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Research Article

On the Projective Description of Weighted (LF)-Spaces of Continuous Functions

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We solve the problem of the topological or algebraic description of countable inductive limits of weighted Fréchet spaces of continuous functions on a cone. This problem is investigated for two families of weights defined by positively homogeneous functions. Weights of this form play the important role in Fourier analysis.

1. Introduction

The problem of the projective description of the topology of inductive limits of weighted spaces of holomorphic and continuous functions has attracted the attention of mathematicians after Ehrenpreis [1] proved the fundamental principle. This problem has applications in partial differential equations and convolution equations, (ultra-) distribution theory, in the theory of quasianalytic functionals, and representations of functions by exponential series. The systematic investigation of it was started by Bierstedt and Meise and Bierstedt et al. [2, 3]. The aim of the projective description is to find the conditions under which an inductive limit coincides algebraically with its projective hull or it is its topological subspace. In the case of an inductive limit of weighted Banach spaces of continuous functions the problem of their projective descriptions is very well studied. By now effective conditions are obtained that the topology of a countable inductive weighted limit $WC(X)$ of continuous functions coincides with the one of its projective hull $C\overline{W}(X)$. The algebraic identity $WC(X) = C\overline{W}(X)$ always holds in this case. The case of (LF)-spaces is more complicated. The development in projective descriptions of countable inductive weighted limits of continuous functions had been surveyed in [4, 5]. Abstract conditions for the algebraic and topological description of weighted (LF)-spaces of continuous functions were obtained by Bierstedt and Bonet [6]. Their realizations in concrete situations are also interesting. In this connection we mention the article of Bonet and Meise [7]. In [7] weighted (LF)-spaces of continuous functions defined by weights which arise in the theory of ultradistributions and qusianalytic functionals of Roumieu type were investigated (see Remark after Theorem 9). Bonet et al. [8] have studied the problem of the projective description of (LF)-spaces of continuous functions for weights which are connected with a convex locally closed set in $\mathbb{C}^N$.

In our article this problem is solved for two families of weights defined by positively homogeneous functions on cones in normed linear spaces. Weights of this form play the important role in Fourier analysis.

2. The Problem of the Projective Description and Notations

We recall the necessary notations and definitions and state the problem of the projective description [3, 6].

Let X be a locally compact Hausdorff space, and let $W = (w_{n,k})_{n,k\in\mathbb{N}}$ be a double sequence of strictly positive continuous functions on X such that

$$w_{n+1,k}(x) \le w_{n,k}(x) \le w_{n,k+1}(x), \quad x \in X, \tag{1}$$

for each $n, k \in \mathbb{N}$. For an upper semicontinuous function $w : X \to (0, +\infty)$ we introduce the Banach space of continuous functions:

$$C(w, X) := \left\{ f \in C(X) \mid \|f\|_w := \sup_{x\in X} |f(x)| w(x) < +\infty \right\}. \tag{2}$$

The weighted inductive limit of Fréchet spaces $WC(X)$ of continuous functions on X is defined by

$$\begin{gathered} W_nC(X) := \operatorname{proj}_k C(w_{n,k}, X), \quad n \in \mathbb{N}; \\ WC(X) := \operatorname{ind}_n W_nC(X). \end{gathered} \tag{3}$$

The space $WC(X)$ is a Hausdorff (LF)-space.

The system $\overline{W}$ of weights assosiated with W consists of all upper semicontinuous functions $\overline{w} : X \to [0, +\infty)$ such that for each n there are $\alpha_n > 0$ and $k = k(n)$ with $\overline{w} \le \alpha_n w_{n,k}$ on X. The projective hull of the inductive limit $WC(X)$ is defined by

$$C\overline{W}(X) := \left\{ f \in C(X) \mid \|f\|_{\overline{w}} := \sup_{x\in X} |f(x)| \overline{w}(x) < +\infty \ \forall \overline{w} \in \overline{W} \right\}. \tag{4}$$

The topology of $C\overline{W}(X)$ is defined by the system of seminorms $\|\cdot\|_{\overline{w}}$, $\overline{w} \in \overline{W}$. The space $WC(X)$ is contained in its projective hull $C\overline{W}(X)$ with continuous inclusion.

