_2^{2n} - r_2^{4n+3}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n+3} - r_2^{4n+3}] + \frac{(r_1r_2)^{2n}}{(r_1 - r_2)} [r_2^3 - r_1^3] \quad (18) \\
 &= F_{k,4n+3} - (-1)^{2n} \left[\frac{r_1 - r_2}{r_1 - r_2} \right] [r_1^2 + r_2^2 + r_1r_2] \\
 &= F_{k,4n+3} - (L_{k,2} - 1) \\
 &= F_{k,4n+3} - (k^2 + 1).
 \end{aligned}$$

□

Theorem 5. $F_{k,2n-1}L_{k,2n+1} = F_{k,4n} + 1$, where $n \geq 1$.

Proof.

$$\begin{aligned}
 & F_{k,2n-1}L_{k,2n+1} \\
 &= \left[\frac{r_1^{2n-1} - r_2^{2n-1}}{r_1 - r_2} \right] [r_1^{2n+1} + r_2^{2n+1}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n} + r_1^{2n-1}r_2^{2n+1} - r_1^{2n+1}r_2^{2n-1} - r_2^{4n}] \quad (19) \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n} - r_2^{4n}] + \frac{(r_1r_2)^{2n}}{(r_1 - r_2)} \left[\frac{r_2}{r_1} - \frac{r_1}{r_2} \right] \\
 &= F_{k,4n} - (r_1r_2)^{2n-1} \\
 &= F_{k,4n} + 1.
 \end{aligned}$$

□

Theorem 6. $F_{k,2n+1}L_{k,2n} = F_{k,4n+1} + 1$, where $n \geq 1$.

Proof.

$$\begin{aligned}
 & F_{k,2n+1}L_{k,2n} \\
 &= \left[\frac{r_1^{2n+1} - r_2^{2n+1}}{r_1 - r_2} \right] [r_1^{2n} + r_2^{2n}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n+1} + r_1^{2n+1}r_2^{2n} - r_1^{2n}r_2^{2n+1} - r_2^{4n+1}] \quad (20) \\
 &= \frac{1}{r_1 - r_2} [r_1^{4n+1} - r_2^{4n+1}] + \frac{(r_1r_2)^{2n}}{(r_1 - r_2)} (r_1 - r_2) \\
 &= F_{k,4n+1} + (-1)^{2n} \\
 &= F_{k,4n+1} + 1.
 \end{aligned}$$

In the same manner, we obtain the following results. □

Theorem 7. $F_{k,2n+2}L_{k,2n} = F_{k,4n+2} + k$, where $n \geq 1$.

Theorem 8. $F_{k,2n+2}L_{k,2n+1} = F_{k,4n+3} - 1$, where $n \geq 1$.

3. Generalized Identities on the Products of k -Fibonacci and k -Lucas Numbers

Theorem 9. $F_{k,m}L_{k,n} = F_{k,m+n} - (-1)^m F_{k,n-m}$, for $n \geq m + 1$, $m \geq 0$.

Proof.

$$\begin{aligned}
 & F_{k,m}L_{k,n} \\
 &= \left[\frac{r_1^m - r_2^m}{r_1 - r_2} \right] [r_1^n + r_2^n] \\
 &= \frac{1}{r_1 - r_2} [r_1^{m+n} + r_1^m r_2^n - r_1^n r_2^m - r_2^{m+n}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{m+n} - r_2^{m+n}] + \frac{1}{r_1 - r_2} [r_1^m r_2^n - r_1^n r_2^m] \\
 &= F_{k,m+n} - \left[\frac{r_1^n r_2^m - r_1^m r_2^n}{r_1 - r_2} \right] \\
 &= F_{k,m+n} - (r_1 r_2)^m \left[\frac{r_1^{n-m} - r_2^{n-m}}{r_1 - r_2} \right] \\
 &= F_{k,m+n} - (-1)^m F_{k,n-m}. \quad (21)
 \end{aligned}$$

For different value of m , we have different results:

If $m = 0$ then $F_{k,0}L_{k,n} = F_{k,n} - F_{k,n} = 0$, $n \geq 1$

If $m = 1$ then $F_{k,1}L_{k,n} = F_{k,n+1} + F_{k,n-1}$, $n \geq 2$

or $L_{k,n} = F_{k,n+1} + F_{k,n-1}$ (22)

If $m = 2$ then $F_{k,2}L_{k,n} = F_{k,n+2} - F_{k,n-2}$, $n \geq 3$

or $L_{k,n} = \frac{F_{k,n+2} - F_{k,n-2}}{k}$ and so on.

□

Theorem 10. $F_{k,n}L_{k,2n+m} = F_{k,3n+m} - (-1)^n F_{k,n+m}$, for $n \geq 1$, $m \geq 0$.

