Research Article

On the p -Version of the Schwab-Borchardt Mean

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This paper deals with a one-parameter generalization of the Schwab-Borchardt mean. The new mean is defined in terms of the inverse functions of the generalized trigonometric and generalized hyperbolic functions. The four new bivariate means are introduced as particular cases of the p -version of the Schwab-Borchardt mean. For the particular value of the parameter p , these means become either the classical logarithmic mean or the Seiffert means or the Neuman-Sándor mean. Wilker- and Huygens-type inequalities involving inverse functions of the generalized trigonometric and the generalized hyperbolic functions are also established.

1. Introduction

The Schwab-Borchardt mean of two numbers $x \geq 0$ and $y > 0$, denoted by $SB(x, y) \equiv SB$, is defined as

$$SB(x, y) = \begin{cases} \frac{\sqrt{y^2 - x^2}}{\cos^{-1}(x/y)}, & 0 \leq x \leq y \\ \frac{\sqrt{x^2 - y^2}}{\cosh^{-1}(x/y)}, & y < x \\ x, & x = y \end{cases} \quad (1)$$

(see [1, Thrm 8.4], [2, (2.3)]). It follows from (1) that $SB(x, y)$ is not symmetric in its arguments and is a homogeneous function of degree 1 in x and y . This mean has been studied extensively in [1–5].

The goal of this paper is to define and investigate a generalization of the Schwab-Borchardt mean SB . The new one has a form which is similar to (1) but depends on a parameter p which is used in definitions of two families of higher transcendental functions called the generalized trigonometric and the generalized hyperbolic functions. Definitions of these functions are given in Section 2. Also, we will use a notion of the R -hypergeometric functions of two variables. Their definition and some basic properties are given in Section 3. Throughout the sequel, the p -version of SB will be denoted by SB_p . The latter is introduced in Section 4.

Therein, some basic properties of the new mean are discussed. In Section 5, we define four new bivariate means which can be considered as the generalized logarithmic mean, the generalized Seiffert means, and the generalized Neuman-Sándor mean. In the last section of this paper, we shall establish inequalities involving new means as well as Wilker- and Huygens-type inequalities involving inverse functions of the generalized trigonometric and the generalized hyperbolic functions.

2. Definitions of Generalized Trigonometric and Hyperbolic Functions

For the reader's convenience, we recall first definition of the celebrated Gauss hypergeometric function $F(a, b; c; z)$:

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)} \frac{z^n}{n!}, \quad |z| < 1, \quad (2)$$

where $(\alpha, n) = \alpha(\alpha+1) \cdots (\alpha+n-1)$ ($n \neq 0$) is the shifted factorial or Appell symbol, with $(\alpha, 0) = 1$ if $\alpha \neq 0$, and $\gamma \neq 0, -1, -2, \dots$.

In what follows, we will always assume that the number p is strictly greater than 1. We will adopt notation and definitions used in [6]. Let

$$\pi_p = 2 \frac{\pi/p}{\sin(\pi/p)}. \quad (3)$$

Further, let

$$\begin{aligned} a_p &= \frac{\pi_p}{2}, & b_p &= 2^{-1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right), \\ c_p &= 2^{-1/p} F\left(1, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right). \end{aligned} \quad (4)$$

Also, let $I = (0, 1)$ and let $J = (1, \infty)$. The generalized trigonometric and hyperbolic functions needed in this paper are the following homeomorphisms:

$$\sin_p : (0, a_p) \rightarrow I, \quad \cos_p : (0, a_p) \rightarrow I, \quad (5)$$

$$\tan_p : (0, b_p) \rightarrow I, \quad \sinh_p : (0, c_p) \rightarrow I,$$

$$\cosh_p : (0, \infty) \rightarrow J, \quad \tanh_p : (0, \infty) \rightarrow J. \quad (6)$$

The inverse functions of $\sin_p^{-1}$ and $\sinh_p^{-1}$ can be represented as follows [7]:

$$\sin_p^{-1} u = \int_0^u (1 - t^p)^{-1/p} dt = u F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; u^p\right), \quad (7)$$

$$\sinh_p^{-1} u = \int_0^u (1 + t^p)^{-1/p} dt = u F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; -u^p\right). \quad (8)$$

