Research Article

Principal Mappings between Posets

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We introduce and study principal mappings between posets which generalize the notion of principal elements in a multiplicative lattice, in particular, the principal ideals of a commutative ring. We also consider some weaker forms of principal mappings such as meet principal, join principal, weak meet principal, and weak join principal mappings which also generalize the corresponding notions on elements in a multiplicative lattice, considered by Dilworth, Anderson and Johnson. The principal mappings between the lattices of powersets and chains are characterized. Finally, for any PID R , it is proved that a mapping $F : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is a contractive principal mapping if and only if there is a fixed ideal $I \in \text{Idl}(R)$ such that $F(J) = IJ$ for all $J \in \text{Idl}(R)$. This exploration also leads to some new problems on lattices and commutative rings.

1. Introduction

A multiplicative lattice [1–3] is a complete lattice L together with a binary operation, called multiplication, that is associative, commutative, and distributive over arbitrary joins and has the greatest element 1_L as the multiplication identity.

The complete lattice $\text{Idl}(R)$ of all ideals of a commutative ring R is a typical example of multiplicative lattices.

If I is a principal ideal of R , then I satisfies the following equations:

$$\begin{aligned} (J \cap [K : I])I &= (JI) \cap K, \\ ([K + (JI)] : I) &= [K : I] + J, \end{aligned} \quad (1)$$

for any $J, K \in \text{Idl}(R)$ where $[K : I]$ is the ideal quotient [4].

In terms of the order and the multiplication on the lattice $\text{Idl}(R)$, the above two equations can be rephrased as

$$\begin{aligned} (J \wedge [K : I])I &= (JI) \wedge K, \\ ([K \vee (JI)] : I) &= [K : I] \vee J. \end{aligned} \quad (2)$$

In his efforts to obtain an abstract ideal theory of commutative rings, Dilworth introduced principal elements, the analogues of principal ideals, in a multiplicative lattice.

The definition of principal elements makes use of the corresponding properties of principal ideals given in (2). Based on this notion of principal elements, Dilworth successfully established Noether's Decomposition Theorems and Krull's Principal Ideal Theorem for multiplicative lattices.

Thereafter, the principal elements have been studied extensively by many people including Anderson, Johnson, and others [5–10]. As pointed out by Anderson and Johnson [2], “principal elements are the cornerstone on which the theory of multiplicative lattices and abstract ideal theory now largely rest.” Dilworth's original definition of principal elements is only valid for a multiplicative lattice as it makes use of the multiplication, meet and join operations in a lattice, and so it does not apply on a general lattice. It is thus natural to wonder whether it is possible to extend this notion to arbitrary lattices or even posets.

Let us relook at the principal ideals of a commutative ring R from the perspective of mappings. Each ideal I of R defines two mappings $F_I, G_I : \text{Idl}(R) \rightarrow \text{Idl}(R)$ by

$$F_I(J) = IJ, \quad G_I(J) = [J : I], \quad (3)$$

for each $J \in \text{Idl}(R)$.

By (2), if I is a principal ideal of R , then F_I and G_I satisfy the following equations for any $J, K \in \text{Idl}(R)$:

$$\begin{aligned} F_I(J \wedge G_I(K)) &= F_I(J) \wedge K, \\ G_I(K \vee F_I(J)) &= G_I(K) \vee J. \end{aligned} \quad (4)$$

Thus, every principal ideal I of R corresponds to a special mapping F_I from the lattice $\text{Idl}(R)$ to itself.

Motivated by these observations, we defined principal mappings between lattices and proved some basic properties of such mappings in [11]. In the current paper, we further generalize the notion of principal mappings to arbitrary posets and systematically explore their properties and investigate various examples. The introduction of principal mappings also provides a new perspective in the study of ideals of commutative rings. For example, for a given ring R , one can consider which principal mapping from $\text{Idl}(R)$ to itself is of the form F_I for some ideal I of R . The notion of principal mapping also provides a platform to compare and link the principal elements in a multiplicative lattice with the corresponding objects in other ordered structures such as the open mappings between topological spaces and the injective or surjective mappings between sets.

The layout of the paper is as follows. In Section 2, we define the principal mappings, meet principal and join principal mappings, between posets and prove some of their basic properties. In Section 3, we study weak principal mappings which generalize the corresponding notions of elements and their properties. In Sections 4 and 5, the principal mappings between some special types of lattices, such as the lattices of power sets and chains, are characterized. We also characterize the principal mappings f satisfying $f(x) \leq x$ (called contractive principal mappings) for some types of lattices such as the lattice of ideals of a principal ideal domain. Some open problems on principal mappings are posed.

In this paper, we assume the readers have basic knowledge on lattices and multiplicative lattices. For posets, lattices and more on adjunctions, see [12–14]. For multiplicative lattices, we refer to [1–3].

2. Principal Mappings between Posets

Throughout this paper, for a poset $(P, \leq)$ and $X \subseteq P$, we denote the set $\{p \in P : p \leq x \text{ for some } x \in X\}$ by $\downarrow X$. In the case $X = \{x\}$, we simply write $\downarrow x$ for $\downarrow X$. The sets $\uparrow X$ and $\uparrow x$ are defined dually.

Definition 1. A mapping $f : P \rightarrow Q$ between two posets is called a *principal mapping* if there is a mapping $g : Q \rightarrow P$ such that the following equations hold for all $x \in P$ and $y \in Q$:

$$f(\downarrow x \cap \downarrow g(y)) = \downarrow f(x) \cap \downarrow y, \quad (5)$$

$$g(\uparrow y \cap \uparrow f(x)) = \uparrow g(y) \cap \uparrow x. \quad (6)$$

The mapping g is then called the *residual* of f .

If f and g satisfy (5), then f is called a *meet principal mapping*.

If f and g satisfy (6), then f is called a *join principal mapping*.

Thus, f is a principal mapping if and only if it is both meet principal and join principal mappings.

The following are some immediate examples.

Examples 1. (1) Every isomorphism $f : P \rightarrow Q$ between posets is a principal mapping. In this case, the residual of f is the inverse mapping of f .

(2) For any set X , the power set $\mathcal{P}(X)$ of X is a poset. Given any $A \in \mathcal{P}(X)$, the mapping $F_A : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a principal mapping, where $F_A(B) = A \cap B$ for any $B \in \mathcal{P}(X)$ (see Corollary 30).