The problem of the projective description for the spaces $WC(X)$ is to determine conditions under which

(A) the spaces $WC(X)$ and $C\overline{W}(X)$ coincide algebraically, or

(T) the space $WC(X)$ is a topological subspace of its projective hull $C\overline{W}(X)$.

3. Case of Weights Defined by Positively Homogeneous Functions

Let Y be a normed (complex or real) linear space with the norm $\|\cdot\|$. We assume that X is a cone in Y; that is, $\lambda x \in X$ for all $\lambda \ge 0$ and $x \in X$. The set X is endowed with the induced topology. It is assumed that X is locally compact. Then each set $\{x \in X \mid \|x\| \le R\}$, $R > 0$, is compact in X. Let $h_n : X \to (0, +\infty)$, $n \in \mathbb{N}$, be continuous and positively homogeneous of degree $\rho > 0$ functions such that $h_n \le h_{n+1}$ for each $n \in \mathbb{N}$. Further $\omega : [0, +\infty) \to [0, +\infty)$ is a continuous function with

$$\lim_{t\to+\infty} \omega(t) = +\infty, \qquad \lim_{t\to+\infty} \frac{\omega(t)}{t^\rho} = 0. \tag{5}$$

Put $\omega(x) := \omega(\|x\|)$, $x \in X$. We define weight functions by

$$w_{n,k}(x) := \exp(-h_n(x) + k\omega(x)), \quad x \in X, \ n, k \in \mathbb{N}. \tag{6}$$

Theorem 1. *The space $WC(X)$ is a topological subspace of $C\overline{W}(X)$.*

Proof. Since $\lim_{t\to+\infty}\omega(t) = +\infty$, then for each n the Fréchet space $W_nC(X)$ coincides algebraically and topologically with the Fréchet space

$$(W_nC)_0(X) := \left\{ f \in C(X) \mid \lim_{\|x\|\to+\infty} |f(x)| w_{n,k}(x) = 0 \text{ for each } k \in \mathbb{N} \right\}. \tag{7}$$

By [3, Theorem 1.3.] $WC(X)$ is a topological subspace of $C\overline{W}(X)$.

We investigate now the algebraic identity $WC(X) = C\overline{W}(X)$. □

Theorem 2. *The following conditions are equivalent.*

(i) *The space $WC(X)$ coincides algebraically with $C\overline{W}(X)$.*

(ii) *The condition (RD) holds:*

$$\begin{gathered} \forall n \, \exists m \quad \forall \mu \, \exists C : \\ h_\mu(x) - h_m(x) \le C(h_m(x) - h_n(x)), \quad x \in X. \end{gathered} \tag{8}$$

Proof. By [6, Proposition 4] algebraic identity $WC(X) = C\overline{W}(X)$ is equivalent to the following condition (wQ) which was introduced by Vogt [9]:

$$\begin{gathered} \forall n \, \exists m, \ k \quad \forall \mu, \ l \, \exists L, C : \\ w_{m,l} \le C \max\{w_{n,k}; w_{\mu,L}\} \text{ on } X, \end{gathered} \tag{9}$$

that is,

$$\begin{aligned} &\exp(-h_m(x) + l\omega(x)) \\ &\le C \max\{\exp(-h_n(x) + k\omega(x)); \exp(-h_\mu(x) + L\omega(x))\}, \\ &\qquad x \in X. \end{aligned} \tag{10}$$

Since $\lim_{t\to+\infty}\omega(t) = +\infty$, the latter is equivalent to

$$\begin{aligned} &\forall n \, \exists m, \ k \quad \forall \mu, \ l \, \exists L, \ A : \forall \|x\| \ge A \\ &-h_m(x) + l\omega(x) \\ &\quad \le \max\{-h_n(x) + k\omega(x); -h_\mu(x) + L\omega(x)\}. \end{aligned} \tag{11}$$

