Research Article

C^* -Algebras Associated with Hilbert C^* -Quad Modules of Finite Type

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Received 30 September 2013; Accepted 2 December 2013; Published 30 January 2014

Academic Editor: A. Zayed

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A Hilbert C^* -quad module of finite type has a multistructure of Hilbert C^* -bimodules with two finite bases. We will construct a C^* -algebra from a Hilbert C^* -quad module of finite type and prove its universality subject to certain relations among generators. Some examples of the C^* -algebras from Hilbert C^* -quad modules of finite type will be presented.

1. Introduction

Robertson and Steger [1] have initiated a certain study of higher-dimensional analogue of Cuntz-Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, Kumjian and Pask [2] have generalized their construction to introduce the notion of higher-rank graphs and its C^* -algebras. Since then, there have been many studies on these C^* -algebras by many authors (see, e.g., [1–6], etc.).

In [7], the author has introduced a notion of C^* -symbolic dynamical system, which is a generalization of a finite labeled graph, a λ -graph system, and an automorphism of a unital C^* -algebra. It is denoted by $(\mathcal{A}, \rho, \Sigma)$ and consists of a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital C^* -algebra $\mathcal{A}$ such that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, $\alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ where $Z_{\mathcal{A}}$ denotes the center of $\mathcal{A}$. It provides a subshift Λ_ρ over Σ and a Hilbert C^* -bimodule $\mathcal{H}_\rho^\rho$ over $\mathcal{A}$ which gives rise to a C^* -algebra $\mathcal{O}_\rho$ as a Cuntz-Pimsner algebra ([7] cf. [8–10]). In [11, 12], the author has extended the notion of C^* -symbolic dynamical system to C^* -textile dynamical system which is a higher-dimensional analogue of C^* -symbolic dynamical system. A C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with common unital C^* -algebra $\mathcal{A}$ and commutation relations κ between the endomorphisms $\rho_\alpha, \alpha \in \Sigma^\rho$

and $\eta_a, a \in \Sigma^\eta$. A C^* -textile dynamical system provides a two-dimensional subshift and a multistructure of Hilbert C^* -bimodules that has multi right actions and multi left actions and multi inner products. Such a multi structure of Hilbert C^* -bimodule is called a Hilbert C^* -quad module. In [12], the author has introduced a C^* -algebra associated with the Hilbert C^* -quad module of C^* -textile dynamical system. It is generated by the quotient images of creation operators on two-dimensional analogue of Fock Hilbert module by module maps of compact operators. As a result, the C^* -algebra has been proved to have a universal property subject to certain operator relations of generators encoded by structure of C^* -textile dynamical system [12].

In this paper, we will generalize the construction of the C^* -algebras of Hilbert C^* -quad modules of C^* -textile dynamical systems. Let $\mathcal{A}, \mathcal{B}_1$, and $\mathcal{B}_2$ be unital C^* -algebras. Assume that $\mathcal{A}$ has unital embeddings into both $\mathcal{B}_1$ and $\mathcal{B}_2$. A Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is a Hilbert C^* -bimodule over $\mathcal{A}$ with $\mathcal{A}$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ which has a multi structure of Hilbert C^* -bimodules over $\mathcal{B}_i$ with right actions φ_i of $\mathcal{B}_i$ and left actions ϕ_i of $\mathcal{B}_i$ and $\mathcal{B}_i$ -valued inner products $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ for $i = 1, 2$ satisfying certain compatibility conditions. A Hilbert C^* -quad module $\mathcal{H}$ is said to be of finite type if there exists a finite basis $\{u_1, \dots, u_M\}$ of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_1$ and

a finite basis $\{v_1, \dots, v_N\}$ of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_2$ such that

$$\begin{aligned} \sum_{i=1}^M \langle u_i | \phi_2(\langle \xi | \eta \rangle_{\mathcal{B}_2}) u_i \rangle_{\mathcal{B}_1} &= \sum_{k=1}^N \langle v_k | \phi_1(\langle \xi | \eta \rangle_{\mathcal{B}_1}) v_k \rangle_{\mathcal{B}_2} \\ &= \langle \xi | \eta \rangle_{\mathcal{A}} \end{aligned} \quad (1)$$

for $\xi, \eta \in \mathcal{H}$ (see [13] for the original definition of finite basis of Hilbert module). For a Hilbert C^* -quad module, we will construct a Fock space $F(\mathcal{H})$ from $\mathcal{H}$, which is a 2-dimensional analogue to the ordinary Fock space of Hilbert C^* -bimodules (cf. [10, 14]). We will then define two kinds of creation operators s_ξ, t_ξ for $\xi \in \mathcal{H}$ on $F(\mathcal{H})$. The C^* -algebra on $F(\mathcal{H})$ generated by them is denoted by $\mathcal{T}_{F(\mathcal{H})}$ and called the Toeplitz quad module algebra. We then define the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ associated with the Hilbert C^* -quad module $\mathcal{H}$ by the quotient C^* -algebra of $\mathcal{T}_{F(\mathcal{H})}$ by the ideal generated by the finite-rank operators. We will then prove that the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ for a C^* -quad module $\mathcal{H}$ of finite type has a universal property in the following way.

Theorem 1 (Theorem 34). *Let $\mathcal{H}$ be a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of finite type with a finite basis $\{u_1, \dots, u_M\}$ of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_1$ and a finite basis $\{v_1, \dots, v_N\}$ of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_2$. Then, the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ generated by the quotients $[s_\xi], [t_\xi]$ of the creation operators s_ξ, t_ξ for $\xi \in \mathcal{H}$ on the Fock spaces $F(\mathcal{H})$ is canonically isomorphic to the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by operators $S_1, \dots, S_M, T_1, \dots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations*

$$\begin{aligned} \sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* &= 1, \quad S_j^* T_l = 0, \\ S_i^* S_j &= \langle u_i | u_j \rangle_{\mathcal{B}_1}, \quad T_k^* T_l = \langle v_k | v_l \rangle_{\mathcal{B}_2}, \\ z S_j &= \sum_{i=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}, \\ z T_l &= \sum_{k=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2}, \\ w S_j &= \sum_{i=1}^M S_i \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}, \\ w T_l &= \sum_{k=1}^N T_k \langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2} \end{aligned} \quad (2)$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \dots, M, k, l = 1, \dots, N$.

The eight relations of the operators above are called the relations $(\mathcal{H})$. As a corollary, we have the following.

Corollary 2 (Corollary 35). *For a C^* -quad module $\mathcal{H}$ of finite type, the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by operators*

$S_1, \dots, S_M, T_1, \dots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations $(\mathcal{H})$ does not depend on the choice of the finite bases $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$.

The paper is organized in the following way. In Section 2, we will define Hilbert C^* -quad module and present some basic properties. In Section 3, we will define a C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ from Hilbert C^* -quad module $\mathcal{H}$ of general type by using creation operators on Fock Hilbert C^* -quad module. In Section 4, we will study algebraic structure of the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ for a Hilbert C^* -quad module $\mathcal{H}$ of finite type. In Section 5, we will prove, as a main result of the paper, that the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ has the universal property stated as in Theorem 1. A strategy to prove Theorem 1 is to show that the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ is regarded as a Cuntz-Pimsner algebra for a Hilbert C^* -bimodule over the C^* -algebra generated by $\phi_1(\mathcal{B}_1)$ and $\phi_2(\mathcal{B}_2)$. We will then prove the gauge invariant universality of the C^* -algebra (Theorem 33). In Section 6, we will present K-theory formulae for the C^* -algebra $\mathcal{O}_{\mathcal{H}}$. In Section 7, we will give examples. In Section 8, we will formulate higher-dimensional analogue of our situations and state a generalized proposition of Theorem 1 without proof.

Throughout the paper, we will denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

2. Hilbert C^* -Quad Modules

Throughout the paper, we fix three unital C^* -algebras $\mathcal{A}, \mathcal{B}_1$, and $\mathcal{B}_2$ such that $\mathcal{A} \subset \mathcal{B}_1, \mathcal{A} \subset \mathcal{B}_2$ with common units. We assume that there exists a right action ψ_i of $\mathcal{A}$ on $\mathcal{B}_i$ so that

$$b_i \psi_i(a) \in \mathcal{B}_i \quad \text{for } b_i \in \mathcal{B}_i, a \in \mathcal{A}, i = 1, 2, \quad (3)$$

which satisfies

$$\|b_i \psi_i(a)\| \leq \|b_i\| \|a\|, \quad b_i \psi_i(aa') = b_i \psi_i(a) \psi_i(a') \quad (4)$$

for $b_i \in \mathcal{B}_i, a, a' \in \mathcal{A}, i = 1, 2$. Hence, $\mathcal{B}_i$ is a right $\mathcal{A}$ -module through ψ_i for $i = 1, 2$. Suppose that $\mathcal{H}$ is a Hilbert C^* -bimodule over $\mathcal{A}$, which has a right action of $\mathcal{A}$, an $\mathcal{A}$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$, and a $*$ -homomorphism $\phi_{\mathcal{A}}$ from $\mathcal{A}$ to the algebra of all bounded adjointable right $\mathcal{A}$ -module maps $\mathcal{L}_{\mathcal{A}}(\mathcal{H})$ satisfying the following.

- (i) $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ is linear in the second variable.
- (ii) $\langle \xi | \eta a \rangle_{\mathcal{A}} = \langle \xi | \eta \rangle_{\mathcal{A}} a$ for $\xi, \eta \in \mathcal{H}, a \in \mathcal{A}$.
- (iii) $\langle \xi | \eta \rangle_{\mathcal{A}}^* = \langle \eta | \xi \rangle_{\mathcal{A}}$ for $\xi, \eta \in \mathcal{H}$.
- (iv) $\langle \xi | \xi \rangle_{\mathcal{A}} \geq 0$, and $\langle \xi | \xi \rangle_{\mathcal{A}} = 0$ if and only if $\xi = 0$.