Inverse functions of the remaining four functions can be expressed in terms of $\sin_p^{-1}$ and $\sinh_p^{-1}$. We have

$$\cos_p^{-1} u = \sin_p^{-1}(\sqrt[p]{1 - u^p}), \quad (9)$$

$$\cosh_p^{-1} u = \sinh_p^{-1}(\sqrt[p]{u^p - 1}), \quad (10)$$

$$\tan_p^{-1} u = \sin_p^{-1}\left(\frac{u}{\sqrt[p]{1 + u^p}}\right), \quad (11)$$

$$\tanh_p^{-1} u = \sinh_p^{-1}\left(\frac{u}{\sqrt[p]{1 - u^p}}\right).$$

The generalized trigonometric functions have been introduced by Lindqvist in [8]. It is obvious that the functions under discussion become classical trigonometric and hyperbolic functions when $p = 2$. It is known that they are eigenfunctions of the Dirichlet problem for the one-dimensional p -Laplacian. For more details concerning generalized trigonometric functions, generalized hyperbolic functions, and inequalities involving these functions, the interested reader is referred to [6–15].

3. The R -Hypergeometric Functions of Two Variables

In this section, we give the definition of the bivariate R -hypergeometric functions which are used in the sequel. Some results for these functions are also included here.

In what follows, the symbols $\mathbb{R}_+$ and $\mathbb{R}_>$ will stand for the nonnegative semiaxis and the set of positive numbers, respectively. Let $b = (b_1, b_2) \in \mathbb{R}_>^2$. By μ_b , where

$$\mu_b(t) = \frac{\Gamma(b_1 + b_2)}{\Gamma(b_1)\Gamma(b_2)} t^{b_1-1} (1-t)^{b_2-1}, \quad (12)$$

we will denote the Dirichlet measure on the interval $[0, 1]$. It is well known that μ_b is the probability measure on its domain.

Also, let $X = (x, y) \in \mathbb{R}_>^2$. In [16, 17], the R -hypergeometric function $R_\alpha(b; X)$ ($\alpha \in \mathbb{R}$) is defined as follows:

$$R_\alpha(b; X) = \int_0^1 (u \cdot X)^\alpha \mu_b(t) dt, \quad (13)$$

where $u = (t, 1-t)$ and $u \cdot X = tx + (1-t)y$ are the dot product of u and X . Many of the important special functions, including Gauss' hypergeometric function F and some elliptic integrals, admit the integral representation (13). For more details, the interested reader is referred to Carlson's monograph [17].

A nice feature of the R -hypergeometric function is its permutation symmetry in both parameters and variables; that is,

$$R_\alpha(b_1, b_2; x, y) = R_\alpha(b_2, b_1; y, x). \quad (14)$$

Another remarkable property of R_α is its homogeneity of degree α in its variables:

$$R_\alpha(b_1, b_2; \gamma x, \gamma y) = \gamma^\alpha R_\alpha(b_1, b_2; x, y) \quad (15)$$

($\gamma > 0$).

For the later use, let us also record Carlson's inequality [18, Theorem 3]:

$$[R_\alpha(b; X)]^{1/\alpha} \leq [R_\beta(b; X)]^{1/\beta} \quad (16)$$

($\alpha, \beta \neq 0, \alpha \leq \beta$).

We will also need the following result which appears in [5, Proposition 2.1]. Let $\alpha < 0$, $b \in \mathbb{R}_+^2$, and let $X, Y \in \mathbb{R}_>^2$. Then the following inequality

$$R_\alpha(b; \lambda X + (1-\lambda)Y) \leq [R_\alpha(b; X)]^\lambda [R_\alpha(b; Y)]^{1-\lambda} \quad (17)$$

holds true for all $0 \leq \lambda \leq 1$.

4. Definition and Basic Properties of the p -Version of SB

Let the numbers x and y have the meaning as in Section 1. For the sake of presentation, we recall first a formula for the mean SB in terms of the R -hypergeometric function:

$$\text{SB}(x, y) \equiv \text{SB} = R_{-1/2}\left(\frac{1}{2}, 1; x^2, y^2\right)^{-1} \quad (18)$$

(see [17, 19]).