Thus the algebraic identity $WC(X) = C\overline{W}(X)$ holds if and only if

$$\begin{gathered}\forall n\,\exists m,\ k \quad \forall \mu,\ l\,\exists L,\ A : \forall \|x\| \geq A \\ (l-k)\,\omega(x) \leq h_m(x) - h_n(x)\end{gathered} \tag{12}$$

or

$$h_\mu(x) - h_m(x) \leq (L-l)\,\omega(x). \tag{13}$$

(ii) $\Rightarrow$ (i). Suppose that (RD) holds. Fix n. By (RD) $\exists m > n\ \forall \mu\ \exists C$:

$$h_\mu(x) - h_m(x) \leq C\left(h_m(x) - h_n(x)\right), \quad x \in X. \tag{14}$$

We choose $k := 1$ and fix l. If

$$(l-1)\,\omega(x) > h_m(x) - h_n(x), \tag{15}$$

then

$$\begin{aligned} h_\mu(x) - h_m(x) &\leq C\left(h_m(x) - h_n(x)\right) \\ &\leq C(l-1)\,\omega(x), \quad x \in X. \end{aligned} \tag{16}$$

Hence the inequality (13) holds with L such that $L \geq l + C(l-1)$. Thus the condition (wQ) is valid and consequently the algebraic identity $WC(X) = C\overline{W}(X)$ is fulfilled.

(i) $\Rightarrow$ (ii). Suppose that $\forall n\,\exists m,\ k\ \forall \mu,\ l\,\exists L,\ A : \forall \|x\| \geq A$ the inequality (12) or (13) holds. Assume that the condition (RD) does not hold. Taking into account the positive homogeneity of functions h_n we obtain that

$$\begin{gathered}\exists n_0\ \forall m > n_0\ \exists \mu\ \exists a_{m,j} \in X : \left\|a_{m,j}\right\| = 1, \\ h_\mu\left(a_{m,j}\right) - h_m\left(a_{m,j}\right) > j\left(h_m\left(a_{m,j}\right) - h_{n_0}\left(a_{m,j}\right)\right), \quad j \in \mathbb{N}.\end{gathered} \tag{17}$$

We select $m_0 > n_0$ and k_0 for n_0 as in (12) and (13). Set $l := k_0 + 1$ and choose $\mu_0 = \mu_0(m_0)$ for m_0 as in (17). We take L and A as in (12) and (13). Put $H_s := \sup_{\|a\|=1} h_s(a)$, $s \in \mathbb{N}$ (from the continuity of h_s it follows that $H_s < +\infty$). For every $j \in \mathbb{N}$ by (17),

$$\begin{aligned} 0 &\leq h_{m_0}\left(a_{m_0,j}\right) - h_{n_0}\left(a_{m_0,j}\right) < \frac{1}{j}\left(h_{\mu_0}\left(a_{m_0,j}\right) - h_{m_0}\left(a_{m_0,j}\right)\right) \\ &\leq \frac{H_{\mu_0} + H_{m_0}}{j}. \end{aligned} \tag{18}$$

Hence,

$$\lim_{j\to\infty}\left(h_{m_0}\left(a_{m_0,j}\right) - h_{n_0}\left(a_{m_0,j}\right)\right) = 0. \tag{19}$$

We set $D := \sup_{t\geq A}\omega(t)/t^\rho$. It is clear that $D \in (0,+\infty)$. In consequence of (17) and (19) there is j_0 such that

$$\begin{aligned} &h_{m_0}\left(a_{m_0,j_0}\right) - h_{n_0}\left(a_{m_0,j_0}\right) \\ &\quad < \min\left\{D; \frac{1}{L-l}\left(h_{\mu_0}\left(a_{m_0,j_0}\right) - h_{m_0}\left(a_{m_0,j_0}\right)\right)\right\}. \end{aligned} \tag{20}$$