A Hilbert C^* -bimodule $\mathcal{H}$ over $\mathcal{A}$ is called a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ if $\mathcal{H}$ has a further structure of a Hilbert C^* -bimodule over $\mathcal{B}_i$ for each $i = 1, 2$ with right

action φ_i of $\mathcal{B}_i$ and left action ϕ_i of $\mathcal{B}_i$ and $\mathcal{B}_i$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ such that for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$,

$$\phi_1(z), \phi_2(w) \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}),$$

$$[\phi_1(z)\xi]\varphi_2(w) = \phi_1(z)[\xi\varphi_2(w)],$$

$$[\phi_2(w)\xi]\varphi_1(z) = \phi_2(w)[\xi\varphi_1(z)],$$

$$\xi\varphi_1(z\psi_1(a)) = [\xi\varphi_1(z)]a, \quad \xi\varphi_2(w\psi_2(a)) = [\xi\varphi_2(w)]a, \quad (5)$$

for $\xi \in \mathcal{H}, z \in \mathcal{B}_1, w \in \mathcal{B}_2, a \in \mathcal{A}$, and

$$\phi_{\mathcal{A}}(a) = \phi_1(a) = \phi_2(a) \quad \text{for } a \in \mathcal{A}, \quad (6)$$

where $\mathcal{A}$ is regarded as a subalgebra of $\mathcal{B}_i$. The left action ϕ_i of $\mathcal{B}_i$ on $\mathcal{H}$ means that $\phi_i(b_i)$ for $b_i \in \mathcal{B}_i$ is a bounded adjointable operator with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ for each $i = 1, 2$. The operator $\phi_i(b_i)$ for $b_i \in \mathcal{B}_i$ is also adjointable with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$. We assume that the adjoint of $\phi_i(b_i)$ with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ coincides with the adjoint of $\phi_i(b_i)$ with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$. Both of them coincide with $\phi_i(b_i^*)$. We assume that the left actions ϕ_i of $\mathcal{B}_i$ on $\mathcal{H}$ for $i = 1, 2$ are faithful. We require the following compatibility conditions between the right $\mathcal{A}$ -module structure of $\mathcal{H}$ and the right $\mathcal{A}$ -module structure of $\mathcal{B}_i$ through ψ_i :

$$\begin{aligned} \langle \xi | \eta a \rangle_{\mathcal{B}_i} &= \langle \xi | \eta \rangle_{\mathcal{B}_i} \psi_i(a) \quad \text{for } \xi, \eta \in \mathcal{H}, a \in \mathcal{A}, \\ i &= 1, 2. \end{aligned} \quad (7)$$

We further assume that $\mathcal{H}$ is a full Hilbert C^* -bimodule with respect to the three inner products $\langle \cdot | \cdot \rangle_{\mathcal{A}}, \langle \cdot | \cdot \rangle_{\mathcal{B}_1}$, and $\langle \cdot | \cdot \rangle_{\mathcal{B}_2}$ for each. This means that the C^* -algebras generated by elements $\{\langle \xi | \eta \rangle_{\mathcal{A}} \mid \xi, \eta \in \mathcal{H}\}$, $\{\langle \xi | \eta \rangle_{\mathcal{B}_1} \mid \xi, \eta \in \mathcal{H}\}$ and $\{\langle \xi | \eta \rangle_{\mathcal{B}_2} \mid \xi, \eta \in \mathcal{H}\}$ coincide with $\mathcal{A}, \mathcal{B}_1$, and $\mathcal{B}_2$, respectively.

For a vector $\xi \in \mathcal{H}$, denote by $\|\xi\|_{\mathcal{A}}, \|\xi\|_{\mathcal{B}_1}$, and $\|\xi\|_{\mathcal{B}_2}$ the norms $\|\langle \xi | \xi \rangle_{\mathcal{A}}\|^{1/2}$, $\|\langle \xi | \xi \rangle_{\mathcal{B}_1}\|^{1/2}$, and $\|\langle \xi | \xi \rangle_{\mathcal{B}_2}\|^{1/2}$ induced by the right inner products, respectively. By definition, $\mathcal{H}$ is complete under the above three norms for each.

Definition 3. (i) A Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is said to be of *general type* if there exists a faithful completely positive map $\lambda_i : \mathcal{B}_i \rightarrow \mathcal{A}$ for $i = 1, 2$ such that

$$\lambda_i(b_i\psi_i(a)) = \lambda_i(b_i)a \quad \text{for } b_i \in \mathcal{B}_i, a \in \mathcal{A}, \quad (8)$$

$$\lambda_i(\langle \xi | \eta \rangle_{\mathcal{B}_i}) = \langle \xi | \eta \rangle_{\mathcal{A}}, \quad \text{for } \xi, \eta \in \mathcal{H}, i = 1, 2. \quad (9)$$

(ii) A Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is said to be of *finite type* if there exist a finite basis $\{u_1, \dots, u_M\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_1$ -module and a finite basis $\{v_1, \dots, v_N\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_2$ -module; that is,

$$\sum_{i=1}^M u_i \varphi_1(\langle u_i | \xi \rangle_{\mathcal{B}_1}) = \sum_{k=1}^N v_k \varphi_2(\langle v_k | \xi \rangle_{\mathcal{B}_2}) = \xi, \quad \xi \in \mathcal{H} \quad (10)$$

such that

$$\langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1} \in \mathcal{A}, \quad i, j = 1, \dots, M, \quad (11)$$

$$\langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} \in \mathcal{A}, \quad k, l = 1, \dots, N$$

for $w \in \mathcal{B}_2, z \in \mathcal{B}_1$ and

$$\begin{aligned} \sum_{i=1}^M \langle u_i | \phi_2(\langle \xi | \eta \rangle_{\mathcal{B}_2}) u_i \rangle_{\mathcal{B}_1} &= \langle \xi | \eta \rangle_{\mathcal{A}}, \\ \sum_{k=1}^N \langle v_k | \phi_1(\langle \xi | \eta \rangle_{\mathcal{B}_1}) v_k \rangle_{\mathcal{B}_2} &= \langle \xi | \eta \rangle_{\mathcal{A}} \end{aligned} \quad (12)$$

for all $\xi, \eta \in \mathcal{H}$. Following [13], $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$ are called finite bases of $\mathcal{H}$, respectively.

(iii) A Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is said to be of *strongly finite type* if it is of finite type and there exist a finite basis $\{e_1, \dots, e_{M'}\}$ of $\mathcal{B}_1$ as a right $\mathcal{A}$ -module through $\psi_1 \circ \lambda$ and a finite basis $\{f_1, \dots, f_{N'}\}$ of $\mathcal{B}_2$ as a right $\mathcal{A}$ -module through $\psi_2 \circ \lambda_2$. This means that the following equalities hold:

$$\begin{aligned} z &= \sum_{j=1}^{M'} e_j \psi_1(\lambda_1(e_j^* z)), \quad z \in \mathcal{B}_1, \\ w &= \sum_{l=1}^{N'} f_l \psi_2(\lambda_2(f_l^* w)), \quad w \in \mathcal{B}_2. \end{aligned} \quad (13)$$

We note that for a Hilbert C^* -quad module of general type, the conditions (9) imply

$$\|\langle \xi | \xi \rangle_{\mathcal{A}}\| \leq \|\lambda_i(1)\| \|\langle \xi | \xi \rangle_{\mathcal{B}_i}\|, \quad \xi \in \mathcal{H}. \quad (14)$$

Put $C_i = \|\lambda_i(1)\|^{1/2} > 0$ so that $\|\xi\|_{\mathcal{A}} \leq C_i \|\xi\|_{\mathcal{B}_i}$. Hence, the identity operators from the Banach spaces $(\mathcal{H}, \|\cdot\|_{\mathcal{B}_i})$ to $(\mathcal{H}, \|\cdot\|_{\mathcal{A}})$ are bounded linear maps. By the inverse mapping theorem, there exist constants C'_i such that $\|\xi\|_{\mathcal{B}_i} \leq C'_i \|\xi\|_{\mathcal{A}}$ for $\xi \in \mathcal{H}$. Therefore, the three norms $\|\cdot\|_{\mathcal{A}}, \|\cdot\|_{\mathcal{B}_1}$, and $\|\cdot\|_{\mathcal{B}_2}$, induced by the three inner products $\langle \cdot | \cdot \rangle_{\mathcal{A}}, \langle \cdot | \cdot \rangle_{\mathcal{B}_1}$, and $\langle \cdot | \cdot \rangle_{\mathcal{B}_2}$ on $\mathcal{H}$, are equivalent to each other.

Lemma 4. Let $\mathcal{H}$ be a Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. If $\mathcal{H}$ is of finite type, then it is of general type.

Proof. Suppose that $\mathcal{H}$ is of finite type with finite bases $\{u_1, \dots, u_M\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_1$ -module and $\{v_1, \dots, v_N\}$ of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_2$ -module as above. We put

$$\begin{aligned} \lambda_1(z) &= \sum_{k=1}^N \langle v_k | \phi_1(z) v_k \rangle_{\mathcal{B}_2} \in \mathcal{A}, \quad z \in \mathcal{B}_1, \\ \lambda_2(w) &= \sum_{i=1}^M \langle u_i | \phi_2(w) u_i \rangle_{\mathcal{B}_1} \in \mathcal{A}, \quad w \in \mathcal{B}_2. \end{aligned} \quad (15)$$

They give rise to faithful completely positive maps $\lambda_i : \mathcal{B}_i \rightarrow \mathcal{A}$, $i = 1, 2$. The equalities (12) imply that

$$\lambda_i(\langle \xi | \eta \rangle_{\mathcal{B}_i}) = \langle \xi | \eta \rangle_{\mathcal{A}}, \quad \text{for } \xi, \eta \in \mathcal{H}, i = 1, 2. \quad (16)$$

It then follows that

$$\begin{aligned}\lambda_i(\langle \xi | \eta \rangle_{\mathcal{B}_i} \psi_i(a)) &= \lambda_i(\langle \xi | \eta a \rangle_{\mathcal{B}_i}) \\ &= \langle \xi | \eta \rangle_{\mathcal{A}} a = \lambda_i(\langle \xi | \eta \rangle_{\mathcal{B}_i}) a, \quad a \in \mathcal{A}.\end{aligned}\quad (17)$$

Since $\mathcal{H}$ is full, the equalities (8) hold. $\square$

Lemma 5. Suppose that a Hilbert C^* -quad module $\mathcal{H}$ of finite type is of strongly finite type with a finite basis $\{e_1, \dots, e_{M'}\}$ of $\mathcal{B}_1$ as a right $\mathcal{A}$ -module through $\psi_1 \circ \lambda_1$ and a finite basis $\{f_1, \dots, f_{N'}\}$ of $\mathcal{B}_2$ as a right $\mathcal{A}$ -module through $\psi_2 \circ \lambda_2$. Let $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$ be finite bases of $\mathcal{H}$ satisfying (10). Then, two families $\{u_i \varphi_1(e_j) \mid i = 1, \dots, M, j = 1, \dots, M'\}$ and $\{v_k \varphi_2(f_l) \mid k = 1, \dots, N, l = 1, \dots, N'\}$ of $\mathcal{H}$ form bases of $\mathcal{H}$ as right $\mathcal{A}$ -modules, respectively.