Proof.

$$\begin{aligned}
 & F_{k,n}L_{k,2n+m} \\
 &= \left[\frac{r_1^n - r_2^n}{r_1 - r_2} \right] [r_1^{2n+m} + r_2^{2n+m}] \\
 &= \frac{1}{r_1 - r_2} [r_1^{3n+m} + r_1^n r_2^{2n+m} - r_1^{2n+m} r_2^n - r_2^{3n+m}]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r_1 - r_2} \left[r_1^{3n+m} - r_2^{3n+m} \right] + (r_1 r_2)^n \left[\frac{r_2^{n+m} - r_1^{n+m}}{r_1 - r_2} \right] \\
&= F_{k,3n+m} - (-1)^n F_{k,n+m} \\
&= F_{k,3n+m} - F_{k,n+m}.
\end{aligned}
\tag{23}$$

For different values of m , we have various results:

$$\text{If } m = 0 \text{ then } F_{k,n} L_{k,2n} = F_{k,3n} - (-1)^n F_{k,n}, \quad n \geq 1$$

$$\text{If } m = 1 \text{ then } F_{k,n} L_{k,2n+1} = F_{k,3n+1} - (-1)^n F_{k,n+1}, \quad n \geq 1$$

and so on.

(24)

□

Similarly we have the following result.

Theorem 11. $F_{k,2n+m} L_{k,n} = F_{k,3n+m} + (-1)^n F_{k,n+m}$, for $n \geq 1$, $m \geq 0$.

Theorem 12. $F_{k,2n} L_{k,2n+m} = F_{k,4n+m} - F_{k,m}$, for $n \geq 1$, $m \geq 0$.

Proof.

$$\begin{aligned}
&F_{k,2n} L_{k,2n+m} \\
&= \left[\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right] \left[r_1^{2n+m} + r_2^{2n+m} \right] \\
&= \frac{1}{r_1 - r_2} \left[r_1^{4n+m} + r_1^{2n} r_2^{2n+m} - r_1^{2n+m} r_2^{2n} - r_2^{4n+m} \right] \\
&= \frac{1}{r_1 - r_2} \left[r_1^{4n+m} - r_2^{4n+m} \right] + (r_1 r_2)^{2n} \left[\frac{r_2^m - r_1^m}{r_1 - r_2} \right] \\
&= F_{k,4n+m} - F_{k,m}.
\end{aligned}
\tag{25}$$

For different values of m , we have various results:

$$\text{If } m = 0 \text{ then } F_{k,2n} L_{k,2n} = F_{k,4n}, \quad n \geq 1$$

$$\text{If } m = 1 \text{ then } F_{k,2n} L_{k,2n+1} = F_{k,4n+1} - 1, \quad n \geq 1 \text{ and so on.}$$

(26)

□

Theorem 13. $F_{k,2n+m} L_{k,2n} = F_{k,4n+m} + F_{k,m}$, for $n \geq 1$, $m \geq 0$.

Proof.

$$\begin{aligned}
&F_{k,2n+m} L_{k,2n} \\
&= \left[\frac{r_1^{2n+m} - r_2^{2n+m}}{r_1 - r_2} \right] \left[r_1^{2n} + r_2^{2n} \right] \\
&= \frac{1}{r_1 - r_2} \left[r_1^{4n+m} + r_1^{2n+m} r_2^{2n} - r_1^{2n} r_2^{2n+m} - r_2^{4n+m} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r_1 - r_2} \left[r_1^{4n+m} - r_2^{4n+m} \right] + (r_1 r_2)^{2n} \left[\frac{r_1^m - r_2^m}{r_1 - r_2} \right] \\
&= F_{k,4n+m} + F_{k,m}.
\end{aligned}
\tag{27}$$

For different values of m , we have various results:

$$\text{If } m = 0 \text{ then } F_{k,2n} L_{k,2n} = F_{k,4n}, \quad n \geq 1$$

$$\text{If } m = 1 \text{ then } F_{k,2n+1} L_{k,2n} = F_{k,4n+1} + 1, \quad n \geq 1$$

$$\text{If } m = 2 \text{ then } F_{k,2n+2} L_{k,2n} = F_{k,4n+2} + k, \quad n \geq 1$$

and so on.

(28)

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

- [1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, NY, USA, 2001.
- [2] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Ellis Horwood, Chichester, UK, 1989.
- [3] S. Falcón and Á. Plaza, "On the Fibonacci k -numbers," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1615–1624, 2007.
- [4] S. Falcon, "On the k -Lucas numbers," *International Journal of Contemporary Mathematical Sciences*, vol. 6, no. 21, pp. 1039–1050, 2011.
- [5] C. Bolat, A. Ipeck, and H. Kose, "On the sequence related to Lucas numbers and its properties," *Mathematica Aeterna*, vol. 2, no. 1, pp. 63–75, 2012.
- [6] M. Thongmoon, "Identities for the common factors of Fibonacci and Lucas numbers," *International Mathematical Forum*, vol. 4, no. 7, pp. 303–308, 2009.
- [7] M. Thongmoon, "New identities for the even and odd Fibonacci and Lucas numbers," *International Journal of Contemporary Mathematical Sciences*, vol. 4, no. 14, pp. 671–676, 2009.
- [8] N. Yilmaz, N. Taskara, K. Uslu, and Y. Yazlik, "On the binomial sums of k -Fibonacci and k -Lucas sequences," in *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics (ICNAAM '11)*, pp. 341–344, September 2011.
- [9] S. Falcón and Á. Plaza, "On the 3-dimensional k -Fibonacci spirals," *Chaos, Solitons and Fractals*, vol. 38, no. 4, pp. 993–1003, 2008.
- [10] S. Falcón and Á. Plaza, "The k -Fibonacci sequence and the Pascal 2-triangle," *Chaos, Solitons and Fractals*, vol. 33, no. 1, pp. 38–49, 2007.