(3) Let $(\text{Idl}(R), \subseteq)$ be the lattice of ideals of a commutative ring R with inclusion as the partial order. For any principal ideal $I_0 \in \text{Idl}(R)$, the mapping $F_{I_0} : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is a principal mapping where $F_{I_0}(J) = I_0 J$ (the multiplication of I_0 with J) for any $J \in \text{Idl}(R)$. The residual of F_{I_0} sends J to the ideal quotient $[J : I_0]$. See Theorem 7 for the general result on lattices.

(4) Let X be a topological space and $\mathcal{O}(X)$ be the complete lattice of open sets of X . Given any $U \in \mathcal{O}(X)$, define the mapping $\phi_U : \mathcal{O}(X) \rightarrow \mathcal{O}(X)$ by $\phi_U(V) = U \cap V$, $V \in \mathcal{O}(X)$. Then, ϕ_U is a meet principal mapping that is not a join principal mapping unless U is closed.

(5) Let $f : X \rightarrow Y$ be a continuous open mapping between topological spaces X and Y (i.e., $f(U)$ is an open set of Y for any open set U of X). Let $f_* : \mathcal{O}(X) \rightarrow \mathcal{O}(Y)$ be defined by $f_*(U) = f(U)$, $U \in \mathcal{O}(X)$. Then, f_* is a meet principal mapping. In general, f_* is not a join principal mapping.

More examples of principal mappings, meet principal mappings, and join principal mappings will be considered in later sections.

Recall that an adjunction is a pair (f, g) of monotone mappings $f : P \rightarrow Q$ and $g : Q \rightarrow P$ between posets such that, for all $x \in P$ and $y \in Q$, $f(x) \leq y$ if and only if $x \leq g(y)$ (see Definition O-3.1 of [14]). In this case, g is called the upper adjoint of f and f is called the lower adjoint of g .

Equivalently, (f, g) is an adjunction if and only if f and g are monotone and $g(f(x)) \geq x$ and $f(g(y)) \leq y$ hold for all $x \in P$, $y \in Q$ (Theorem O-3.6 of [14]).

Lemma 2. Let $f : P \rightarrow Q$ be a principal mapping between posets with $g : Q \rightarrow P$ as the residual. Then, (f, g) is an adjunction.

Proof. Putting $x = g(y)$ in (5) and $y = f(x)$ in (6) in Definition 1, we obtain

$$\begin{aligned} f(\downarrow g(y)) &= \downarrow f(g(y)) \cap \downarrow y, \\ g(\uparrow f(x)) &= \uparrow g(f(x)) \cap \uparrow x. \end{aligned} \quad (7)$$

These imply that $f(g(y)) \leq y$ and $g(f(x)) \geq x$ for all $x \in P$ and $y \in Q$.

For any $x_1, x_2 \in P$ with $x_1 \leq x_2$, $\downarrow x_1 \subseteq \downarrow x_2 \subseteq \downarrow g(f(x_2))$. As f is a principal mapping, so

$$f(\downarrow x_1) = f(\downarrow x_1 \cap \downarrow g(f(x_2))) = \downarrow f(x_1) \cap \downarrow f(x_2), \quad (8)$$

implying $f(x_1) \leq f(x_2)$, and thus f is monotone. Similarly, we can show that g is monotone. Thus, (f, g) is an adjunction. $\square$

As the upper adjoint of a mapping is unique, the residual of a principal mapping is also unique.

The following theorem can be proved using Definition 1 and Lemma 2.

Theorem 3. Let $f : P \rightarrow Q$ be a mapping between posets. Then, f is a principal mapping if and only if

- (i) f has an upper adjoint $g : Q \rightarrow P$;
- (ii) for all $x \in P$ and $y \in Q$,

$$\downarrow f(x) = f(\downarrow x), \quad \uparrow g(y) = g(\uparrow y). \quad (9)$$

Remark 4. If $f : P \rightarrow Q$ has an upper adjoint $g : Q \rightarrow P$, then both f and g are monotone. Hence, for any $x \in P$, $y \in Q$ and $f(\downarrow x) \subseteq \downarrow f(x)$, $g(\uparrow y) \subseteq \uparrow g(y)$. Therefore, to prove that f is a principal mapping, it suffices to show that $\downarrow f(x) \subseteq f(\downarrow x)$ and $\uparrow g(y) \subseteq g(\uparrow y)$.

In particular, to prove that f is a meet (join) principal mapping, it suffices to show that $\downarrow f(x) \subseteq f(\downarrow x)$ ($\uparrow g(y) \subseteq g(\uparrow y)$).

The example below gives a lower adjoint mapping that is not principal.

Example 5. Let $[0, 1]$ be the unit interval of real numbers with the usual order of numbers. Define $f : [0, 1] \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases} \quad (10)$$

Then, f has an upper adjoint $g : [0, 1] \rightarrow [0, 1]$ given by $g(1) = 1$, $g(x) = 0$ ($x \neq 1$). But $\downarrow f(1) \neq f(\downarrow 1)$; hence, f is not a principal mapping.

In the case where P and Q are semilattices or lattices, we have a neater characterization of the types of mappings defined in Definition 1.

Theorem 6. Let $f : P \rightarrow Q$ be a mapping between posets with an upper adjoint.

(i) If P is a meet semilattice, then f is a meet principal mapping if and only if, for all $x \in P$ and $y \in Q$,

$$f(x \wedge g(y)) = f(x) \wedge y. \quad (11)$$

(ii) If Q is a join semilattice, then f is a join principal mapping if and only if, for all $x \in P$ and $y \in Q$,

$$g(y \vee f(x)) = g(y) \vee x. \quad (12)$$

Theorem 7. Let $f : L \rightarrow M$ be a mapping between two lattices. Then, f is a principal mapping if and only if there is a mapping $g : M \rightarrow L$ such that, for all $x \in L$, $y \in M$,

$$\begin{aligned} f(x \wedge g(y)) &= f(x) \wedge y, \\ g(y \vee f(x)) &= g(y) \vee x. \end{aligned} \quad (13)$$

The following proposition easily follows from the definition of principal mappings.

Proposition 8. If $f_1 : P_1 \rightarrow P_2$ and $f_2 : P_2 \rightarrow P_3$ are principal mappings between posets, then $f_2 \circ f_1 : P_1 \rightarrow P_3$ is a principal mapping.

For any two elements x and y in a multiplicative lattice L , the residual $[y : x]$ is defined as

$$[y : x] = \bigvee \{b \in L : bx \leq y\}. \quad (14)$$

Given an element a in a multiplicative lattice L , let $\phi_a : L \rightarrow L$ be the mapping such that $\phi_a(x) = ax$ and $\phi^a : L \rightarrow L$ be the mapping given by

$$\phi^a(y) = \bigvee \{x \in L : ax \leq y\} = [y : a]. \quad (15)$$

Then, (ϕ_a, ϕ^a) forms an adjunction.