Since the function $\omega(t)/t^\rho$ is continuous on $[A,+\infty)$ and $\lim_{t\to+\infty}\omega(t)/t^\rho = 0$, there is $t_0 \geq A$ such that

$$\begin{aligned} &\frac{1}{l-k_0}\left(h_{m_0}\left(a_{m_0,j_0}\right) - h_{n_0}\left(a_{m_0,j_0}\right)\right) \\ &\quad = h_{m_0}\left(a_{m_0,j_0}\right) - h_{n_0}\left(a_{m_0,j_0}\right) \\ &\quad < \frac{\omega(t_0)}{t_0^\rho} \\ &\quad < \frac{1}{L-l}\left(h_{\mu_0}\left(a_{m_0,j_0}\right) - h_{m_0}\left(a_{m_0,j_0}\right)\right). \end{aligned} \tag{21}$$

We put $x_0 := t_0 a_{m_0,j_0}$. Then $x_0 \in X$ and $\|x_0\| \geq A$ and also for $x := x_0$ both inequalities (12) and (13) do not hold. It is a contradiction. □

Example 3. (a) Let K_n be convex compact subsets of $X = Y := \mathbb{R}^N$ such that K_n is contained in the interior of K_{n+1} for each $n \in \mathbb{N}$. We denote by $h_n(x) := \sup_{y\in K_n}\langle x, y\rangle$, $x \in \mathbb{R}^N$, $\langle x, y\rangle := \sum_{j=1}^N x_j y_j$, the supporting function of K_n. The functions h_n are positively homogeneous of order 1 and convex on $\mathbb{R}^N$. We put $\|x\| := \langle x, x\rangle^{1/2}$.

The sequence $(h_n)_{n\in\mathbb{N}}$ satisfies the condition (RD) of Theorem 2. Indeed, we can take in the condition (ii) of Theorem 2 $m := n+1$ for $n \in \mathbb{N}$. The necessary inequality follows from $\inf_{\|a\|=1}(h_{n+1}(a) - h_n(a)) > 0$, from the positive homogeneity and continuity of functions h_s.

(b) Let $h : X \to [0,+\infty)$ be a continuous and positively homogeneity of order $\rho > 0$ function and $h_n := \alpha_n h$, $n \in \mathbb{N}$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive numbers. The sequence $(h_n)_{n\in\mathbb{N}}$ satisfies the condition (RD).

We will show that the condition (RD) is equivalent to the condition of the regular decrease of the sequence $(\exp(-h_n))_{n\in\mathbb{N}}$ which plays the important role in projective descriptions in the (LB) case [3].

Definition 4 (see [3, Definition 2.1]). The sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions $\varphi_n : X \to (0,+\infty)$ such that $\varphi_{n+1} \leq \varphi_n$ on $X\forall n \in \mathbb{N}$ is called regularly decreasing if $\forall n\,\exists m \geq n\forall \mu > m\forall Q \subseteq X$

$$\inf_{y\in Q}\frac{\varphi_m(y)}{\varphi_n(y)} > 0 \Longrightarrow \inf_{y\in Q}\frac{\varphi_\mu(y)}{\varphi_m(y)} > 0. \tag{22}$$

Theorem 5. *Let $h_n : X \to \mathbb{R}$ be continuous functions which are positively homogeneous of degree $\rho > 0$, $h_n \leq h_{n+1}$ on $\mathbb{C}^N$, $n \in \mathbb{N}$. The following assertions are equivalent:*

(i) *The condition (RD) holds.*

(ii) *The sequence $(\exp(-h_n))_{n\in\mathbb{N}}$ is regularly decreasing.*

Proof. (ii) $\Rightarrow$ (i). Suppose that the sequence $(\exp(-h_n))_{n\in\mathbb{N}}$ decreases regularly. Then $\forall n\,\exists m \geq n\ \forall \mu > m\ \forall Q \subseteq X$,

$$\begin{aligned} &\sup_{y\in Q}\left(h_m(y) - h_n(y)\right) \\ &\quad < +\infty \Longrightarrow \sup_{y\in Q}\left(h_\mu(y) - h_m(y)\right) < +\infty. \end{aligned} \tag{23}$$