Proof. For $\xi \in \mathcal{H}$, by the equalities

$$\begin{aligned}\xi &= \sum_{i=1}^M u_i \varphi_1(\langle u_i | \xi \rangle_{\mathcal{B}_1}), \\ \langle u_i | \xi \rangle_{\mathcal{B}_1} &= \sum_{j=1}^{M'} e_j \psi_1(\lambda_1(e_j^* \langle u_i | \xi \rangle_{\mathcal{B}_1})),\end{aligned}\quad (18)$$

it follows that

$$\begin{aligned}\xi &= \sum_{i=1}^M u_i \varphi_1\left(\sum_{j=1}^{M'} e_j \psi_1(\lambda_1(e_j^* \langle u_i | \xi \rangle_{\mathcal{B}_1}))\right) \\ &= \sum_{i=1}^M \sum_{j=1}^{M'} u_i \varphi_1(e_j) \cdot \lambda_1(e_j^* \langle u_i | \xi \rangle_{\mathcal{B}_1}) \\ &= \sum_{i=1}^M \sum_{j=1}^{M'} u_i \varphi_1(e_j) \cdot \lambda_1(\langle u_i \varphi_1(e_j) | \xi \rangle_{\mathcal{B}_1}) \\ &= \sum_{i=1}^M \sum_{j=1}^{M'} u_i \varphi_1(e_j) \cdot \langle u_i \varphi_1(e_j) | \xi \rangle_{\mathcal{A}}.\end{aligned}\quad (19)$$

We similarly have

$$\xi = \sum_{k=1}^N \sum_{l=1}^{N'} v_k \varphi_2(f_l) \cdot \langle v_k \varphi_2(f_l) | \xi \rangle_{\mathcal{A}}. \quad (20) \quad \square$$

We present some examples.

Examples. (1) Let α, β be automorphisms of a unital C^* -algebra $\mathcal{A}$ satisfying $\alpha \circ \beta = \beta \circ \alpha$. We set $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{A}$. Define right actions ψ_i of $\mathcal{A}$ on $\mathcal{B}_i$ by

$$b_1 \psi_1(a) = b_1 \alpha(a), \quad b_2 \psi_2(a) = b_2 \beta(a) \quad (21)$$

for $b_i \in \mathcal{B}_i$, $a \in \mathcal{A}$. We put $\mathcal{H}_{\alpha, \beta} = \mathcal{A}$ and equip it with Hilbert C^* -quad module structure over $(\mathcal{A}; \mathcal{A}, \mathcal{A})$ in the following way. For $\xi = x, \xi' = x' \in \mathcal{H}_{\alpha, \beta} = \mathcal{A}$, $a \in \mathcal{A}$, $z \in \mathcal{B}_1 = \mathcal{A}$, and $w \in \mathcal{B}_2 = \mathcal{A}$, define the right $\mathcal{A}$ -module structure and the right $\mathcal{A}$ -valued inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ by

$$\xi \cdot a = x a, \quad \langle \xi | \xi' \rangle_{\mathcal{A}} = x^* x'. \quad (22)$$

Define the right actions φ_i of $\mathcal{B}_i$ with right $\mathcal{B}_i$ -valued inner products $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ and the left actions ϕ_i of $\mathcal{B}_i$ by setting

$$\begin{aligned}\xi \varphi_1(z) &= x \alpha(z), & \xi \varphi_2(w) &= x \beta(w), \\ \langle \xi | \xi' \rangle_{\mathcal{B}_1} &= \alpha^{-1}(x^* x'), & \langle \xi | \xi' \rangle_{\mathcal{B}_2} &= \beta^{-1}(x^* x'), \\ \phi_1(z) \xi &= \beta(\alpha(z)) x, & \phi_2(w) \xi &= \alpha(\beta(w)) x.\end{aligned}\quad (23)$$

It is straightforward to see that $\mathcal{H}_{\alpha, \beta}$ is a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{A}, \mathcal{A})$ of strongly finite type.

(2) We fix natural numbers $1 < N, M \in \mathbb{N}$. Consider finite-dimensional commutative C^* -algebras $\mathcal{A} = \mathbb{C}$, $\mathcal{B}_1 = \mathbb{C}^N$, and $\mathcal{B}_2 = \mathbb{C}^M$. The right actions ψ_i of $\mathcal{A}$ on $\mathcal{B}_i$ are naturally defined as right multiplications of $\mathbb{C}$. The algebras $\mathcal{B}_1, \mathcal{B}_2$ have the ordinary product structure and the inner product structure which we denote by $\langle \cdot | \cdot \rangle_N$ and $\langle \cdot | \cdot \rangle_M$, respectively. Let us denote by $\mathcal{H}_{M, N}$ the tensor product $\mathbb{C}^M \otimes \mathbb{C}^N$. Define the right actions φ_i of $\mathcal{B}_i$ with $\mathcal{B}_i$ -valued right inner products $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ and the left actions ϕ_i of $\mathcal{B}_i$ on $\mathcal{H}_{M, N} = \mathbb{C}^M \otimes \mathbb{C}^N$ for $i = 1, 2$ by setting

$$\begin{aligned}(\xi \otimes \eta) \varphi_1(z) &= \xi \otimes (\eta \cdot z), & (\xi \otimes \eta) \varphi_2(w) &= (\xi \cdot w) \otimes \eta, \\ \langle \xi \otimes \eta | \xi' \otimes \eta' \rangle_{\mathcal{B}_1} &= \langle \xi | \xi' \rangle_M \eta^* \cdot \eta' \in \mathcal{B}_1, \\ \langle \xi \otimes \eta | \xi' \otimes \eta' \rangle_{\mathcal{B}_2} &= \xi^* \cdot \xi' \langle \eta | \eta' \rangle_N \in \mathcal{B}_2, \\ \phi_1(z) (\xi \otimes \eta) &= \xi \otimes (z \cdot \eta), & \phi_2(w) (\xi \otimes \eta) &= (w \cdot \xi) \otimes \eta\end{aligned}\quad (24)$$

for $z \in \mathcal{B}_1$, $\xi, \xi' \in \mathbb{C}^N$, $w \in \mathcal{B}_2$, and $\eta, \eta' \in \mathbb{C}^M$. Let e_i , $i = 1, \dots, M$ and f_k , $k = 1, \dots, N$ be the standard bases of $\mathbb{C}^M$ and $\mathbb{C}^N$, respectively. Put the finite bases

$$\begin{aligned}u_i &= e_i \otimes 1 \in \mathcal{H}_{M, N}, & i &= 1, \dots, M, \\ v_k &= 1 \otimes f_k \in \mathcal{H}_{M, N}, & i &= 1, \dots, N.\end{aligned}\quad (25)$$

It is straightforward to see that $\mathcal{H}_{M, N}$ is a Hilbert C^* -quad module over $(\mathbb{C}; \mathbb{C}^N, \mathbb{C}^M)$ of strongly finite type.

(3) Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^* -textile dynamical system which means that for $j \in \Sigma^\eta$, $l \in \Sigma^\rho$ endomorphisms η_j, ρ_l of $\mathcal{A}$ are given with commutation relations $\eta_j \circ \rho_l = \rho_k \circ \eta_i$ if $\kappa(l, j) = (i, k)$. In [12], a Hilbert C^* -quad module $\mathcal{H}_{\kappa}^{\rho, \eta}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is constructed (see [12] for its detail construction). The two triplets $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ are C^* -symbolic dynamical systems [7], that yield C^* -algebras $\mathcal{O}_\rho$ and $\mathcal{O}_\eta$, respectively. The C^* -algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined as the C^* -subalgebra of $\mathcal{O}_\eta$ generated by elements $T_j y T_j^*$, $j \in \Sigma^\eta$, $y \in \mathcal{A}$ and that of $\mathcal{O}_\rho$ generated by $S_k y S_k^*$, $k \in \Sigma^\rho$, $y \in \mathcal{A}$, respectively. Define the maps $\psi_i : \mathcal{A} \rightarrow \mathcal{B}_i$, $i = 1, 2$, by

$$\psi_1(a) = \sum_{j \in \Sigma^\eta} T_j a T_j^*, \quad \psi_2(a) = \sum_{l \in \Sigma^\rho} S_l a S_l^*, \quad a \in \mathcal{A}, \quad (26)$$

which yield the right actions of $\mathcal{A}$ on $\mathcal{B}_i$, $i = 1, 2$. Define the maps $\lambda_i : \mathcal{B}_i \rightarrow \mathcal{A}$, $i = 1, 2$ by

$$\lambda_1(z) = \sum_{j \in \Sigma^\eta} T_j^* z T_j, \quad \lambda_2(w) = \sum_{l \in \Sigma^p} S_l^* w S_l, \quad (27)$$

$$z \in \mathcal{B}_1, \quad w \in \mathcal{B}_2.$$

Put $e_j = T_j T_j^* \in \mathcal{B}_1$, $j \in \Sigma^\eta$. Let $z = \sum_{j \in \Sigma^\eta} T_j z_j T_j^*$ be an element of $\mathcal{B}_1$ for $z_j \in \mathcal{A}$ with $T_j^* T_j z_j T_j^* T_j = z_j$. As $\lambda_1(e_j^* z) = \lambda_1(T_j z_j T_j^*) = z_j$, one sees that

$$z = \sum_{j \in \Sigma^\eta} T_j T_j^* T_j z_j T_j^* = \sum_{j \in \Sigma^\eta} T_j T_j^* \psi_1(z_j) \\ = \sum_{j \in \Sigma^\eta} e_j \psi_1(\lambda_1(e_j^* z)). \quad (28)$$

We similarly have by putting $f_l = S_l S_l^* \in \mathcal{B}_2$,

$$w = \sum_{l \in \Sigma^p} f_l \psi_2(\lambda_2(f_l^* w)) \quad \text{for } w \in \mathcal{B}_2. \quad (29)$$

We see that $\mathcal{H}^{\rho, \eta}$ is a Hilbert C^* -quad module of strongly finite type. In particular, two nonnegative commuting matrices A, B with a specification κ coming from the equality $AB = BA$ yield a C^* -textile dynamical system and hence a Hilbert C^* -quad module of strongly finite type, which are studied in [15].