By Dilworth [1], an element a of a multiplicative lattice L is called a principal element if and only if, for any $x, y \in L$,

$$\begin{aligned} (x \wedge [y : a])a &= xa \wedge y, \\ [(y \vee xa) : a] &= [y : a] \vee x. \end{aligned} \quad (16)$$

The element a is called a meet principal element if it satisfies $(x \wedge [y : a])a = xa \wedge y$ and a join principal element if it satisfies $[(y \vee xa) : a] = [y : a] \vee x$ for any $x, y \in L$.

The above two equations can be rewritten in terms of ϕ_a and ϕ^a as follows:

$$\begin{aligned} \phi_a(x \wedge \phi^a(y)) &= \phi_a(x) \wedge y, \\ \phi^a(y \vee \phi_a(x)) &= \phi^a(y) \vee x. \end{aligned} \quad (17)$$

Therefore, by Theorem 7, we have the following.

Proposition 9. (i) An element a of a multiplicative lattice L is a meet (join) principal element if and only if the mapping $\phi_a : L \rightarrow L$ is meet (join) principal; (ii) a is a principal element if and only if the mapping $\phi_a : L \rightarrow L$ is principal.

Applying Theorem 3 to the mappings ϕ_a and ϕ^a defined above, we deduce the following result where (i) appeared in [2].

Corollary 10. Let L be a multiplicative lattice and $a \in L$.

(i) a is meet principal if and only if, for any $x, y \in L$ with $y \leq xa$, there is $p \in L$ and $p \leq x$ such that $y = pa$.

(ii) a is join principal if and only if, for any $x, y \in L$ with $x \geq [y : a]$, there is $q \in L$ and $q \geq y$ such that $x = [q : a]$.

Note that for any two elements a and b in a multiplicative lattice L , $\phi_a \circ \phi_b = \phi_{ab}$. Thus, by Propositions 8 and 9, we deduce the following result which first appeared in [1].

Corollary 11. *The product of two principal elements of a multiplicative lattice is a principal element.*

Remark 12. Let $\text{PRMP}(L, L)$ be the set of all principal mappings from the multiplicative lattice L to itself, and let $\text{PRE}(L)$ be the set of all principal elements of L . Then, $(\text{PRMP}(L, L), \circ)$ ($\circ$ denotes the composition operation) is a semigroup with id_L as the identity, and $(\text{PRE}(L), \cdot)$ is also a semigroup with 1_L as the identity, and $\cdot$ as the multiplication on L . Now the mapping $a \mapsto \phi_a$ defines an embedding of $(\text{PRE}(L), \cdot)$ into $(\text{PRMP}(L, L), \circ)$.

In a multiplicative lattice L , $ab \leq b$ holds for any elements a, b . Hence, the mapping $\phi_a : L \rightarrow L$ satisfies the following condition:

$$\phi_a(x) \leq x \quad \forall x \in L. \quad (18)$$

This is equivalent to

$$x \leq \phi^a(x) = [x : a] \quad \forall x \in L. \quad (19)$$

For any poset P , a mapping $f : P \rightarrow P$ will be called *contractive* if $f(x) \leq x$ for all $x \in P$. If f has an upper adjoint $g : P \rightarrow P$, then f is contractive if and only if $x \leq g(x)$ for all $x \in P$.

A mapping $f : P \rightarrow P$ from a poset to itself is called a *contractive principal mapping* if f is both principal and contractive.

One of the problems on multiplicative lattices (in particular, $\text{Idl}(R)$ for a commutative ring R) we shall address is as follows. Given a multiplicative lattice L , under what condition every contractive principal mapping $f : L \rightarrow L$ is of the form ϕ_a for some principal element $a \in L$?

3. Weak Principal Mappings between Posets

In this section, we consider weak principal mappings and their some links to modular lattices.

Let L be a multiplicative lattice with bottom element 0_L and top element 1_L . By [2], an element a of L is called

(i) a *weak meet principal element* if, for all $x \in L$,

$$[x : a]a = a \wedge x; \quad (20)$$

(ii) a *weak join principal element* if, for all $x \in L$,

$$[xa : a] = [0_L : a] \vee x. \quad (21)$$

We now define the corresponding notions for mappings between posets, whose definitions appear more natural than the corresponding ones for elements.

Definition 13. Let $f : P \rightarrow Q$ be a mapping between posets with $g : Q \rightarrow P$ as the upper adjoint of f .

(i) f is called a *weak meet principal mapping* if

$$\downarrow f(P) = f(P). \quad (22)$$

(ii) f is called a *weak join principal mapping* if

$$\uparrow g(Q) = g(Q). \quad (23)$$

If f is both a weak meet principal mapping and a weak join principal mapping, then f is called a *weak principal mapping*.

Remark 14. (1) To prove that f is a weak meet (weak join) principal mapping, it suffices to show that $\downarrow f(P) \subseteq f(P)$ ($\uparrow g(Q) \subseteq g(Q)$, resp.).

(2) If $f : P \rightarrow Q$ has an upper adjoint $g : Q \rightarrow P$ and P has a top element 1_P (Q has a bottom element 0_Q), then f is weak meet (weak join) principal if and only if $\downarrow f(1_P) = f(\downarrow 1_P) = f(P)$ ($\uparrow g(0_Q) = g(\uparrow 0_Q) = g(Q)$).

Examples 2. (1) Let L be a frame with 0_L and 1_L as the bottom and top elements, respectively. For any $a \in L$, the mapping $\psi_a : L \rightarrow L$, defined by $\psi_a(x) = a \wedge x$ for all $x \in L$, is always a weak meet principal mapping. ψ_a is a weak join principal mapping if and only if there exists $a^\perp \in L$ such that $a \wedge a^\perp = 0_L$ and $a \vee a^\perp = 1_L$ (i.e., a has a complement).

(2) Let $h : X \rightarrow Y$ be a mapping from set X to set Y . Then, the mapping $F_h : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ between lattices of power sets of Y and X , respectively, defined by $F_h(V) = h^{-1}(V)$ for any $V \in \mathcal{P}(Y)$, is a weak join principal mapping. F_h is a weak meet principal mapping if and only if h is injective.

Proposition 15. Let $f : P \rightarrow Q$ be a mapping between posets with $g : Q \rightarrow P$ as the upper adjoint.