Assume that the condition (RD) does not hold. Then

$$\exists n_0\ \forall m > n_0\ \exists \mu > m\ \forall j\ \exists y_{m,j} \in X: \\ h_\mu\left(y_{m,j}\right) - h_m\left(y_{m,j}\right) > j\left(h_m\left(y_{m,j}\right) - h_{n_0}\left(y_{m,j}\right)\right). \tag{24}$$

For n_0 we choose m as in the condition (23) and for this m select μ as in the condition (24).

We will prove that from the equality $h_m(y) = h_{n_0}(y)$ it follows that $h_\mu(y) = h_m(y)$. Indeed, if $h_m(y) = h_{n_0}(y)$ for some $y \in X$, then also for each $t > 0$ the equality $h_m(ty) = h_{n_0}(ty)$ holds. Therefore

$$\begin{aligned} +\infty &> \sup_{t>0}\left(h_\mu(ty) - h_m(ty)\right) \\ &= \sup_{t>0}\left(t^\rho\left(h_\mu(y) - h_m(y)\right)\right). \end{aligned} \tag{25}$$

From this it follows that $h_\mu(y) = h_m(y)$. Hence $c_j := h_m(y_{m,j}) - h_{n_0}(y_{m,j}) > 0$ for each $j \in \mathbb{N}$. Since for all $j \in \mathbb{N}$

$$1 = \frac{h_m\left(y_{m,j}\right) - h_{n_0}\left(y_{m,j}\right)}{c_j} = h_m\left(\frac{y_{m,j}}{c_j^{1/\rho}}\right) - h_{n_0}\left(\frac{y_{m,j}}{c_j^{1/\rho}}\right), \tag{26}$$

by (24)

$$\sup_{j\in\mathbb{N}}\left(h_\mu\left(\frac{y_{m,j}}{c_j^{1/\rho}}\right) - h_m\left(\frac{y_{m,j}}{c_j^{1/\rho}}\right)\right) = +\infty. \tag{27}$$

It is a contradiction with (23).

The implication (i) $\Rightarrow$ (ii) is obvious. □

4. Case of Weights Defined by a Composition with Positively Homogeneous Functions

Let Γ be a cone in a normed (real or complex) linear space Z, that is, $\lambda x \in \Gamma$ for all $\lambda \geq 0$ and $x \in \Gamma$. Set for $R > 0$

$$\begin{aligned} S_R(X) &:= \{x \in X \mid \|x\| = R\}; \\ \overline{B}_R(\Gamma) &:= \{x \in \Gamma \mid \|x\|_Z \leq R\}, \end{aligned} \tag{28}$$

where $\|\cdot\|_Z$ is the norm in Z. We endow Γ with the induced topology.

Let $h_n : \Gamma \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous and positively homogeneous of degree $\rho > 0$ functions such that $h_n \leq h_{n+1}$ on Γ for each $n \in \mathbb{N}$. Suppose that $\omega : [0, +\infty) \to [0, +\infty)$ is a continuous function such that

$$\lim_{t\to+\infty}\omega(t) = +\infty, \qquad \lim_{t\to+\infty}\frac{\omega(t)}{t^\rho} = 0. \tag{29}$$

We put $\omega(x) := \omega(\|x\|)$, $x \in X$.

Further we will use functions $v : X \to \Gamma$ with a covering property.

Definition 6. A function $v : X \to \Gamma$ satisfies the condition (SJ) if there is $r > 0$ such that $\overline{B}_{rR}(\Gamma) \subseteq v(S_R(X))$ for all $R > 0$.

Example 7. The following functions satisfy the condition (SJ) with $r = 1$:

(a) $v(z) := \operatorname{Im} z$ if $X = Y = \mathbb{C}^N$, $\Gamma = Z = \mathbb{R}^N$, $N \in \mathbb{N}$,

(b) $v(z) := (z_1, \ldots, z_k)$, $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, if $X = Y = \mathbb{C}^N$, $\Gamma = Z = \mathbb{C}^k$, $k, N \in \mathbb{N}$, $k < N$.