3. Fock Hilbert C^* -Quad Modules and Creation Operators

In this section, we will construct a C^* -algebra from a Hilbert C^* -quad module $\mathcal{H}$ of general type by using two kinds of creation operators on Fock space of Hilbert C^* -quad module. We first consider relative tensor products of Hilbert C^* -quad modules and then introduce Fock space of Hilbert C^* -quad modules which is a two-dimensional analogue of Fock space of Hilbert C^* -bimodules. We fix a Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of general type as in the preceding section. The Hilbert C^* -quad module $\mathcal{H}$ is originally a Hilbert C^* -right module over $\mathcal{A}$ with $\mathcal{A}$ -valued inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$. It has two other structures of Hilbert C^* -bimodules: the Hilbert C^* -bimodule $(\phi_1, \mathcal{H}, \varphi_1)$ over $\mathcal{B}_1$ and the Hilbert C^* -bimodule $(\phi_2, \mathcal{H}, \varphi_2)$ over $\mathcal{B}_2$, where ϕ_i is a left action of $\mathcal{B}_i$ on $\mathcal{H}$ and φ_i is a right action of $\mathcal{B}_i$ on $\mathcal{H}$ with $\mathcal{B}_i$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ for each $i = 1, 2$. This situation is written as in the figure

$$\begin{array}{ccc} & \mathcal{B}_2 & \\ & \downarrow \phi_2 & \\ \mathcal{B}_1 & \xrightarrow{\phi_1} \mathcal{H} \xleftarrow{\varphi_1} & \mathcal{B}_1 \\ & \uparrow \varphi_2 & \\ & \mathcal{B}_2 & \end{array} \quad (30)$$

We will define two kinds of relative tensor products

$$\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}, \quad \mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H} \quad (31)$$

as Hilbert C^* -quad modules over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. The latter one should be written vertically as

$$\begin{array}{c} \mathcal{H} \\ \otimes_{\mathcal{B}_2} \\ \mathcal{H} \end{array} \quad (32)$$

rather than horizontally $\mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H}$. The first relative tensor product $\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}$ is defined as the relative tensor product as Hilbert C^* -modules over $\mathcal{B}_1$, where the left $\mathcal{H}$ is a right $\mathcal{B}_1$ -module through φ_1 and the right $\mathcal{H}$ is a left $\mathcal{B}_1$ -module through ϕ_1 . It has a right $\mathcal{B}_1$ -valued inner product and a right $\mathcal{B}_2$ -valued inner product defined by

$$\langle \xi \otimes_{\mathcal{B}_1} \zeta | \xi' \otimes_{\mathcal{B}_1} \zeta' \rangle_{\mathcal{B}_1} := \langle \zeta | \phi_{\mathcal{B}_1}(\langle \xi | \xi' \rangle_{\mathcal{B}_1}) \zeta' \rangle_{\mathcal{B}_1}, \\ \langle \xi \otimes_{\mathcal{B}_1} \zeta | \xi' \otimes_{\mathcal{B}_1} \zeta' \rangle_{\mathcal{B}_2} := \langle \zeta | \phi_{\mathcal{B}_1}(\langle \xi | \xi' \rangle_{\mathcal{B}_1}) \zeta' \rangle_{\mathcal{B}_2}, \quad (33)$$

respectively. It has two right actions: $\text{id} \otimes \varphi_1$ from $\mathcal{B}_1$ and $\text{id} \otimes \varphi_2$ from $\mathcal{B}_2$. It also has two left actions: $\phi_1 \otimes \text{id}$ from $\mathcal{B}_1$ and $\phi_2 \otimes \text{id}$ from $\mathcal{B}_2$. By these operations, $\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}$ is a Hilbert C^* -bimodule over $\mathcal{B}_1$ as well as a Hilbert C^* -bimodule over $\mathcal{B}_2$. It also has a right $\mathcal{A}$ -valued inner product defined by

$$\langle \xi \otimes_{\mathcal{B}_1} \zeta | \xi' \otimes_{\mathcal{B}_1} \zeta' \rangle_{\mathcal{A}} \\ := \lambda_1(\langle \xi \otimes_{\mathcal{B}_1} \zeta | \xi' \otimes_{\mathcal{B}_1} \zeta' \rangle_{\mathcal{B}_1}) \\ = \lambda_2(\langle \xi \otimes_{\mathcal{B}_1} \zeta | \xi' \otimes_{\mathcal{B}_1} \zeta' \rangle_{\mathcal{B}_2}), \quad (34)$$

a right $\mathcal{A}$ -action $\text{id} \otimes a$ for $a \in \mathcal{A}$ and a left $\mathcal{A}$ -action $\phi_{\mathcal{A}} \otimes \text{id}$. By these structure $\mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H}$ is a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$:

$$\begin{array}{ccccc} & & \mathcal{B}_2 & & \\ & & \downarrow \phi_2 \otimes \text{id} & & \\ \mathcal{B}_1 & \xrightarrow{\phi_1 \otimes \text{id}} & \mathcal{H} \otimes_{\mathcal{B}_1} \mathcal{H} & \xleftarrow{\text{id} \otimes \varphi_1} & \mathcal{B}_1 \\ & & \uparrow \text{id} \otimes \varphi_2 & & \\ & & \mathcal{B}_2 & & \end{array} \quad (35)$$

We denote the above operations $\phi_1 \otimes \text{id}$, $\phi_2 \otimes \text{id}$, $\text{id} \otimes \varphi_1$, and $\text{id} \otimes \varphi_2$ still by ϕ_1 , ϕ_2 , φ_1 , and φ_2 , respectively. Similarly, we consider the other relative tensor product $\mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H}$ defined by the relative tensor product as Hilbert C^* -modules over $\mathcal{B}_2$, where the left $\mathcal{H}$ is a right $\mathcal{B}_2$ -module through φ_2 and the right $\mathcal{H}$ is a left $\mathcal{B}_2$ -module through ϕ_2 . By a symmetric discussion to the above, $\mathcal{H} \otimes_{\mathcal{B}_2} \mathcal{H}$ is a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. The following lemma is routine.

Lemma 6. Let $\mathcal{H}_i = \mathcal{H}$, $i = 1, 2, 3$. The correspondences

$$\begin{aligned} (\xi_1 \otimes_{\mathcal{B}_1} \xi_2) \otimes_{\mathcal{B}_2} \xi_3 &\in (\mathcal{H}_1 \otimes_{\mathcal{B}_1} \mathcal{H}_2) \otimes_{\mathcal{B}_2} \mathcal{H}_3 \\ &\longrightarrow \xi_1 \otimes_{\mathcal{B}_1} (\xi_2 \otimes_{\mathcal{B}_2} \xi_3) \in \mathcal{H}_1 \otimes_{\mathcal{B}_1} (\mathcal{H}_2 \otimes_{\mathcal{B}_2} \mathcal{H}_3), \\ (\xi_1 \otimes_{\mathcal{B}_2} \xi_2) \otimes_{\mathcal{B}_1} \xi_3 &\in (\mathcal{H}_1 \otimes_{\mathcal{B}_2} \mathcal{H}_2) \otimes_{\mathcal{B}_1} \mathcal{H}_3 \\ &\longrightarrow \xi_1 \otimes_{\mathcal{B}_2} (\xi_2 \otimes_{\mathcal{B}_1} \xi_3) \in \mathcal{H}_1 \otimes_{\mathcal{B}_2} (\mathcal{H}_2 \otimes_{\mathcal{B}_1} \mathcal{H}_3) \end{aligned} \quad (36)$$

yield isomorphisms of Hilbert C^* -quad modules, respectively.

We write the isomorphism class of the former Hilbert C^* -quad modules as $\mathcal{H}_1 \otimes_{\mathcal{B}_1} \mathcal{H}_2 \otimes_{\mathcal{B}_2} \mathcal{H}_3$ and that of the latter ones as $\mathcal{H}_1 \otimes_{\mathcal{B}_2} \mathcal{H}_2 \otimes_{\mathcal{B}_1} \mathcal{H}_3$, respectively.

Note that the direct sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ has a structure of a pre-Hilbert C^* -right module over $\mathcal{A}$ by the following operations. For $b_1 \oplus b_2, b'_1 \oplus b'_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$, and $a \in \mathcal{A}$, set

$$\begin{aligned} (b_1 \oplus b_2) \psi_{\mathcal{A}}(a) &:= b_1 \psi_1(a) \oplus b_2 \psi_2(a) \in \mathcal{B}_1 \oplus \mathcal{B}_2, \\ \langle b_1 \oplus b_2 \mid b'_1 \oplus b'_2 \rangle_{\mathcal{A}} &:= \lambda_1 \langle b_1 \mid b'_1 \rangle + \lambda_2 \langle b_2 \mid b'_2 \rangle \in \mathcal{A}. \end{aligned} \quad (37)$$

By (8), the equality

$$\langle b_1 \oplus b_2 \mid (b'_1 \oplus b'_2) \psi_{\mathcal{A}}(a) \rangle_{\mathcal{A}} = \langle b_1 \oplus b_2 \mid b'_1 \oplus b'_2 \rangle_{\mathcal{A}} \cdot a \quad (38)$$

holds so that $\mathcal{B}_1 \oplus \mathcal{B}_2$ is a pre-Hilbert C^* -right module over $\mathcal{A}$. We denote by $F_0(\mathcal{H})$ the completion of $\mathcal{B}_1 \oplus \mathcal{B}_2$ by the norm induced by the inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}$. It has right $\mathcal{B}_i$ -actions φ_i and left $\mathcal{B}_i$ -action ϕ_i by

$$\begin{aligned} (b_1 \oplus b_2) \varphi_1(z) &= b_1 z \oplus 0, & (b_1 \oplus b_2) \varphi_2(w) &= 0 \oplus b_2 w, \\ \phi_1(z) (b_1 \oplus b_2) &= z b_1 \oplus 0, & \phi_2(w) (b_1 \oplus b_2) &= 0 \oplus w b_2 \end{aligned} \quad (39)$$

for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2, z \in \mathcal{B}_1$, and $w \in \mathcal{B}_2$.