(i) If P has a top element 1_P , then f is a weak meet principal mapping if and only if, for all $y \in Q$,

$$f(g(y)) = f(1_P) \wedge y. \quad (24)$$

(ii) If Q has a bottom element 0_Q , then f is a weak join principal mapping if and only if, for all $x \in P$,

$$g(f(x)) = g(0_Q) \vee x. \quad (25)$$

Proof. (i) Suppose f is a weak meet principal mapping. Clearly, for any $y \in Q$, $f(g(y))$ is a lower bound of $\{f(1_P), y\}$. Now, if u is a lower bound of $\{f(1_P), y\}$, then $u \in \downarrow f(P) = f(P)$, implying that $u = f(z)$ for some $z \in P$ which implies that $z \leq g(y)$. Hence, $u = f(z) \leq f(g(y))$ and so $f(g(y)) = f(1_P) \wedge y$.

Conversely, suppose $f(g(y)) = f(1_P) \wedge y$ holds for all $y \in Q$. For any $v \in \downarrow f(P)$, there exists $a \in P$ such that $v \leq f(a) \leq f(1_P)$. So, $v = f(1_P) \wedge v = f(g(v)) \in f(P)$. Thus, $\downarrow f(P) \subseteq f(P)$, so f is a weak meet principal.

(ii) follows a dual proof to that of (i). $\square$

Using the relation between an element a in a multiplicative lattice L and the mapping $\phi_a : L \rightarrow L$ defined by $\phi_a(x) = ax$ for all $x \in L$, we obtain the following results.

Corollary 16. An element a in a multiplicative lattice L is weak meet (weak join) principal if and only if the mapping $\phi_a : L \rightarrow L$ is weak meet (weak join) principal.

Corollary 17. *An element a in a multiplicative lattice L is*

- (i) *weak meet principal if and only if, for any $y \in L$ with $y \leq a$, there is $p \in L$ such that $y = pa$;*
- (ii) *weak join principal if and only if, for any $x \in L$ with $x \geq [0 : a]$, there is $q \in L$ such that $x = [q : a]$.*

Corollary 17 (i) is Lemma 1 (a) in [2].

Remark 18. If $f : P \rightarrow Q$ is a meet principal mapping between posets, then $f(\downarrow x) = \downarrow f(x)$ holds for all $x \in P$, and so $\downarrow f(P) = f(P)$. Thus, a meet principal mapping is weak meet principal. Similarly, every join principal mapping between posets is weak join principal.

A weak meet (weak join) principal mapping need not be meet (join) principal. A counterexample can be easily constructed by considering the mappings from the nonmodular lattice, the pentagon N_5 , to itself.

Recall that a lattice L is said to be *modular* if, for any $x, a, b \in L$ with $b \leq x$, $x \wedge (a \vee b) = (x \wedge a) \vee b$ (see, e.g., [12, 13]).

Proposition 19. *Let $f : L \rightarrow M$ be a mapping between bounded lattices L and M with an upper adjoint such that f is a weak principal mapping.*

- (i) *If L is modular, then f is a meet principal mapping.*
- (ii) *If M is modular, then f is a join principal mapping.*

Proof. Let $g : M \rightarrow L$ be the upper adjoint of f .

(i) For any $a \in M$ with $a \leq f(b)$ for some $b \in L$, we have $u = g(a) \leq g(f(b)) = b \vee g(0_M)$ since f is weak join principal. Also, $a \leq f(b) \leq f(1_L)$ which implies that $a = a \wedge f(1_L) = f(g(a)) = f(u)$ as f is weak meet principal. Since $u \leq b \vee g(0_M)$, L is modular and $u = g(a) \geq g(0_M)$,

$$f(u) = f(u \wedge (b \vee g(0_M))) = f((u \wedge b) \vee g(0_M)). \quad (26)$$

As a lower adjoint, f preserves joins. Furthermore, $f(g(0_M)) \leq 0_M$ which implies that $f(g(0_M)) = 0_M$. Now,

$$\begin{aligned} a &= f(u) = f(u \wedge b) \vee f(g(0_M)) = f(u \wedge b) \vee 0_M \\ &= f(u \wedge b) \in f(\downarrow b). \end{aligned} \quad (27)$$

Hence, f is meet principal.

(ii) follows a proof dual to that of (i). $\square$

The following theorem now follows from Proposition 19.

Theorem 20. *Let $f : L \rightarrow M$ be a mapping between bounded modular lattices with an upper adjoint. Then, f is a principal mapping if and only if f is a weak principal mapping.*

Corollary 21. *If L is a bounded modular lattice, then every weak principal mapping $f : L \rightarrow L$ is principal.*

Using Theorem 20, Proposition 9, and Corollary 16, we obtain the following result in [2, 7].

Corollary 22. *An element in a modular multiplicative lattice L is principal if and only if it is weak principal.*

A natural question arising here is whether the converse of Corollary 21 is true. If L is a bounded lattice and every weak principal mapping $f : L \rightarrow L$ is principal, must L be modular?

Example 23. Consider the lattice $L = \{0, a, b, c, e, f, g, 1\}$ where $0 < a < b < d$, $0 < c < d < e < 1$, and $d < f < 1$. One can check that every weak principal mapping $f : L \rightarrow L$ is principal. However, L is not modular.

Another counterexample, suggested by Dr. Peter Jipsen, is the 6-element nonmodular lattice $H = \{0, a, b, c, d, 1\}$ where $0 < a < d < 1$, $0 < b < d$, and $0 < c < 1$.

The following problem is still open.

Problem 24. Let L be a bounded lattice such that for any bounded lattice M , every weak principal mapping $f : L \rightarrow M$ is meet principal. Must L be modular?

The composition of two weak meet (weak join) principal mappings need not be weak meet (weak join) principal. A counterexample can be easily constructed by considering the composition of two weak meet principal mappings from the nonmodular lattice, the pentagon N_5 , to itself.

Proposition 25. *Let $f_1 : P_1 \rightarrow P_2$ and $f_2 : P_2 \rightarrow P_3$ be mappings between posets.*

- (1) *If f_1 is weak meet principal and f_2 is meet principal, then $f_2 \circ f_1 : P_1 \rightarrow P_3$ is weak meet principal.*
- (2) *If f_1 is join principal and f_2 is weak join principal, then $f_2 \circ f_1 : P_1 \rightarrow P_3$ is weak join principal.*

Using Propositions 9 and 25 and Corollary 16, we obtain the following result which is Proposition 1 (a) in [2].

Proposition 26. *If d is a weak meet principal element and e is a meet principal element in a multiplicative lattice L , then de is a weak meet principal element in L .*

For bounded semilattices, we have another characterization of weak join principal and weak meet principal mappings.