We fix $\alpha \geq 0$ and define weight functions by

$$w_{n,k}(x) := \exp\left(-h_n(v(x)) - \left(\alpha + \frac{1}{k}\right)\omega(x)\right), \\ x \in X, \quad n, k \in \mathbb{N}, \tag{30}$$

where $v : X \to \Gamma$ is a continuous function which satisfies the condition (SJ).

Theorem 8. *The space $WC(X)$ is a topological subspace of $C\overline{W}(X)$.*

The assertion follows from [3, Theorem 1.3] (see the proof of Theorem 1).

We will investigate further when the algebraic identity $WC(X) = C\overline{W}(X)$ holds.

Theorem 9. *The following assertions are equivalent:*

(i) *$WC(X) = C\overline{W}(X)$ holds algebraically.*

(ii) *There is $m \in \mathbb{N}$ such that $h_\mu = h_m$ for each $\mu > m$.*

Proof. By [3, Proposition 4] the algebraic identity $WC(X) = C\overline{W}(X)$ is equivalent to the condition (wQ):

$$\forall n\ \exists m,\ k \quad \forall \mu, l\ \exists L, C: \\ w_{m,l} \leq C \max\left\{w_{n,k}; w_{\mu,L}\right\} \text{ on } X; \tag{31}$$

that is,

$$\begin{aligned} &\exp\left(-h_m(v(x)) - \left(\alpha + \frac{1}{l}\right)\omega(x)\right) \\ &\quad \leq C \max\left\{\exp\left(-h_n(v(x)) - \left(\alpha + \frac{1}{k}\right)\omega(x)\right);\right. \\ &\qquad \left.\exp\left(-h_\mu(v(x)) - \left(\alpha + \frac{1}{L}\right)\omega(x)\right)\right\}, \\ &\qquad x \in X. \end{aligned} \tag{32}$$

Since $\lim_{t\to+\infty}\omega(t) = +\infty$, this condition is equivalent to

$$\forall n\ \exists m,\ k \quad \forall \mu,\ l\ \exists L,\ A: \forall \|x\| \geq A \tag{33}$$

$$\begin{aligned} &-h_m(v(x)) - \left(\alpha + \frac{1}{l}\right)\omega(x) \\ &\quad \leq \max\left\{-h_n(v(x)) - \left(\alpha + \frac{1}{k}\right)\omega(x);\right. \\ &\qquad \left.-h_\mu(v(x)) - \left(\alpha + \frac{1}{L}\right)\omega(x)\right\}. \end{aligned} \tag{34}$$

Thus the algebraic equality $WC(X) = C\overline{W}(X)$ holds if and only if the following condition (wQ1) is fulfilled:

$$\begin{gathered}\forall n\,\exists m,\ k \quad \forall \mu,\ l\,\exists L,\ A : \forall \|x\| \geq A \\ \left(\frac{1}{k} - \frac{1}{l}\right)\omega(x) \leq h_m(v(x)) - h_n(v(x))\end{gathered} \tag{35}$$

or

$$h_\mu(v(x)) - h_m(v(x)) \leq \left(\frac{1}{l} - \frac{1}{L}\right)\omega(x). \tag{36}$$

(i) $\Rightarrow$ (ii). Assume that the condition (i) holds but (ii) is not satisfied. We select m and k for $n = 1$ as in (wQ1). Since (ii) does not hold, there are $\mu > m$ and $a \in \Gamma$ such that $\|a\|_Z = 1$ and $h_\mu(a) > h_m(a)$. Choose $l > k$ so large that

$$\begin{aligned}\alpha &:= \frac{1}{(1/k) - (1/l)}\left(h_m(a) - h_1(a)\right) \\ &< l\left(h_\mu(a) - h_m(a)\right) =: \beta.\end{aligned} \tag{37}$$

We fix some $\gamma \in (\alpha, \beta)$. Define L and A for l as in (wQ1). Since $\lim_{t\to+\infty}\omega(t) = +\infty$ and $\lim_{t\to+\infty}(\omega(t)/t^\rho) = 0$, there is $R \geq A$ such that

$$\omega(R) > \beta, \qquad \frac{\omega(R)}{R^\rho} < r^\rho\gamma, \tag{38}$$

where $r > 0$ is a constant as in condition (SJ).