We denote the relative tensor product $\mathcal{H} \otimes_{\mathcal{B}_i} \mathcal{H}$ and elements $\xi \otimes_{\mathcal{B}_i} \eta$ by $\mathcal{H} \otimes_i \mathcal{H}$ and $\xi \otimes_i \eta$, respectively, for $i = 1, 2$. Let us define the Fock Hilbert C^* -quad module as a two-dimensional analogue of the Fock space of Hilbert C^* -bimodules. Put $\Gamma_0 = \{\emptyset\}$ and $\Gamma_n = \{(i_1, \dots, i_n) \mid i_j = 1, 2\}, n = 1, 2, \dots$. We set

$$\begin{aligned} F_1(\mathcal{H}) &= \mathcal{H}, \\ F_2(\mathcal{H}) &= (\mathcal{H} \otimes_1 \mathcal{H}) \oplus (\mathcal{H} \otimes_2 \mathcal{H}), \\ F_3(\mathcal{H}) &= (\mathcal{H} \otimes_1 \mathcal{H} \otimes_1 \mathcal{H}) \oplus (\mathcal{H} \otimes_1 \mathcal{H} \otimes_2 \mathcal{H}) \\ &\quad \oplus (\mathcal{H} \otimes_2 \mathcal{H} \otimes_1 \mathcal{H}) \oplus (\mathcal{H} \otimes_2 \mathcal{H} \otimes_2 \mathcal{H}) \\ &\quad \vdots \\ F_n(\mathcal{H}) &= \oplus_{(i_1, \dots, i_{n-1}) \in \Gamma_{n-1}} \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \dots \otimes_{i_{n-1}} \mathcal{H} \\ &\quad \vdots \end{aligned} \quad (40)$$

as Hilbert C^* -bimodules over $\mathcal{A}$. We will define the Fock Hilbert C^* -module $F(\mathcal{H})$ by setting

$$F(\mathcal{H}) := \overline{\oplus_{n=0}^{\infty} F_n(\mathcal{H})}, \quad (41)$$

which is the completion of the algebraic direct sum $\oplus_{n=0}^{\infty} F_n(\mathcal{H})$ of the Hilbert C^* -right module over $\mathcal{A}$ under the norm $\|\xi\|_{\mathcal{A}}$ on $\oplus_{n=0}^{\infty} F_n(\mathcal{H})$ induced by the $\mathcal{A}$ -valued right inner product on $\oplus_{n=0}^{\infty} F_n(\mathcal{H})$. Then, $F(\mathcal{H})$ is a Hilbert C^* -right module over $\mathcal{A}$. It has a natural left $\mathcal{B}_i$ -action defined by ϕ_i for $i = 1, 2$.

For $\xi \in \mathcal{H}_{\kappa}$, we define two operators

$$\begin{aligned} s_{\xi} : F_n(\mathcal{H}) &\longrightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \dots, \\ t_{\xi} : F_n(\mathcal{H}) &\longrightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \dots \end{aligned} \quad (42)$$

by setting for $n = 0$,

$$\begin{aligned} s_{\xi}(b_1 \oplus b_2) &= \xi \varphi_1(b_1), \quad b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2, \\ t_{\xi}(b_1 \oplus b_2) &= \xi \varphi_2(b_2), \quad b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2, \end{aligned} \quad (43)$$

and for $n = 1, 2, \dots$,

$$\begin{aligned} s_{\xi}(\xi_1 \otimes_{\pi_1} \dots \otimes_{\pi_{n-1}} \xi_n) &= \xi \otimes_1 \xi_1 \otimes_{\pi_1} \dots \otimes_{\pi_{n-1}} \xi_n, \\ t_{\xi}(\xi_1 \otimes_{\pi_1} \dots \otimes_{\pi_{n-1}} \xi_n) &= \xi \otimes_2 \xi_1 \otimes_{\pi_1} \dots \otimes_{\pi_{n-1}} \xi_n \end{aligned} \quad (44)$$

for $\xi_1 \otimes_{\pi_1} \dots \otimes_{\pi_{n-1}} \xi_n \in F_n(\mathcal{H})$ with $(\pi_1, \dots, \pi_{n-1}) \in \Gamma_{n-1}$.

Lemma 7. For $\xi \in \mathcal{H}$, the two operators

$$\begin{aligned} s_{\xi} : F_n(\mathcal{H}) &\longrightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \dots, \\ t_{\xi} : F_n(\mathcal{H}) &\longrightarrow F_{n+1}(\mathcal{H}), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (45)$$

are both right $\mathcal{A}$ -module maps.

Proof. We will show the assertion for s_{ξ} . For $n = 0$, we have for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$ and $a \in \mathcal{A}$,

$$\begin{aligned} s_{\xi}((b_1 \oplus b_2) \psi_{\mathcal{A}}(a)) &= \xi \varphi_1(b_1 \psi_1(a)) = (\xi \varphi_1(b_1)) a \\ &= (s_{\xi}(b_1 \oplus b_2)) a. \end{aligned} \quad (46)$$

For $n = 1, 2, \dots$, one has

$$\begin{aligned} s_{\xi}((\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n) a) &= s_{\xi}(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} (\xi_n a)) \\ &= \xi \otimes_1 \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} (\xi_n a) \\ &= (\xi \otimes_1 \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n) a \\ &= [s_{\xi}(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n)] a. \end{aligned} \quad (47)$$

□

It is clear that the two operators s_{ξ}, t_{ξ} yield bounded right $\mathcal{A}$ -module maps on $F(\mathcal{H})$ having its adjoints with respect to the $\mathcal{A}$ -valued right inner product on $F(\mathcal{H})$. The operators are still denoted by s_{ξ}, t_{ξ} , respectively. The adjoints of $s_{\xi}, t_{\xi} : F(\mathcal{H}) \rightarrow F(\mathcal{H})$ with respect to the $\mathcal{A}$ -valued right inner product on $F(\mathcal{H})$ map $F_{n+1}(\mathcal{H})$ to $F_n(\mathcal{H})$, $n = 0, 1, 2, \dots$

Lemma 8. (i) For $\xi, \xi' \in \mathcal{H} = F_1(\mathcal{H})$, one has

$$\begin{aligned} s_{\xi}^* \xi' &= \langle \xi | \xi' \rangle_{\mathcal{B}_1} \oplus 0 \in \mathcal{B}_1 \oplus \mathcal{B}_2, \\ t_{\xi}^* \xi' &= 0 \oplus \langle \xi | \xi' \rangle_{\mathcal{B}_2} \in \mathcal{B}_1 \oplus \mathcal{B}_2. \end{aligned} \quad (48)$$

(ii) For $\xi \in \mathcal{H}$ and $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \in F_{n+1}(\mathcal{H})$, $n = 1, 2, \dots$, we have

$$\begin{aligned} s_{\xi}^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) &= \begin{cases} \phi_1 (\langle \xi | \xi_1 \rangle_{\mathcal{B}_1}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2, \end{cases} \\ t_{\xi}^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) &= \begin{cases} 0 & \text{if } i_1 = 1, \\ \phi_2 (\langle \xi | \xi_1 \rangle_{\mathcal{B}_2}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} & \text{if } i_1 = 2. \end{cases} \end{aligned} \quad (49)$$

Proof. We will show the assertions (i) and (ii) for s_{ξ}^* . (i) For $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$, we have

$$\begin{aligned} \langle b_1 \oplus b_2 | s_{\xi}^* \xi' \rangle_{\mathcal{A}} &= \langle \xi \varphi_1(b_1) | \xi' \rangle_{\mathcal{A}} = \lambda_1 (b_1^* \langle \xi | \xi' \rangle_{\mathcal{B}_1}) \\ &= \langle b_1 \oplus b_2 | \langle \xi | \xi' \rangle_{\mathcal{B}_1} \oplus 0 \rangle_{\mathcal{A}} \end{aligned} \quad (50)$$

so that $s_{\xi}^* \xi' = \langle \xi | \xi' \rangle_{\mathcal{B}_1} \oplus 0$.

(ii) For $\zeta_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \zeta_n \in F_n(\mathcal{H})$ with $n = 1, 2, \dots$, we have

$$\begin{aligned} \langle \zeta_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \zeta_n | s_{\xi}^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1}) \rangle_{\mathcal{A}} &= \langle \xi \otimes_1 \zeta_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \zeta_n | \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \rangle_{\mathcal{A}} \\ &= \lambda_1 (\langle \xi \otimes_1 \zeta_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \zeta_n | \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \rangle_{\mathcal{B}_1}) \\ &= \begin{cases} \langle \zeta_1 \otimes_{j_1} \cdots \otimes_{j_{n-1}} \zeta_n | \phi_1 (\langle \xi | \xi_1 \rangle_{\mathcal{B}_1}) \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_{n+1} \rangle_{\mathcal{A}} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2. \end{cases} \end{aligned} \quad (51)$$

Denote by $\bar{\phi}_i$ the left actions of $\mathcal{B}_i$, $i = 1, 2$ on $F_n(\mathcal{H})$ and hence on $F(\mathcal{H})$, respectively. They satisfy the following equalities

$$\begin{aligned} \bar{\phi}_1(z)(b_1 \oplus b_2) &= zb_1 \oplus 0, \\ \bar{\phi}_2(w)(b_1 \oplus b_2) &= 0 \oplus wb_2, \\ \bar{\phi}_1(z)(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n) &= (\phi_1(z) \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n, \\ \bar{\phi}_2(w)(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n) &= (\phi_2(w) \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_n} \xi_n \end{aligned} \quad (52)$$

for $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$, $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$, and $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$. More generally let us denote by $\mathcal{L}_{\mathcal{A}}(\mathcal{H})$ and $\mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ the C^* -algebras of all bounded adjointable right $\mathcal{A}$ -module maps on $\mathcal{H}$ and on $F(\mathcal{H})$ with respect to their right $\mathcal{A}$ -valued inner products, respectively. For $L \in \mathcal{L}_{\mathcal{A}}(\mathcal{H})$, define $\bar{L} \in \mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ by

$$\begin{aligned} \bar{L}(b_1 \oplus b_2) &= 0 \quad \text{for } b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2 \subset F_0(\mathcal{H}), \\ \bar{L}(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) &= (L\xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \end{aligned} \quad (53)$$

for $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$.

Lemma 9. Both the maps $\bar{\phi}_i : \mathcal{B}_i \rightarrow \mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ for $i = 1, 2$ are faithful $*$ -homomorphisms.