Proposition 27. *Let $f : L \rightarrow M$ be a mapping between two posets with $g : M \rightarrow L$ as the upper adjoint of f .*

- (1) *If L and M are join semilattices with 0_M as the bottom element of M , then f is a weak join principal mapping if and only if, for any $b, c \in L$, $f(b) \leq f(c)$ implies that $b \leq c \vee g(0_M)$.*
- (2) *If L and M are meet semilattices with 1_L as the top element of L , then f is a weak meet principal mapping if and only if, for any $a, d \in M$, $g(a) \geq g(d)$ implies that $a \geq d \wedge f(1_L)$.*

Proof. As before, we just gave the proof of (1).

Suppose f is weak join principal and $f(b) \leq f(c)$ where $b, c \in L$. Then, $b \leq g(f(b)) \leq g(f(c)) = c \vee g(0_M)$.

Conversely, suppose $f : L \rightarrow M$ satisfies the given condition. Clearly, $b \vee g(0_M) \leq g(f(b))$ for any $b \in L$. For any $b \in L$, $f(g(f(b))) \leq f(b)$, so, by the assumption on f , we have $g(f(b)) \leq b \vee g(0_M)$. $\square$

By [14], a lower adjoint is injective if and only if its upper adjoint is surjective. Now, if $f : L \rightarrow M$ is an injective weak join principal mapping between two join semilattices with bottom elements 0_L and 0_M , respectively, then we must have $g(0_M) = 0_L$, where $g : M \rightarrow L$ is the upper adjoint of f . Then, $f(b) \leq f(c)$ implies that $b \leq c \vee g(0_M) = c \vee 0_L = c$, so f is an order embedding.

Corollary 28. *If $f : L \rightarrow M$ is an injective weak join principal mapping between join semilattices with bottom elements, then f is an order embedding.*

4. Principal Mappings between Lattices of Powersets and Chains

In this section, we investigate the principal mappings and their weaker versions between the lattices of powersets and chains.

For any set X , the power set $\mathcal{P}(X)$ of X is a complete lattice with respect to the inclusion order.

Theorem 29. *Let X and Y be two nonempty sets. For any mapping $F : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, the following statements are equivalent.*

- (1) F is a principal mapping.
- (2) There is a subset X_1 of X and an injective mapping $t : X_1 \rightarrow Y$ such that $F(A) = t(A \cap X_1)$ for all $A \in \mathcal{P}(X)$.

Proof. (1) implies (2). Let F be principal and let G be its upper adjoint. As every lower adjoint preserves arbitrary joins, it sends bottom element to the bottom element. Thus, $F(\emptyset) = \emptyset$.

(i) For any $x \in X$, if $F(\{x\})$ is nonempty, then it is a singleton set. As a matter of fact, assume $u \in F(\{x\})$. Then, $\{u\} \in \downarrow F(\{x\})$, where $\downarrow F(\{x\}) = \{A \subseteq X : A \subseteq F(\{x\})\}$. Since F is meet principal, there is a subset $B \subseteq \{x\}$ such that $F(B) = \{u\}$. Clearly, $B \neq \emptyset$, so $B = \{x\}$, which implies that $F(\{x\}) = \{u\}$ is a singleton set.

(ii) Let $X_1 = \{x \in X : F(\{x\}) \neq \emptyset\}$. Then, $F(X - X_1) = \bigcup \{F(\{z\}) : z \in X - X_1\} = \emptyset$ because F preserves arbitrary joins. It then follows that $G(\emptyset) = X - X_1$. Now, by Proposition 15 (ii), for any $u \in X_1$, $G(F(\{u\})) = G(\emptyset) \cup \{u\} = (X - X_1) \cup \{u\}$. Thus, $u \neq v$ and $u, v \in X_1$ imply that $F(\{u\}) \neq F(\{v\})$.

(iii) Define $t : X_1 \rightarrow Y$ by $t(x) = y$, where $\{y\} = F(\{x\})$. By (ii), t is an injective mapping. For any $A \in \mathcal{P}(X)$, note that F preserves joins (i.e., unions in this case), so $F(A) = F(A - X_1) \cup F(A \cap X_1) = \emptyset \cup t(A \cap X_1)$.

(2) implies (1). Assume the condition in (2) is satisfied. Then, F has an upper adjoint G given by $G(B) = t^{-1}(B) \cup (X - X_1)$ for each $B \in \mathcal{P}(Y)$. For any $A \subseteq X$ and $B \subseteq Y$, if $y \in F(A) \cap B$, then $y = t(x)$ for some $x \in A \cap X_1$. Also, $y = t(x) \in B$ which implies that $x \in t^{-1}(B) \subseteq G(B)$. It follows that $y = t(x) \in F(G(B) \cap A)$. Hence, $F(G(B) \cap A) = F(A) \cap B$.

If $x \in G(B \cup F(A))$, then $F(\{x\}) \subseteq B \cup F(A)$. But $F(\{x\}) = t(\{x\} \cap X_1)$ is at most a nonempty singleton set. If $F(\{x\}) \subseteq B$, then $x \in G(B)$. If $F(\{x\}) \not\subseteq B$, then $F(\{x\}) \neq \emptyset$ and $F(\{x\}) = t(\{x\} \cap X_1) \subseteq F(A) = t(A \cap X_1)$, implying that $x \in A$ since t is injective. It follows that $G(B \cup F(A)) = G(B) \cup A$. Therefore, F is principal. $\square$

For any subset X_1 of a set X , let $t : X_1 \rightarrow X$ be the embedding mapping; that is, $t(x) = x$, for any $x \in X_1$. Then, for any $A \in \mathcal{P}(X)$, $t(A \cap X_1) = A \cap X_1$. By Theorem 29 (2), we have the following.

Corollary 30. *For any fixed subset X_1 of a set X , the mapping $F_{X_1} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a principal mapping, where $F_{X_1}(B) = X_1 \cap B$ for any $B \subseteq X$.*

Corollary 31. *Let $f : X \rightarrow Y$ be a mapping between nonempty sets X and Y . Then, the mapping $F : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, defined by $F(A) = f(A) = \{f(x) : x \in A\}$ for any $A \in \mathcal{P}(X)$, is a principal mapping if and only if f is injective.*

For contractive principal mappings on lattices of powersets, we have the following.