We define the p -version ($p > 1$) of the mean SB as follows:

$$\text{SB}_p(x, y) \equiv \text{SB}_p = R_{-1/p}\left(\frac{1}{p}, 1; x^p, y^p\right)^{-1}. \quad (19)$$

The rightmost member of (19) is a special case of what is called in mathematical literature the R -hypergeometric mean (see [2, 17, 18]). Using elementary properties in the R -hypergeometric functions, we see that $\text{SB}_p(x, y)$ is the mean

value of x and y . Moreover, this mean is nonsymmetric and homogeneous of degree 1 in its variables. The well-known results on the R -hypergeometric means lead to the conclusion that SB_p is a strongly increasing function of the parameter p .

For the brevity of notation, let us introduce a particular R -hypergeometric function:

$$R_S(x, y) = R_{-1/p}\left(\frac{1}{p}, 1; x, y\right). \quad (20)$$

Clearly function R_S is nonsymmetric and homogeneous of degree $-1/p$ in its variables. Comparison with (19) yields

$$SB_p(x, y) = R_S(x^p, y^p)^{-1}. \quad (21)$$

We shall demonstrate now that SB_p can also be expressed in terms of $\cos_p^{-1}$ and $\cosh_p^{-1}$:

$$SB_p(x, y) = \begin{cases} \sqrt[p]{y^p - x^p} \cos_p^{-1}(x/y), & 0 \leq x \leq y \\ \sqrt[p]{x^p - y^p} \cosh_p^{-1}(x/y), & y < x \\ x, & x = y. \end{cases} \quad (22)$$

For the proof of the first part of (22), let us record a formula which shows that the Gauss hypergeometric function F can be expressed in terms of the bivariate R -hypergeometric function:

$$F(\alpha, \beta; \gamma; z) = R_{-\alpha}(\beta, \gamma - \beta; 1 - z, 1) \quad (23)$$

(see, e.g., [17, ((5.9)–(12))]). Application of the last formula to (7) yields

$$\sin_p^{-1} u = u R_S(1 - u^p, 1), \quad (24)$$

where $0 < u < 1$. This, in conjunction with (9), gives

$$\cos_p^{-1} u = \sqrt[p]{1 - u^p} R_S(u^p, 1). \quad (25)$$

Letting above $u = x/y$ and utilizing homogeneity of the function R_S , we obtain

$$\begin{aligned} \cos_p^{-1}\left(\frac{x}{y}\right) &= \sqrt[p]{y^p - x^p} R_S(x^p, y^p) \\ &= \sqrt[p]{y^p - x^p} [SB_p(x, y)]^{-1}, \end{aligned} \quad (26)$$

where in the last step we have utilized formula (19). This yields the first part of (22). The second part can be established in an analogous manner. A key formula needed here reads as follows:

$$\cosh_p^{-1} u = \sqrt[p]{u^p - 1} R_S(u^p, 1), \quad (27)$$

$u > 1$. We omit further details.

Function R_S admits an integral representation:

$$R_S(x, y) = \frac{1}{p} \int_0^\infty (t+x)^{-1/p} (t+y)^{-1} dt. \quad (28)$$

This follows from the known result [20, (19.16.9)]

$$R_{-\alpha}(\beta_1, \beta_2; x, y) = \frac{1}{B(\alpha, \alpha')} \int_0^\infty t^{\alpha'-1} (t+x)^{-\beta_1} (t+y)^{-\beta_2} dt, \quad (29)$$

where B stands for the beta function and $\alpha' = \beta_1 + \beta_2 - \alpha$. Letting $\alpha = \beta_1 = 1/p$ and $\beta_2 = 1$, we obtain $\alpha' = 1$. Formula (28) now follows because $B(1/p, 1) = p$.

5. Four New Bivariate Means Derived from SB_p

The goal of this section is to define and investigate four new bivariate means. They are defined in terms of the SB_p and the bivariate power mean A_r ($r \in \mathbb{R}$). Recall that

$$A_r(x, y) \equiv A_r = \begin{cases} \sqrt[r]{\frac{x^r + y^r}{2}} & r \neq 0 \\ \sqrt{xy}, & r = 0, \end{cases} \quad (30)$$

where $x, y > 0$. The power mean A_0 of order 0 is usually denoted by G and is called the geometric mean. It is well known that the power mean A_r is a strictly increasing function of r .