Proof. By assumption, the $*$ -homomorphisms $\phi_i : \mathcal{B}_i \rightarrow \mathcal{L}_{\mathcal{A}}(\mathcal{H})$, $i = 1, 2$ are faithful, so that the $*$ -homomorphisms $\bar{\phi}_i : \mathcal{B}_i \rightarrow \mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$, $i = 1, 2$ are both faithful. $\square$

Lemma 10. For $\xi, \zeta \in \mathcal{H}$, $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$, $L \in \mathcal{L}_{\mathcal{A}}(\mathcal{H})$, and $c, d \in \mathbb{C}$, the following equalities hold on $F(\mathcal{H})$:

$$s_{c\xi+d\zeta} = cs_{\xi} + ds_{\zeta}, \quad t_{c\xi+d\zeta} = ct_{\xi} + dt_{\zeta}, \quad (54)$$

$$s_{L\xi\varphi_1(z)} = \bar{L}s_{\xi}\bar{\phi}_1(z), \quad t_{L\xi\varphi_2(w)} = \bar{L}t_{\xi}\bar{\phi}_2(w), \quad (55)$$

$$s_{\xi}^* \bar{L}s_{\xi} = \bar{\phi}_1(\langle \xi | L\xi \rangle_{\mathcal{B}_1}), \quad t_{\xi}^* \bar{L}t_{\xi} = \bar{\phi}_2(\langle \xi | L\xi \rangle_{\mathcal{B}_2}). \quad (56)$$

Proof. The equalities (54) are obvious. We will show the equalities (55) and (56) for s_{ξ} . We have for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$

$$\begin{aligned} s_{L\xi\varphi_1(z)}(b_1 \oplus b_2) &= [L\xi\varphi_1(z)]\varphi_1(b_1) \\ &= \bar{L}[\xi\varphi_1(zb_1)] = \bar{L}[s_{\xi}(zb_1 \oplus 0)] \\ &= \bar{L}[s_{\xi}[\bar{\phi}_1(z)(b_1 \oplus b_2)]] \\ &= [\bar{L}s_{\xi}\bar{\phi}_1(z)](b_1 \oplus b_2). \end{aligned} \quad (57)$$

For $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$, $n = 1, 2, \dots$, we have

$$\begin{aligned} s_{L\xi\varphi_1(z)}(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) &= (L\xi\varphi_1(z)) \otimes_{i_1} \xi_1 \otimes_{i_2} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \\ &= \bar{L}[(\xi\varphi_1(z)) \otimes_{i_1} \xi_1 \otimes_{i_2} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n] \\ &= \bar{L}[\xi \otimes_1 (\phi_1(z) \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n] \\ &= \bar{L}[s_{\xi}((\phi_1(z) \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n)] \\ &= \bar{L}s_{\xi}\bar{\phi}_1(z)[\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n] \end{aligned} \quad (58)$$

so that $s_{L\xi\varphi_1(z)} = \bar{L}s_{\xi}\bar{\phi}_1(z)$ on $F_n(\mathcal{H})$, $n = 0, 1, \dots$. Hence, the equalities (55) hold.

For $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$, we have

$$\begin{aligned} s_\xi^* \bar{L} s_\xi (b_1 \oplus b_2) &= s_\xi^* (L \xi \varphi_1(b_1)) \\ &= \langle \zeta \mid L \xi \varphi_1(b_1) \rangle_{\mathcal{B}_1} \oplus 0 \\ &= \langle \zeta \mid L \xi \rangle_{\mathcal{B}_1} b_1 \oplus 0 \\ &= \bar{\phi}_1(\langle \zeta \mid L \xi \rangle_{\mathcal{B}_1})(b_1 \oplus b_2). \end{aligned} \quad (59)$$

For $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$, we have

$$\begin{aligned} s_\xi^* \bar{L} s_\xi (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) \\ &= s_\xi^* \bar{L} (\xi \otimes_1 \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) \\ &= (\phi_1(\langle \zeta \mid L \xi \rangle_{\mathcal{B}_1}) \xi_1) \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \\ &= \bar{\phi}_1(\langle \zeta \mid L \xi \rangle_{\mathcal{B}_1})(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) \end{aligned} \quad (60)$$

so that $s_\xi^* \bar{L} s_\xi = \bar{\phi}_1(\langle \zeta \mid L \xi \rangle_{\mathcal{B}_1})$ on $F_n(\mathcal{H})$ for $n = 0, 1, 2, \dots$. Hence, the equalities (56) hold. $\square$

The C^* -subalgebra of $\mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ generated by the operators s_ξ, t_ξ for $\xi \in \mathcal{H}$ is denoted by $\mathcal{T}_{F(\mathcal{H})}$ and is called the Toeplitz quad module algebra for $\mathcal{H}$.

Lemma 11. *The C^* -algebra $\mathcal{T}_{F(\mathcal{H})}$ contains the operators $\bar{\phi}_1(z), \bar{\phi}_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.*

Proof. By (56) in the preceding lemma, one sees that

$$\begin{aligned} s_\xi^* s_\xi &= \bar{\phi}_1(\langle \zeta \mid \xi \rangle_{\mathcal{B}_1}), & t_\xi^* t_\xi &= \bar{\phi}_2(\langle \zeta \mid \xi \rangle_{\mathcal{B}_2}), \\ & & \zeta, \xi &\in \mathcal{H}. \end{aligned} \quad (61)$$

Since $\mathcal{H}$ is a full C^* -quad module, the inner products $\langle \zeta \mid \xi \rangle_{\mathcal{B}_1}, \langle \zeta \mid \xi \rangle_{\mathcal{B}_2}$ for $\zeta, \xi \in \mathcal{H}$ generate the C^* -algebras $\mathcal{B}_1, \mathcal{B}_2$, respectively. Hence, $\bar{\phi}_1(\mathcal{B}_1), \bar{\phi}_2(\mathcal{B}_2)$ are contained in $\mathcal{T}_{F(\mathcal{H})}$. $\square$

Lemma 12. *There exists an action γ of $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ on $\mathcal{T}_{F(\mathcal{H})}$ such that*

$$\begin{aligned} \gamma_r(s_\xi) &= e^{2\pi\sqrt{-1}r} s_\xi, & \gamma_r(t_\xi) &= e^{2\pi\sqrt{-1}r} t_\xi, & \xi &\in \mathcal{H}, \\ \gamma_r(\bar{\phi}_1(z)) &= \bar{\phi}_1(z), & z &\in \mathcal{B}_1, \\ \gamma_r(\bar{\phi}_2(w)) &= \bar{\phi}_2(w), & w &\in \mathcal{B}_2 \end{aligned} \quad (62)$$

for $r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$.

Proof. We will first define a one-parameter unitary group $u_r, r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ on $F(\mathcal{H})$ with respect to the right $\mathcal{A}$ -valued inner product as in the following way.

For $n = 0 : u_r : F_0(\mathcal{H}) \rightarrow F_0(\mathcal{H})$ is defined by

$$u_r(b_1 \oplus b_2) = b_1 \oplus b_2 \quad \text{for } b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2. \quad (63)$$

For $n = 1, 2, \dots : u_r : F_n(\mathcal{H}) \rightarrow F_n(\mathcal{H})$ is defined by

$$u_r(\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n) = e^{2\pi\sqrt{-1}nr} \xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \quad (64)$$

for $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$. We therefore have a one-parameter unitary group u_r on $F(\mathcal{H})$. We then define an automorphism γ_r on $\mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ for $r \in \mathbb{R}/\mathbb{Z}$ by

$$\gamma_r(T) = u_r T u_r^* \quad (65)$$

for $T \in \mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$, $r \in \mathbb{R}/\mathbb{Z}$.

It then follows that for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$

$$\begin{aligned} \gamma_r(s_\xi)(b_1 \oplus b_2) &= u_r s_\xi u_r^*(b_1 \oplus b_2) = u_r(\xi(\varphi_1(b_1))) \\ &= e^{2\pi\sqrt{-1}r} s_\xi(b_1 \oplus b_2), \end{aligned} \quad (66)$$

and for $\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$, $n = 1, 2, \dots$,

$$\begin{aligned} \gamma_r(s_\xi)(\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n) \\ &= e^{-2\pi\sqrt{-1}nr} u_r(\xi \otimes_1 \xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n) \\ &= e^{2\pi\sqrt{-1}r} s_\xi(\xi_1 \otimes_{i_1} \cdots \otimes_{i_{n-1}} \xi_n). \end{aligned} \quad (67)$$

Therefore, we conclude that $\gamma_r(s_\xi) = e^{2\pi\sqrt{-1}r} s_\xi$ on $F(\mathcal{H})$ and similarly $\gamma_r(t_\xi) = e^{2\pi\sqrt{-1}r} t_\xi$ on $F(\mathcal{H})$. It is direct to see that

$$\begin{aligned} \gamma_r(\bar{\phi}_1(z)) &= \bar{\phi}_1(z), & \gamma_r(\bar{\phi}_2(w)) &= \bar{\phi}_2(w), \\ & & \text{for } z \in \mathcal{B}_1, w \in \mathcal{B}_2. \end{aligned} \quad (68)$$

It is also obvious that $\gamma_r(\mathcal{T}_{F(\mathcal{H})}) = \mathcal{T}_{F(\mathcal{H})}$ for $r \in \mathbb{R}/\mathbb{Z}$. $\square$

Denote by $J(\mathcal{H})$ the C^* -subalgebra of $\mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$ generated by the elements

$$\mathcal{L}_{\mathcal{A}}\left(\bigoplus_{n=0}^{\text{finite}} F_n(\mathcal{H})\right). \quad (69)$$

The algebra $J(\mathcal{H})$ is a closed two-sided ideal of $\mathcal{L}_{\mathcal{A}}(F(\mathcal{H}))$.

Definition 13. The C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ associated with the Hilbert C^* -quad module $\mathcal{H}$ of general type is defined by the quotient C^* -algebra of $\mathcal{T}_{F(\mathcal{H})}$ by the ideal $\mathcal{T}_{F(\mathcal{H})} \cap J(\mathcal{H})$.