Proposition 32. *A mapping $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is a contractive principal mapping if and only if there exists a subset X_1 of X such that $F(A) = A \cap X_1$ for all $A \in \mathcal{P}(X)$.*

Proof. Assume that F is a principal mapping that is also contractive. By Theorem 29, there is a subset X_1 of X and an injective mapping $t : X_1 \rightarrow X$ such that $F(A) = t(A \cap X_1)$ for all $A \in \mathcal{P}(X)$. Since F is contractive, for any $x \in X_1$, $F(\{x\}) = t(\{x\} \cap X_1) \subseteq \{x\}$ which implies that $t(x) = x$ for all $x \in X_1$. Hence, $F(A) = A \cap X_1$ for all $A \in \mathcal{P}(X)$. $\square$

Problem 33. Determine the complete lattices L such that for any contractive principal mapping $F : L \rightarrow L$, there is an element $a \in L$ such that $F(x) = a \wedge x$ for all $x \in L$.

Recall that a chain is a poset in which every two elements are comparable.

Theorem 34. *Let C be a chain and let $f : C \rightarrow C$ be a mapping. Then, the following statements are equivalent.*

- (1) f is a principal mapping.
- (2) f is a meet principal mapping and, for any $x, y \in C$, $x < y$ and $f(x) = f(y)$ imply that $f(z) = f(y)$ for all $z \leq y$.

If C does not have a bottom element, then (2) is equivalent to the following.

- (3) f is a meet principal mapping that is strictly monotone.

Proof. (1) implies (2). Let f be principal and let g be its residual. Suppose $x, y \in C$ such that $x < y$ and $f(x) = f(y)$. Clearly, for any z with $x < z < y$, $f(z) = f(y)$.

If there exists a $z_0 \in C$ such that $z_0 \leq x < y$, and $f(x) \neq f(z_0)$, then, as f is monotone, we have $f(z_0) < f(x)$, which implies that $g(f(z_0)) \not\geq x$; hence, $g(f(z_0)) < x$. Thus, $x \in \uparrow g(f(z_0)) = g(\uparrow f(z_0))$. So there exists $k \geq f(z_0)$ such that $x = g(k)$. Then, $f(y) = f(x) = f(g(k)) \leq k$, so $y \leq g(k) = x$, which contradicts $x < y$. Hence, for all $z \leq x < y$, we have $f(z) = f(x) = f(y)$.

(2) implies (1). We only need to show that f is join principal. Let g be the upper adjoint of f . For any $a \in C$, let $y > g(a)$. Then, $f(y) \geq f(g(a))$. If $f(y) = f(g(a))$, then $y \leq g(f(y)) = g(f(g(a))) = g(a)$ (note that, for any adjunction (f, g) , $g \circ f \circ g = g$ and $f \circ g \circ f = f$),

which contradicts $y > g(a)$. Hence, $f(y) > f(g(a))$. Also, if $f(y) \leq a$, then $y \leq g(a)$ which contradicts $y > g(a)$ again. It follows that $a \in \downarrow f(y) = f(\downarrow y)$ since f is a meet principal mapping. There exists $u \in \downarrow y$ such that $a = f(u)$. Hence, $f(g(a)) = f(g(f(u))) = f(u) = a$.

We now show that $g(f(y)) = y$. For this, we only need to show $y \geq g(f(y))$ as $g(f(y)) \geq y$ always holds. Suppose $y \not\geq g(f(y))$; then $y < g(f(y))$. As $f(y) = f(g(f(y)))$ and $g(a) < y < g(f(y))$, so by the assumption in (2), $a = f(g(a)) = f(g(f(y))) = f(y)$, a contradiction. It follows that $y = g(f(y))$ and $f(y) \in \uparrow a$. Thus, $\uparrow g(a) \subseteq g(\uparrow a)$ and so f is join principal. Hence, f is principal.

Now assume C does not have a bottom element and that (3) implying (2) is trivial.

(2) implies (3). Suppose f is not strictly monotone. Then, there exist $a, b \in C$, $a \neq b$, such that $f(a) = f(b)$. Without loss of generality, assume that $a < b$. By (2), for all $x \leq b$, $f(x) = f(b)$. Since C does not have a bottom element, there exists $y_0 \in C$ such that $y_0 < f(b)$. So, $y_0 \in \downarrow f(b) = f(\downarrow b)$ since f is a meet principal mapping. Thus, there exists $x_0 \leq b$ such that $y_0 = f(x_0)$. But then $f(x_0) = f(b)$, and we have $y_0 = f(b)$, which contradicts $y_0 < f(b)$. Hence, f must be strictly monotone. $\square$

Let $C = \{c_1, \dots, c_n\}$ be a finite chain where $c_1 < c_2 < \dots < c_n$. For each k , $1 \leq k \leq n$, let $f_k : C \rightarrow C$ be the mapping given by $f_k(c_n) = c_k$, $f_k(c_{n-1}) = c_{k-1}, \dots, f_k(c_{n-k+1}) = f_k(c_{n-k}) = \dots = f_k(c_1) = c_1$. Then, using Theorem 34, one can verify that f_k is a principal mapping. Conversely, if $f : C \rightarrow C$ is a principal mapping, then $f = f_k$, where $f(c_n) = c_k$.

Corollary 35. For any chain C with n elements, there are exactly n principal mappings $f : C \rightarrow C$.

One of the main tasks we are interested in is to determine all the principal mappings $f : \text{Idl}(R) \rightarrow \text{Idl}(R)$ where $\text{Idl}(R)$ is the lattice of ideals of a commutative ring R .

Corollary 36. For any prime p and $n \in \mathbb{Z}^+$, there are exactly $n+1$ principal mappings $f : \text{Idl}(\mathbb{Z}_{p^n}) \rightarrow \text{Idl}(\mathbb{Z}_{p^n})$ ($\mathbb{Z}_{p^n}$ denotes the ring of integers modulo p^n). Thus, for every principal mapping $f : \text{Idl}(\mathbb{Z}_{p^n}) \rightarrow \text{Idl}(\mathbb{Z}_{p^n})$, there exists $I \in \text{Idl}(\mathbb{Z}_{p^n})$ such that $f = F_I$, where $F_I(J) = IJ$ for any $J \in \text{Idl}(\mathbb{Z}_{p^n})$.

Proof. $(\mathbb{Z}_{p^n}, +, \cdot)$ is a principal ideal domain and there are exactly $n+1$ ideals forming a chain. So the first part follows from Corollary 35. For the second part, just note that each F_I ($I \in \text{Idl}(\mathbb{Z}_{p^n})$) is principal and that $I \neq K$ for I and $K \in \text{Idl}(\mathbb{Z}_{p^n})$ implies that $F_I \neq F_K$. $\square$

An element a of a complete lattice L is called a *pseudo-principal element* if there is a contractive principal mapping $f : L \rightarrow L$ such that $f(1_L) = a$.

If a is a principal element of a multiplicative lattice L , then $a = \phi_a(1_L)$, so a is pseudoprincipal. The proof of Corollary 35 also shows that every element in a finite chain is pseudoprincipal.