We are in a position to define four new means of positive numbers x and y . In what follows, these means will be denoted by L_p, P_p, M_p and T_p , where $p > 1$. They are defined as follows:

$$L_p(x, y) \equiv L_p = SB_p(A_{p/2}, G), \quad (31)$$

$$P_p(x, y) \equiv P_p = SB_p(G, A_{p/2}), \quad (32)$$

$$T_p(x, y) \equiv T_p = SB_p(A_{p/2}, A_p), \quad (33)$$

$$M_p(x, y) \equiv M_p = SB_p(A_p, A_{p/2}). \quad (34)$$

In the case when $p = 2$, these means become the classical logarithmic mean L , two Seiffert means P and T (see [21, 22]), and the Neuman-Sándor mean M introduced in [4].

For the later use, we introduce quantity v_p , where

$$v_p = \left(\frac{|x^{p/2} - y^{p/2}|}{x^{p/2} + y^{p/2}} \right)^{2/p}. \quad (35)$$

Clearly, $0 < v_p < 1$.

The main result of this section reads as follows.

Theorem 1 (let $x, y > 0$). Then the following formulas

$$L_p = A_{p/2} \frac{v_p}{\tanh_p^{-1} v_p}, \quad (36)$$

$$P_p = A_{p/2} \frac{v_p}{\sin_p^{-1} v_p}, \quad (37)$$

$$T_p = A_{p/2} \frac{v_p}{\tan_p^{-1} v_p}, \quad (38)$$

$$M_p = A_{p/2} \frac{v_p}{\sinh_p^{-1} v_p} \quad (39)$$

are valid.

Proof. We begin with the proof of (36). Making use of (31) and (22), we obtain

$$L_p = \frac{\sqrt[p]{A_{p/2}^p - G^p}}{\cosh_p^{-1}(A_{p/2}/G)}. \quad (40)$$

Elementary computations yield

$$A_{p/2}^p - G^p = \left(\frac{x^{p/2} + y^{p/2}}{2} \right)^2 - (xy)^{p/2} = \left(\frac{x^{p/2} - y^{p/2}}{2} \right)^2. \quad (41)$$

Multiplying and dividing by

$$\left(\frac{x^{p/2} + y^{p/2}}{2} \right)^2, \quad (42)$$

we obtain, using (35),

$$A_{p/2}^p - G^p = (A_{p/2} v_p)^p. \quad (43)$$

We shall write the denominator of (40) using (10) and (43) as follows:

$$\begin{aligned} \cosh_p^{-1} \left(\frac{A_{p/2}}{G} \right) &= \tanh_p^{-1} \sqrt[p]{1 - \left(\frac{G}{A_{p/2}} \right)^p} \\ &= \tanh_p^{-1} \sqrt[p]{\frac{A_{p/2}^p - G^p}{A_{p/2}^p}} = \tanh_p^{-1} v_p. \end{aligned} \quad (44)$$

This, in conjunction with (40) and (43), gives the desired result (36).

We shall provide now a sketch of the proof of formula (39). It follows from (22) and (34) that

$$M_p = \frac{\sqrt[p]{A_p^p - A_{p/2}^p}}{\cosh_p^{-1}(A_p/A_{p/2})}. \quad (45)$$

Elementary computations yield

$$A_p^p - A_{p/2}^p = (A_{p/2} v_p)^p. \quad (46)$$

Application of (10) with $u = A_p/A_{p/2}$ gives $\cosh_p^{-1}(A_p/A_{p/2}) = \sinh_p^{-1} v_p$. This in conjunction with (45) and (46) yields the asserted result (39). The remaining two formulas for the p -versions P_p and T_p of the Seiffert means can be established in an analogous manner using (32) or (33), (22), and (9). We leave it to the interested reader. The proof is complete. $\square$

6. Inequalities Involving the SB_p Means

This section deals with inequalities involving the SB_p means. In particular, inequalities for the four means introduced in Section 5 are established. Also, we shall prove Wilker-type and Huygens-type inequalities involving inverse functions of the generalized trigonometric functions and the generalized hyperbolic functions.