Let $t := (\omega(R)/\gamma)^{1/\rho}$. Then $t < rR$. By the condition (SJ) there is $x_0 \in X$ such that $\|x_0\| = R$ and $v(x_0) = ta$. From

$$\begin{aligned}\frac{1}{(1/k) - (1/l)}\left(h_m(a) - h_1(a)\right) &< \frac{\omega(R)}{t^\rho} < \beta \\ &< \frac{1}{(1/l) - (1/L)}\left(h_\mu(a) - h_m(a)\right)\end{aligned} \tag{39}$$

it follows that

$$\begin{aligned}\frac{1}{(1/k) - (1/l)}\left(h_m(v(x_0)) - h_1(v(x_0))\right) &< \omega(x_0) \\ &< \frac{1}{(1/l) - (1/L)}\left(h_\mu(v(x_0)) - h_m(v(x_0))\right).\end{aligned} \tag{40}$$

There is a contradiction with (wQ1).

The implication (ii) $\Rightarrow$ (i) is obvious.

We give a corollary of Theorem 9. Let $X = Y = \mathbb{C}^N$, $\Gamma = Z = \mathbb{R}^N$; $\|z\| := (\sum_{j=1}^N |z_j|^2)^{1/2}$, $z \in \mathbb{C}^N$; $\|x\|_Z := (\sum_{j=1}^N |x_j|^2)^{1/2}$, $x \in \mathbb{R}^N$; $h_n : \mathbb{R}^N \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous and positively homogeneous of degree $\rho > 0$ functions such that $h_n \leq h_{n+1}$ for each $n \in \mathbb{N}$; let function ω be such as above. We put $v(z) := \operatorname{Im} z$, $z \in \mathbb{C}^N$. The function $v : \mathbb{C}^N \to \mathbb{R}^N$ satisfies the condition (SJ).

Put

$$\begin{gathered}w_{n,k}(z) := \exp\left(-h_n(\operatorname{Im} z) - \left(\alpha + \frac{1}{k}\right)\omega(z)\right), \\ z \in \mathbb{C}^N, \quad n, k \in \mathbb{N},\end{gathered} \tag{41}$$

where $\alpha \in [0, +\infty)$ is fixed. □

Corollary 10. *The following conditions are equivalent:*

(i) $WC(\mathbb{C}^N) = C\overline{W}(\mathbb{C}^N)$ *holds algebraically.*

(ii) *There is* $m \in \mathbb{N}$ *such that* $h_\mu = h_m$ *for each* $\mu > m$.

Remark 11. Let Ω be a convex open subset of $\mathbb{R}^N$; let $(K_n)_{n\in\mathbb{N}}$ be a fundamental sequence of convex compact subsets of Ω such that $K_n \subset \operatorname{int} K_{n+1}$ for each n. For each n we denote by $h_n(x) := \sup_{y\in K_n}\langle x, y\rangle$, $x \in \mathbb{R}^N$, the supporting functions of K_n. The functions h_n are convex and positively homogeneous of order 1. Let ω be a quasianalytic weight function as in Braun et al. [10] (see [7, p. 125] too). In [7, Proposition 7] it was proved that the algebraic identity $WC(\mathbb{C}^N) = C\overline{W}(\mathbb{C}^N)$ does not hold for the functions $w_{n,k}(z) := \exp(-h_n(\operatorname{Im} z) - (1/k)\omega(z))$, $z \in \mathbb{C}^N$. This assertion follows from Corollary 10, since $h_n(x) < h_{n+1}(x)$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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