5. Principal Mappings between Lattices of Ideals of a Principal Ideal Domain

We now investigate the principal mappings between the complete lattices of ideals of some special types of commutative rings, in particular, principal ideal domains. The main purpose is to identify those principal mappings that are defined by a principal ideal.

For any ideal I of a commutative ring R , the mapping $F_I : \text{Idl}(R) \rightarrow \text{Idl}(R)$ given by $F_I(J) = IJ$ for all $J \in \text{Idl}(R)$ will be called the *mapping defined by I* . Note that F_I is always a lower adjoint of the mapping $G_I : \text{Idl}(R) \rightarrow \text{Idl}(R)$ defined by $G_I(K) = [K : I]$ for any $K \in \text{Idl}(R)$. By Proposition 9 (ii), F_I is a principal mapping if and only if I is a principal element of the multiplicative lattice $\text{Idl}(R)$.

Definition 37. A multiplicative lattice L has the *divisibility order* if, for all $a, b \in L$ and $a \leq b$ if and only if b divides a (i.e., $a = bc$ for some $c \in L$).

Recall that a *cancellation ideal ring* [15] is a commutative ring R in which every nonzero ideal I of R is a cancellation ideal [16]; that is, whenever $IJ = IK$ for any $J, K \in \text{Idl}(R)$, and then $J = K$. It is easy to see that the above condition that is equivalent to $IJ \subseteq IK$ implies that $J \subseteq K$.

Note that every principal ideal domain R is a cancellation ideal ring and the corresponding multiplicative lattice $\text{Idl}(R)$ has the divisibility order.

Theorem 38. Let R be a cancellation ideal ring such that $\text{Idl}(R)$ has the divisibility order. Then, $\phi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is an injective principal mapping if and only if there is an order isomorphism $\psi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ and a nonzero principal element I of $\text{Idl}(R)$ such that $\phi = F_I \circ \psi$.

Proof. Only the necessity needs verification because every isomorphism is a principal mapping and the composition of principal mappings is principal.

Let $\phi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ be an injective principal mapping and let ϕ^* be its upper adjoint. Since ϕ is injective, $\phi(R) \neq \{0_R\}$. For any $J \in \text{Idl}(R)$, since $\phi(J) \subseteq \phi(R)$ and $\text{Idl}(R)$ has the divisibility order, there exists $H \in \text{Idl}(R)$ such that $\phi(J) = H\phi(R)$. As $\phi(R) \neq \{0_R\}$, R is a cancellation ideal ring, such an H is unique for each J , which will be denoted by $\psi(J)$. Now, $\phi(J) = \psi(J)\phi(R) = (F_{\phi(R)} \circ \psi)(J)$.

The mapping ψ is injective because ϕ is injective. For any $I \in \text{Idl}(R)$, since $I\phi(R) \subseteq \phi(R)$ and ϕ is a principal mapping, by Theorem 3, there exists $J \in \text{Idl}(R)$ such that $\phi(J) = I\phi(R)$. But $\phi(J) = \psi(J)\phi(R)$. So $\psi(J)\phi(R) = I\phi(R)$, which implies that $I = \psi(J)$ as R is a cancellation ideal ring and so ψ is surjective. Suppose $\psi(J) \subseteq \psi(K)$. Then, $\phi(J) = \phi(R)\psi(J) \subseteq \phi(R)\psi(K) = \phi(K)$. It follows that

$$J \subseteq \phi^*(\phi(J)) \subseteq \phi^*(\phi(K)) = K \vee \phi^*(\{0_R\}) = K; \quad (28)$$

the first equality follows from that ϕ is join principal and Proposition 15 (ii) and the second equality follows from the assumption that ϕ is injective (so its upper adjoint ϕ^* is surjective implying $\phi^*\{0_R\} = \{0_R\}$). Thus, $J \subseteq K$.

We now show that ψ is monotone. For any $J, K \in \text{Idl}(R)$ such that $J \subseteq K$, since ϕ is monotone, $\phi(J) \subseteq \phi(K)$.

So $\psi(J)\phi(R) \subseteq \psi(K)\phi(R)$ and since $\phi(R) \neq \{0_R\}$, it follows that $\psi(J) \subseteq \psi(K)$ as R is a cancellation ideal ring. Hence, ψ is indeed an order isomorphism.

Finally, $F_{\phi(R)} = \phi \circ \psi^{-1}$, as a composition of principal mappings is principal; thus $\phi(R)$ is a principal element of $\text{Idl}(R)$. $\square$

In the following, for any element x of a commutative ring, $\langle x \rangle$ denotes the principal ideal generated by x .

The following fact will be used several times in later arguments.

Lemma 39. *Let R be a principal ideal domain. For any nonzero $a \in R$, any prime element p in R and $t \in \mathbb{Z}^+$, $\langle p^t \rangle \subseteq \langle a \rangle$ if and only if $a = up^s$ for some unit u and some $0 \leq s \leq t$.*

Recall that an element a of a lattice L is called a (meet) irreducible element if, for any $u, v \in L$, $a = u \wedge v$ implies that $a = u$ or $a = v$.

Lemma 40. *Let R be a principal ideal domain. Then, $\langle p^t \rangle$, where p is a prime element in R and $t \in \mathbb{Z}^+$, are exactly the irreducible elements of the lattice $\text{Idl}(R)$.*

Proof. Let p be a prime element in R and $t \in \mathbb{Z}^+$. If $\langle p^t \rangle = \langle a \rangle \cap \langle b \rangle$, then, by Lemma 39, $a = up^s$ and $b = vp^k$ for some $s \leq t$ and $k \leq t$ and units $u, v \in R$. Thus, $\langle p^t \rangle = \langle p^s \rangle \cap \langle p^k \rangle$. But clearly $\langle p^s \rangle \cap \langle p^k \rangle = \langle p^l \rangle$, where $l = \max\{s, k\}$. Therefore, $\langle p^t \rangle = \langle p^s \rangle = \langle a \rangle$ or $\langle p^t \rangle = \langle p^k \rangle = \langle b \rangle$, showing that $\langle p^t \rangle$ is an irreducible element of the lattice $\text{Idl}(R)$.