Our first result reads as follows.

Theorem 2. *Let the positive numbers x and y be such that $x > y$. Then*

$$SB_p(x, y) < SB_p(y, x). \quad (47)$$

Proof. We shall prove the assertion using integral formula (28) and formula (21). Let $u > 1$ and let $t > 0$. Then $u^p > 1$ and

$$(t + u^p)^{1-1/p} > (t + 1)^{1-1/p} \quad (48)$$

or what is the same that

$$(t + u^p)^{-1/p} (t + 1)^{-1} > (t + u^p)^{-1} (t + 1)^{-1/p} \quad (49)$$

because $1 - 1/p > 0$. Integration yields

$$\begin{aligned} &\frac{1}{p} \int_0^\infty (t + u^p)^{-1/p} (t + 1)^{-1} dt \\ &> \frac{1}{p} \int_0^\infty (t + u^p)^{-1} (t + 1)^{-1/p} dt \end{aligned} \quad (50)$$

or what is the same that the inequality

$$R_S(u^p, 1) > R_S(1, u^p). \quad (51)$$

Raising both sides to the power of -1 and next applying formula (21), we obtain

$$SB_p(u, 1) < SB_p(1, u). \quad (52)$$

Letting $u = x/y$ and next utilizing homogeneity of SB_p , we obtain the desired result. $\square$

The four new means defined in Section 5 and the power means are comparable. We have the following.

Corollary 3. *Let G, A_p, L_p, P_p, T_p and M_p be the mean values of two positive and unequal numbers. Then the following chain of inequalities*

$$G < L_p < P_p < A_{p/2} < M_p < T_p < A_p \quad (53)$$

is valid.

Proof. The first, third, fourth, and the sixth inequalities in (53) follow from their definitions (see (31), (32), (34), and (33)). The second and the fifth inequalities can be obtained using Theorem 2 applied to two pairs of defining equations (31)-(32) and (34)-(33). $\square$

Our next result reads as follows.

Theorem 4. *Let $\alpha = 1/p$ and $\beta = 1 - 1/p$. Then the inequality*

$$[SB_p(y, x)]^\alpha y^\beta < SB_p(x, y) \quad (54)$$

holds true for positive and unequal numbers x and y .

Proof. We shall obtain the desired result utilizing the following transformation for the R -hypergeometric functions [17, ((5.5)–(19))]:

$$R_{-\beta_1}(\alpha_1, \alpha_2; x, y) y^{\beta_1 - \alpha_1} = R_{-\alpha_1}(\beta_1, \beta_2; x, y), \quad (55)$$

where $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$. Letting $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 1/p$ and making substitutions $x := x^p$ and $y := y^p$, we obtain

$$R_{-1/p}\left(1, \frac{1}{p}; x^p, y^p\right) y^{1-p} = R_{-1}\left(\frac{1}{p}, 1; y^p, x^p\right). \quad (56)$$

Applying the permutation symmetry (see (14)) to the R -hypergeometric function on the left-hand side and next raising both sides of the resulting formula to the power of -1 , we obtain

$$R_{-1/p}\left(\frac{1}{p}, 1; y^p, x^p\right)^{-1} y^{p-1} = R_{-1}\left(\frac{1}{p}, 1; x^p, y^p\right)^{-1}. \quad (57)$$

Since $-1 < -1/p$, Carlson's inequality (16) yields

$$R_{-1}\left(\frac{1}{p}, 1; x^p, y^p\right)^{-1} < R_{-1/p}\left(\frac{1}{p}, 1; x^p, y^p\right)^{-p}. \quad (58)$$

Combining this with (57), we obtain

$$R_{-1/p}\left(\frac{1}{p}, 1; y^p, x^p\right)^{-1} y^{p-1} < R_{-1/p}\left(\frac{1}{p}, 1; x^p, y^p\right)^{-p}. \quad (59)$$

Utilizing (20) and (21), we can write the last inequality in the form

$$SB_p(y, x) y^{p-1} < SB_p(x, y)^p. \quad (60)$$

The desired inequality (54) now follows. $\square$

Corollary 5. Let the numbers α and β be the same as in Theorem 4. Then the following inequalities

$$P_p^\alpha G^\beta < L_p, \quad (61)$$

$$L_p^\alpha A_{p/2}^\beta < P_p, \quad (62)$$

$$T_p^\alpha A_{p/2}^\beta < M_p, \quad (63)$$

$$M_p^\alpha A_p^\beta < T_p$$

hold true.