We denote by $[x]$ the quotient image of an element $x \in \mathcal{T}_{F(\mathcal{H})}$ under the ideal $\mathcal{T}_{F(\mathcal{H})} \cap J(\mathcal{H})$. We set the elements of $\mathcal{O}_{F(\mathcal{H})}$

$$\begin{aligned} S_\xi &= [s_\xi], & T_\xi &= [t_\xi], \\ \Phi_1(z) &= [\bar{\phi}_1(z)], & \Phi_2(w) &= [\bar{\phi}_2(w)] \end{aligned} \quad (70)$$

for $\xi \in \mathcal{H}$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. By the preceding lemmas, we have the following.

Proposition 14. The C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ is generated by the family of operators S_ξ, T_ξ for $\xi \in \mathcal{H}$. It contains the operators $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. They satisfy the following equalities,

$$S_{c\xi+d\zeta} = cS_\xi + dS_\zeta, \quad T_{c\xi+d\zeta} = cT_\xi + dT_\zeta, \quad (71)$$

$$S_{\phi_1(z')\xi\phi_1(z)} = \Phi_1(z') S_\xi \Phi_1(z), \quad (72)$$

$$T_{\phi_1(z')\xi\phi_2(w)} = \Phi_1(z') T_\xi \Phi_2(w),$$

$$S_{\phi_2(w')\xi\phi_1(z)} = \Phi_2(w') S_\xi \Phi_1(z), \quad (73)$$

$$T_{\phi_2(w')\xi\phi_2(w)} = \Phi_2(w') T_\xi \Phi_2(w),$$

$$S_\zeta^* S_\xi = \Phi_1(\langle \zeta | \xi \rangle_{\mathcal{B}_1}), \quad T_\zeta^* T_\xi = \Phi_2(\langle \zeta | \xi \rangle_{\mathcal{B}_2}) \quad (74)$$

for $\xi, \zeta \in \mathcal{H}, c, d \in C$ and $z, z' \in \mathcal{B}_1, w, w' \in \mathcal{B}_2$.

4. The C^* -Algebras of Hilbert C^* -Quad Modules of Finite Type

In what follows, we assume that a Hilbert C^* -quad module $\mathcal{H}$ is of finite type. In this section, we will study the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ for a Hilbert C^* -quad module $\mathcal{H}$ of finite type. Let $\{u_1, \dots, u_M\}$ be a finite basis of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_1$ and $\{v_1, \dots, v_N\}$ a finite basis of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_2$. Keep the notations as in the previous section. We set

$$s_i = s_{u_i} \quad \text{for } i = 1, \dots, M, \quad (75)$$

$$t_k = t_{v_k} \quad \text{for } k = 1, \dots, N.$$

By (10) and Lemma 10, we have for $\xi \in \mathcal{H}$

$$s_\xi = \sum_{i=1}^M s_i \bar{\phi}_1(\langle u_i | \xi \rangle_{\mathcal{B}_1}), \quad (76)$$

$$t_\xi = \sum_{k=1}^N t_k \bar{\phi}_2(\langle v_k | \xi \rangle_{\mathcal{B}_2}).$$

Let P_n be the projection on $F(\mathcal{H})$ onto $F_n(\mathcal{H})$ for $n = 0, 1, \dots$ so that $\sum_{n=0}^\infty P_n = 1$ on $F(\mathcal{H})$.

Lemma 15. For $\xi, \zeta \in \mathcal{H}$, one has

$$(i) \quad s_\xi^* t_\zeta P_n = 0 \text{ for } n = 1, 2, \dots \text{ and hence } s_\xi^* t_\zeta = s_\xi^* t_\zeta P_0.$$

$$(ii) \quad t_\zeta^* s_\xi P_n = 0 \text{ for } n = 1, 2, \dots \text{ and hence } t_\zeta^* s_\xi = t_\zeta^* s_\xi P_0.$$

Proof. (i) For $n = 1, 2, \dots$, we have

$$s_\xi^* t_\zeta (\xi_1 \otimes_{i_1} \dots \otimes_{i_{n-1}} \xi_n) = s_\xi^* (\zeta \otimes_2 \xi_1 \otimes_{i_1} \dots \otimes_{i_{n-1}} \xi_n) = 0. \quad (77)$$

(ii) Is similar to (i). $\square$

Define two projections on $F(\mathcal{H})$ by

P_s = The projection onto

$$\bigoplus_{n=0}^\infty \sum_{(i_1, \dots, i_n) \in \Gamma_n} \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \dots \otimes_{i_n} \mathcal{H}, \quad (78)$$

P_t = The projection onto

$$\bigoplus_{n=0}^\infty \sum_{(i_1, \dots, i_n) \in \Gamma_n} \mathcal{H} \otimes_2 \mathcal{H} \otimes_{i_1} \mathcal{H} \otimes_{i_2} \dots \otimes_{i_n} \mathcal{H}.$$

Lemma 16. Keep the above notations.

$$\sum_{i=1}^M s_i s_i^* = P_1 + P_s, \quad \sum_{k=1}^N t_k t_k^* = P_1 + P_t. \quad (79)$$

Hence,

$$\sum_{i=1}^M s_i s_i^* + \sum_{k=1}^N t_k t_k^* + P_0 = 1_{F(\mathcal{H})} + P_1. \quad (80)$$

Proof. For $\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n \in F_n(\mathcal{H})$ with $2 \leq n \in N$, we have

$$s_i s_i^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n) = \begin{cases} u_i \otimes_1 \phi_1(\langle u_i | \xi_1 \rangle_\eta) \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2. \end{cases} \quad (81)$$

As $u_i \otimes_1 \phi_1(\langle u_i | \xi_1 \rangle_{\mathcal{B}_1}) \xi_2 = u_i \phi_1(\langle u_i | \xi_1 \rangle_{\mathcal{B}_1}) \otimes_1 \xi_2$, and $\sum_{i=1}^M u_i \phi_1(\langle u_i | \xi_1 \rangle_{\mathcal{B}_1}) = \xi_1$, we have

$$\sum_{i=1}^M s_i s_i^* (\xi_1 \otimes_{i_1} \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n) = \begin{cases} \xi_1 \otimes_1 \xi_2 \otimes_{i_2} \dots \otimes_{i_{n-1}} \xi_n & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 = 2 \end{cases} \quad (82)$$

and hence

$$\sum_{i=1}^M s_i s_i^* \Big|_{\bigoplus_{n=2}^\infty F_n(\mathcal{H})} = P_s \Big|_{\bigoplus_{n=2}^\infty F_n(\mathcal{H})}. \quad (83)$$

For $\xi \in F_1(\mathcal{H}) = \mathcal{H}$, we have $s_i s_i^* \xi = s_i(\langle u_i | \xi \rangle_{\mathcal{B}_1} \oplus 0) = u_i \phi_1(\langle u_i | \xi \rangle_{\mathcal{B}_1})$ so that

$$\sum_{i=1}^M s_i s_i^* \xi = \sum_{i=1}^M u_i \phi_1(\langle u_i | \xi \rangle_{\mathcal{B}_1}) = \xi \quad (84)$$

and hence

$$\sum_{i=1}^M s_i s_i^* \Big|_{F_1(\mathcal{H})} = 1_{F_1(\mathcal{H})}. \quad (85)$$

As $s_i s_i^* (b_1 \oplus b_2) = 0$ for $b_1 \oplus b_2 \in \mathcal{B}_1 \oplus \mathcal{B}_2$, we have

$$\sum_{i=1}^M s_i s_i^* \Big|_{F_0(\mathcal{H})} = 0. \quad (86)$$

Therefore, we conclude that

$$\sum_{i=1}^M s_i s_i^* = P_s + P_1 \quad \text{and similarly} \quad \sum_{k=1}^N t_k t_k^* = P_t + P_1. \quad (87)$$

As $P_s + P_t + P_0 + P_1 = 1_{F(\mathcal{H})}$, one obtains (80). $\square$

We set the operators

$$\begin{aligned} S_i &= S_{u_i} (= [s_i]) \quad \text{for } i = 1, \dots, M, \\ T_k &= T_{v_k} (= [t_k]) \quad \text{for } k = 1, \dots, N \end{aligned} \quad (88)$$

in the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$. As two operators $\sum_{i=1}^M s_i s_i^*$ and $\sum_{k=1}^N t_k t_k^*$ are projections by (79), so are $\sum_{i=1}^M S_i S_i^*$ and $\sum_{k=1}^N T_k T_k^*$. Since $P_1 - P_0 \in J(\mathcal{H})$, the identity (80) implies

$$\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1. \quad (89)$$

Therefore, we have the following.

Theorem 17. *Let $\mathcal{H}$ be a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of finite type with finite basis $\{u_1, \dots, u_M\}$ as a right $\mathcal{B}_1$ -module and $\{v_1, \dots, v_N\}$ as a right $\mathcal{B}_2$ -module. Then, one has the following.*

(i) *The C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ is generated by the operators $S_1, \dots, S_M, T_1, \dots, T_N$ and the elements $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.*

(ii) *They satisfy the following operator relations:*

$$\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1, \quad S_j^* T_l = 0, \quad (90)$$

$$S_i^* S_j = \Phi_1(\langle u_i | u_j \rangle_{\mathcal{B}_1}), \quad T_k^* T_l = \Phi_2(\langle v_k | v_l \rangle_{\mathcal{B}_2}), \quad (91)$$

$$\Phi_1(z) S_j = \sum_{i=1}^M S_i \Phi_1(\langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}), \quad (92)$$

$$\Phi_1(z) T_l = \sum_{k=1}^N T_k \Phi_2(\langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2}),$$

$$\Phi_2(w) S_j = \sum_{i=1}^M S_i \Phi_1(\langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}), \quad (93)$$

$$\Phi_2(w) T_l = \sum_{k=1}^N T_k \Phi_2(\langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2})$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \dots, M, k, l = 1, \dots, N$.

(iii) *There exists an action γ of $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ on $\mathcal{O}_{F(\mathcal{H})}$ such that*

$$\begin{aligned} \gamma_r(S_i) &= e^{2\pi\sqrt{-1}r} S_i, & \gamma_r(T_k) &= e^{2\pi\sqrt{-1}r} T_k, \\ \gamma_r(\Phi_1(z)) &= \Phi_1(z), & \gamma_r(\Phi_2(w)) &= \Phi_2(w) \end{aligned} \quad (94)$$

for $r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}, i = 1, \dots, M, k = 1, \dots, N$, and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.

Proof. (i) The assertion comes from the equalities (76).