Conversely, let $\langle a \rangle$ be an irreducible element of the lattice $\text{Idl}(R)$ and let $a = up_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ be a prime factorization of a (note that, in any UFD, in particular, a PID, prime elements coincide with (ring) irreducible elements), where u is a unit and p_i are distinct prime elements of R . Then, $\langle a \rangle = \langle p_1^{t_1} \rangle \cap \langle p_2^{t_2} \rangle \cap \dots \cap \langle p_k^{t_k} \rangle$. Thus, $\langle a \rangle = \langle p_i^{t_i} \rangle$ for some $1 \leq i \leq k$. $\square$

Lemma 41. *Let R be a principal ideal domain. If $\psi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is an order isomorphism, then, for any prime element p of R , there is a prime element q of R such that for any $k \in \mathbb{Z}^+$, $\psi(\langle p^k \rangle) = \langle q^k \rangle$.*

Proof. Since $\langle p^k \rangle$ is an irreducible element of the lattice $\text{Idl}(R)$, $\psi(\langle p^k \rangle)$ is also an irreducible element, and so, by Lemma 40, $\psi(\langle p^k \rangle) = \langle q^m \rangle$ for some prime element q and some $m \in \mathbb{Z}^+$.

By Lemma 39, in $\text{Idl}(R)$, there are $k - 1$ distinct elements lying between $\langle p^k \rangle$ and $\langle 1 \rangle$; that is, $\langle p^k \rangle \subseteq \langle p^{k-1} \rangle \subseteq \dots \subseteq \langle p^2 \rangle \subseteq \langle p \rangle \subseteq \langle 1 \rangle$. Since ψ is an isomorphism, there are also $k - 1$ distinct elements lying between $\psi(\langle p^k \rangle) = \langle q^m \rangle$ and $\psi(\langle 1 \rangle) = \langle 1 \rangle$, which deduces that $m = k$. It is clear that the prime element q is uniquely determined by p . $\square$

We are now able to prove the main result of this section.

Theorem 42. *Let R be a principal ideal domain. Then, $\phi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is an injective contractive principal mapping if and only if $\phi = F_I$ for some nonzero $I \in \text{Idl}(R)$.*

Proof. By Theorem 38, there exist an order isomorphism $\psi : \text{Idl}(R) \rightarrow \text{Idl}(R)$ and a nonzero ideal I of R such that $\phi = F_I \circ \psi$ where F_I is the mapping defined by I .

(1) We first show that $\psi(\langle p \rangle) = \langle q \rangle$ for any prime element $p \in R$. By Lemma 41, $\psi(\langle p \rangle) = \langle q \rangle$ for some prime element q .

Since $\phi(\langle p \rangle) \subseteq \langle p \rangle$, so $I\psi(\langle p \rangle) \subseteq \langle p \rangle$. As $\langle p \rangle$ is a prime ideal, either $\psi(\langle p \rangle) \subseteq \langle p \rangle$ or $I \subseteq \langle p \rangle$.

(a) If $\psi(\langle p \rangle) \subseteq \langle p \rangle$ holds, then $\langle q \rangle \subseteq \langle p \rangle$, which implies that $p = q$ and so $\psi(\langle p \rangle) = \langle p \rangle$.

(b) Now assume that $\psi(\langle p \rangle) \not\subseteq \langle p \rangle$. Then, $I \subseteq \langle p \rangle$ and so $I = \langle cp^k \rangle$ for some $k \in \mathbb{Z}^+$, and $c \in R$ with $c \neq 0_R$ such that p does not divide c . Now, $\psi(\langle p^{k+1} \rangle) = \langle q^{k+1} \rangle$ by Lemma 41. Since ϕ is contractive, $\phi(\langle p^{k+1} \rangle) \subseteq \langle p^{k+1} \rangle$. Then, $\langle p^{k+1} \rangle$ divides $\phi(\langle p^{k+1} \rangle) = I\psi(\langle p^{k+1} \rangle) = \langle cp^k \rangle \langle q^{k+1} \rangle = \langle cp^k q^{k+1} \rangle$, which is not possible because $p \neq q$ and p does not divide c .

From both (a) and (b), we can deduce that $\psi(\langle p \rangle) = \langle p \rangle$.

(2) By Lemma 41 and part (1), for any $t \in \mathbb{Z}^+$, $\psi(\langle p^t \rangle) = \langle p^t \rangle$.

(3) Now, for any $J \in \text{Idl}(R)$, with $J \neq \{0_R\}$ and $J \neq R$, $J = \langle p_1^{j_1} p_2^{j_2} \dots p_t^{j_t} \rangle$ for some distinct prime elements p_i and $j_i \in \mathbb{Z}^+$, $i \in \{1, 2, 3, \dots, t\}$, for some $t \in \mathbb{Z}^+$. Consider

$$\begin{aligned} \psi(J) &= \psi \langle p_1^{j_1} p_2^{j_2} \dots p_t^{j_t} \rangle \\ &= \psi(\langle p_1^{j_1} \rangle \cap \langle p_2^{j_2} \rangle \cap \dots \cap \langle p_t^{j_t} \rangle) \\ &= \psi(\langle p_1^{j_1} \rangle) \cap \psi(\langle p_2^{j_2} \rangle) \cap \dots \cap \psi(\langle p_t^{j_t} \rangle) \\ &= \langle p_1^{j_1} \rangle \cap \langle p_2^{j_2} \rangle \cap \dots \cap \langle p_t^{j_t} \rangle = J. \end{aligned} \quad (29)$$

Also $\psi(R) = R$ and $\psi(\{0_R\}) = \{0_R\}$ because every isomorphism sends the bottom (top) element to the bottom (top) element. It follows that $\psi(J) = J$ for all $J \in \text{Idl}(R)$ and hence $\phi = F_I$. $\square$

Problem 43. Does Theorem 42 hold for every commutative cancellation ideal ring R where $\text{Idl}(R)$ has the divisibility order?

6. Conclusion and Future Works

In this paper, we introduce and study the principal mappings between posets which generalize the notion of principal elements in multiplicative lattices. A number of results on principal elements in multiplicative lattices have been generalized for such mappings. The principal mappings between some special posets have been characterized.

Besides the three concrete problems posed above, the following are some more general problems for further studies.

(1) Given a commutative ring R , determine all the principal mappings from the lattice $\text{Idl}(R)$ to itself.

(2) Determine, for which ring R , every injective contractive principal mapping from $\text{Idl}(R)$ to itself is defined by an ideal of R .

Let $\mathbb{Z}[X]$ be the polynomial ring over the ring of integers. Is there an injective contractive principal mapping $f : \text{Idl}(\mathbb{Z}[X]) \rightarrow \text{Idl}(\mathbb{Z}[X])$ that is not defined by any ideal of $\mathbb{Z}[X]$?

(3) Is it true that for any semiring R , if every weak principal mapping $f : \text{Idl}(R) \rightarrow \text{Idl}(R)$ is principal, then the lattice $\text{Idl}(R)$ of all ideals of R is modular?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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