Proof. Inequality (61) can be obtained using Theorem 4 with $x := A_{p/2}$ and $y := G$. Next, we utilize formulas (31) and (32) to obtain the desired result. In a similar fashion, one can prove inequality (62) using Theorem 4 with $x := G$ and $y := A_{p/2}$. Making use of formulas (31) and (32) yields the assertion. The remaining inequalities (63) can be proven in an analogous manner. We omit further details. $\square$

Another inequality for the SB_p mean is contained in the following.

Theorem 6. Let $x_1, x_2, y_1, y_2 > 0$. Then

$$SB_p(x_1, y_1) SB_p(x_2, y_2) \leq SB_p^2[A_p(x_1, x_2), A_p(y_1, y_2)]. \quad (64)$$

Proof. First, we make the substitutions in (17)

$$\alpha = -\frac{1}{p}, \quad b = \left(\frac{1}{p}, 1\right), \quad X = (x_1^p, y_1^p), \quad (65)$$

$$Y = (x_2^p, y_2^p), \quad \lambda = \frac{1}{2},$$

next we use (20), and finally we raise both sides of the resulting inequality to the power of -2 . This gives

$$R_S(x_1^p, y_1^p)^{-1} R_S(x_2^p, y_2^p)^{-1} \leq \left[R_S(A_p^p(x_1, x_2), A_p^p(y_1, y_2))^{-1} \right]^2. \quad (66)$$

Application of (21) gives the desired inequality (64). $\square$

Corollary 7. Assume that the positive numbers x and y are not equal. Then the following inequalities

$$P_p M_p < A_{p/2}^2, \quad (67)$$

$$L_p T_p < A_{p/2}^2 \quad (68)$$

hold true.

Proof. For the proof of (67), we let in (64) $x_1 = G, x_2 = A_p$ and $y_1 = y_2 = A_{p/2}$. Then, the left-hand side of (67) follows from (32) and (34). To obtain the right-hand side of the inequality in question, let us notice that $A_p(G, A_p) = A_{p/2}$ and obviously $A_p(A_{p/2}, A_{p/2}) = A_{p/2}$. Inequality (68) can be established in a similar manner. We use (64) with $x_1 = x_2 = A_{p/2}, y_1 = G$, and $y_2 = A_p$ followed by application of (31) and (33). $\square$

We will close this section proving Wilker-type and Huygens-type inequalities which involve inverse functions of the generalized trigonometric and hyperbolic functions. To this aim, we shall employ the following result [23].

Theorem A. Let u, v, λ, μ be positive numbers. Assume that u and v satisfy the separation condition:

$$u < 1 < v. \quad (69)$$

Then the inequality

$$1 < \frac{\lambda}{\lambda + \mu} u^r + \frac{\mu}{\lambda + \mu} v^s \quad (70)$$

holds true if either

$$1 < u^\gamma v^\delta, \quad s > 0, \quad r\lambda \leq \frac{s\mu\gamma}{\delta} \quad (71)$$

or

$$u^\gamma v^\delta < 1, \quad s < 0, \quad r\lambda \leq \frac{s\mu\gamma}{\delta}, \quad (72)$$

for some $\gamma, \delta \geq 0$ with $\gamma + \delta = 1$. If u and v satisfy the separation condition (69) together with

$$1 < \gamma \frac{1}{u} + \delta \frac{1}{v}, \quad (73)$$

then inequality (70) is also valid if

$$r \leq s \leq -1, \quad \mu\gamma \leq \lambda\delta. \quad (74)$$

As in the previous sections, the letters x and y will stand for positive and unequal numbers. Also, $G, A_{p/2}$ and A_p denote the power means of x and y .

For the sake of notation, we define

$$\tau = pt, \quad \text{where } t = \frac{1}{2} \log \left(\frac{x}{y} \right). \quad (75)$$

We are in a position to prove the following.

Theorem 8. Let v_p be the same as in (35) and let $\sigma_1 = (\cosh^2(\tau/2))^{1/p}$. Assume that $\lambda, \mu, s > 0$. Also, let

$$(u, v) = \left(\frac{\sin_p^{-1} v_p}{\tanh_p^{-1} v_p}, \sigma_1 \frac{v_p}{\tanh_p^{-1} v_p} \right) \quad (76)$$

or

$$(u, v) = \left(\frac{\tanh_p^{-1} v_p}{\sinh_p^{-1} v_p}, \frac{v_p}{\sinh_p^{-1} v_p} \right). \quad (77)$$

If

$$(p-1)r\lambda \leq s\mu, \quad (78)$$

then inequality (70) is satisfied if (u, v) is defined either in (76) or in (77). Inequality (70) is also satisfied if either

$$(u, v) = \left(\frac{v_p}{\sin_p^{-1} v_p}, \frac{\tanh_p^{-1} v_p}{\sin_p^{-1} v_p} \right) \quad (79)$$

or

$$(u, v) = \left(\sigma_2 \frac{v_p}{\tan_p^{-1} v_p}, \frac{\sinh_p^{-1} v_p}{\tan_p^{-1} v_p} \right) \quad (80)$$

provided that

$$r\lambda \leq (p-1)s\mu. \quad (81)$$

Here, $\sigma_2 = \sigma_1 / (\cosh \tau)^{1/p}$.

Proof. For the proof of validity of (70) with (u, v) as defined in (76), we let

$$u = \frac{L_p}{P_p}, \quad v = \frac{L_p}{G}. \quad (82)$$

It follows from (53) that the separation condition (69) is satisfied. To complete the proof of (76), we utilize a well-known fact about bivariate means. Let $N(x, y) \equiv N$ be a mean which is homogeneous of degree 1 in its variables. Then

$$N(x, y) = G(x, y) N(e^t, e^{-t}). \quad (83)$$

In particular, we have

$$\frac{A_{p/2}}{G} = A_{p/2}(e^t, e^{-t}) = \left(\cosh \left(\frac{pt}{2} \right) \right)^{2/p} = \left(\cosh^2 \left(\frac{\tau}{2} \right) \right)^{1/p}. \quad (84)$$

This together with (36) and (37) gives the explicit formula (76) for (u, v) . We will show now that the first inequality in (71) is satisfied if $\gamma = 1/p$ and $\delta = 1 - 1/p$. To this aim, we utilize (82) and write inequality (61) as follows:

$$1 < u^\alpha v^\beta, \quad (85)$$

where $\alpha = 1/p$ and $\beta = 1 - 1/p$. This yields $\gamma = \alpha$ and $\delta = \beta$. To obtain condition (78) of validity of (70), we substitute $\gamma/\delta = 1/(p-1)$ into the last inequality in (71). This completes the first part of the proof. Assume now that (u, v) is the same as is defined in (79). We will prove that (70) holds true provided that condition (81) is satisfied. First, we define

$$u = \frac{P_p}{A_{p/2}}, \quad v = \frac{P_p}{L_p}. \quad (86)$$

Again, we appeal to (53) to claim that u and v satisfy the separation condition (69). Making use of (36) and (37), we obtain an explicit formula (79) for (u, v) . We will show that the first inequality in (71) is satisfied if $\gamma = 1 - 1/p$ and $\delta = 1/p$. To this aim, we utilize (86) and write inequality (62) as follows:

$$1 < u^\beta v^\alpha, \quad (87)$$

where $\alpha = 1/p$ and $\beta = 1 - 1/p$. To prove that (70) holds true if (81) is satisfied, we substitute $\gamma/\delta = p-1$ into the last inequality in (71). The assertion now follows. The remaining two cases when (u, v) is defined in (77) or in (80) can be established in the analogous manner. In these cases, we have either

$$(u, v) = \left(\frac{M_p}{T_p}, \frac{M_p}{A_{p/2}} \right) \quad (88)$$

or

$$(u, v) = \left(\frac{T_p}{A_p}, \frac{T_p}{M_p} \right). \quad (89)$$

We leave it to the reader to complete the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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