(ii) The first equality of (90) is (89). As the projection P_0 belongs to $J(\mathcal{H})$, Lemma 15 ensures us the second equality of (90). The equalities (91) come from (74). For $z \in \mathcal{B}_1$ and $j = 1, \dots, M$, we have $\phi_1(z) u_j = \sum_{i=1}^M u_i \phi_1(\langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1})$ so that

$$\bar{\phi}_1(z) s_j = s_{\phi_1(z) u_j} = \sum_{i=1}^M s_{u_i} \bar{\phi}_1(\langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}) \quad (95)$$

which goes to the first equality of (92). The other equalities of (92) and (93) are similarly shown.

(iii) The assertion is direct from Lemma 12. $\square$

The action γ of $\mathbb{T}$ on $\mathcal{O}_{F(\mathcal{H})}$ defined in the above theorem (iii) is called the gauge action.

5. The Universal C^* -Algebras Associated with Hilbert C^* -Quad Modules

In this section, we will prove that the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ associated with a Hilbert C^* -quad module of finite type is the universal C^* -algebra subject to the operator relations stated in Theorem 17 (ii). Throughout this section, we fix a Hilbert C^* -quad module $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of finite type with finite basis $\{u_1, \dots, u_M\}$ as a right Hilbert $\mathcal{B}_1$ -module and $\{v_1, \dots, v_N\}$ as a right Hilbert $\mathcal{B}_2$ -module as in the previous section.

Let $\mathcal{P}_{\mathcal{H}}$ be the universal $*$ -algebra generated by operators $S_1, \dots, S_M, T_1, \dots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations

$$\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1, \quad S_j^* T_l = 0, \quad (96)$$

$$S_i^* S_j = \langle u_i | u_j \rangle_{\mathcal{B}_1}, \quad T_k^* T_l = \langle v_k | v_l \rangle_{\mathcal{B}_2}, \quad (97)$$

$$z S_j = \sum_{i=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}, \quad (98)$$

$$z T_l = \sum_{k=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2},$$

$$w S_j = \sum_{i=1}^M S_i \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}, \quad (99)$$

$$w T_l = \sum_{k=1}^N T_k \langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2}$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \dots, M, k, l = 1, \dots, N$. The above four relations (96), (97), (98), and (99) are called the relations $(\mathcal{H})$. In what follows, we fix operators $S_1, \dots, S_M, T_1, \dots, T_N$ satisfying the relations $(\mathcal{H})$.

Lemma 18. *The sums $\sum_{i=1}^M S_i S_i^*$ and $\sum_{k=1}^N T_k T_k^*$ are both projections.*

Proof. Put $P = \sum_{i=1}^M S_i S_i^*$ and $Q = \sum_{k=1}^N T_k T_k^*$. By the relations (96), one sees that $0 \leq P, Q \leq 1$, $P + Q = 1$, and $PQ = 0$. It is easy to see that both P and Q are projections. $\square$

Lemma 19. *Keep the above notations.*

(i) For $i, j = 1, \dots, M$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$, one has

$$\begin{aligned} S_i^* z S_j &= \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}, \\ S_i^* w S_j &= \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}. \end{aligned} \quad (100)$$

(ii) For $k, l = 1, \dots, N$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$, one has

$$\begin{aligned} T_k^* z T_l &= \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2}, \\ T_k^* w T_l &= \langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2}. \end{aligned} \quad (101)$$

Proof. (i) By (98), we have

$$\begin{aligned} S_i^* z S_j &= \sum_{h=1}^M S_i^* S_h \langle u_h | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \\ &= \sum_{h=1}^M \langle u_i | u_h \rangle_{\mathcal{B}_1} \langle u_h | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \\ &= \left\langle u_i | \sum_{h=1}^M u_h \langle u_h | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \right\rangle_{\mathcal{B}_1} \\ &= \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}. \end{aligned} \quad (102)$$

The other equality $S_i^* w S_j = \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}$ is similar to the above equalities.

(ii) Is similar to (i). $\square$

By the equalities (12), we have the following.

Lemma 20. *Keep the above notations.*

(i) For $w \in \mathcal{B}_2, j = 1, \dots, M$, the element $S_j^* w S_j$ belongs to $\mathcal{A}$ and the formula holds:

$$\sum_{j=1}^M S_j^* \langle \xi | \eta \rangle_{\mathcal{B}_2} S_j = \langle \xi | \eta \rangle_{\mathcal{A}} \quad \text{for } \xi, \eta \in \mathcal{H}. \quad (103)$$

(ii) For $z \in \mathcal{B}_1, l = 1, \dots, N$, the element $T_l^* z T_l$ belongs to $\mathcal{A}$ and the formula holds:

$$\sum_{l=1}^N T_l^* \langle \xi | \eta \rangle_{\mathcal{B}_1} T_l = \langle \xi | \eta \rangle_{\mathcal{A}} \quad \text{for } \xi, \eta \in \mathcal{H}. \quad (104)$$

Lemma 21. *The following equalities for $z \in \mathcal{B}_1$ and $w \in \mathcal{B}_2$ hold:*

(i)

$$\begin{aligned} z &= \sum_{i,j=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} T_l^*, \end{aligned} \quad (105)$$

$$\begin{aligned} w &= \sum_{i,j=1}^M S_i \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2} T_l^*. \end{aligned} \quad (106)$$

(ii)

$$\begin{aligned} z^* &= \sum_{i,j=1}^M S_i \langle u_i | \phi_1(z)^* u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_1(z)^* v_l \rangle_{\mathcal{B}_2} T_l^*, \end{aligned} \quad (107)$$

$$\begin{aligned} w^* &= \sum_{i,j=1}^M S_i \langle u_i | \phi_2(w)^* u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_2(w)^* v_l \rangle_{\mathcal{B}_2} T_l^*. \end{aligned} \quad (108)$$

(iii)

$$\begin{aligned} zw &= \sum_{i,j=1}^M S_i \langle u_i | \phi_1(z) \phi_2(w) u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_1(z) \phi_2(w) v_l \rangle_{\mathcal{B}_2} T_l^*, \end{aligned} \quad (109)$$

$$\begin{aligned} wz &= \sum_{i,j=1}^M S_i \langle u_i | \phi_2(w) \phi_1(z) u_j \rangle_{\mathcal{B}_1} S_j^* \\ &\quad + \sum_{k,l=1}^N T_k \langle v_k | \phi_2(w) \phi_1(z) v_l \rangle_{\mathcal{B}_2} T_l^*. \end{aligned}$$

Proof. (i) By (98) and (99), we have

$$\begin{aligned} z S_j S_j^* &= \sum_{i=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} S_j^*, \\ z T_l T_l^* &= \sum_{k=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} T_l^* \end{aligned} \quad (110)$$

so that by (96)

$$z = \sum_{i,j=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} S_j^* + \sum_{k,l=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} T_l^*. \quad (111)$$

Similarly, we have (106).

(ii) All the adjoints of $\phi_1(z)$, $\phi_2(w)$ for $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$ by the three inner products $\langle \cdot | \cdot \rangle_{\mathcal{B}_1}$, $\langle \cdot | \cdot \rangle_{\mathcal{B}_2}$, and $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ on $\mathcal{H}$ coincide with $\phi_1(z^*)$, $\phi_2(w^*)$, respectively. Hence, the assertions are clear.

(iii) By (i), we have

$$zw = \left(\sum_{i,j=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} S_j^* + \sum_{k,l=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} T_l^* \right) \cdot \left(\sum_{g,h=1}^M S_g \langle u_g | \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* + \sum_{m,n=1}^N T_m \langle v_m | \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^* \right). \quad (112)$$

As $S_j^* T_m = T_l^* S_g = 0$ for any $j, g = 1, \dots, M$, $l, m = 1, \dots, N$, it follows that

$$\begin{aligned} zw &= \sum_{i,j,g,h=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \times S_j^* S_g \langle u_g | \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* \\ &+ \sum_{k,l,m,n=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} \times T_l^* T_m \langle v_m | \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^* \\ &= \sum_{i,j,g,h=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \times \langle u_j | u_g \rangle_{\mathcal{B}_1} \langle u_g | \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* \\ &+ \sum_{k,l,m,n=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} \times \langle v_l | v_m \rangle_{\mathcal{B}_2} \langle v_m | \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^* \\ &= \sum_{i,g,h=1}^M S_i \langle u_i | \phi_1(z) u_g \rangle_{\mathcal{B}_1} \langle u_g | \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* \\ &+ \sum_{k,m,n=1}^N T_k \left\langle v_k | \phi_1(z) \sum_{l=1}^N v_l \langle v_l | v_m \rangle_{\mathcal{B}_2} \right\rangle_{\mathcal{B}_2} \end{aligned}$$

$$\begin{aligned} &\times \langle v_m | \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^* \\ &= \sum_{i,g,h=1}^M S_i \langle u_i | \phi_1(z) u_g \rangle_{\mathcal{B}_1} \langle u_g | \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* \\ &+ \sum_{k,m,n=1}^N T_k \langle v_k | \phi_1(z) v_m \rangle_{\mathcal{B}_2} \langle v_m | \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^* \\ &= \sum_{i,h=1}^M S_i \langle u_i | \phi_1(z) \phi_2(w) u_h \rangle_{\mathcal{B}_1} S_h^* \\ &+ \sum_{k,n=1}^N T_k \langle v_k | \phi_1(z) \phi_2(w) v_n \rangle_{\mathcal{B}_2} T_n^*. \quad (113) \end{aligned}$$

□

Lemma 22. Let $p(z, w)$ be a polynomial of elements of $\mathcal{B}_1$ and $\mathcal{B}_2$. Then, one has

- (i) $p(z, w) S_j = \sum_{i=1}^M S_i z_i$ for some $z_i \in \mathcal{B}_1$.
- (ii) $p(z, w) T_l = \sum_{k=1}^N T_k w_k$ for some $w_k \in \mathcal{B}_2$.

Proof. For $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$, and $i, j = 1, \dots, M$, by putting $z_{i,j} = \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \in \mathcal{B}_1$ and $w_{i,j} = \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1} \in \mathcal{B}_1$, the relations (98), (99) imply

$$z S_j = \sum_{i=1}^M S_i z_{i,j}, \quad w S_j = \sum_{i=1}^M S_i w_{i,j} \quad (114)$$

so that the assertion of (i) holds. (ii) is similar to (i). □

Lemma 23. Let $p(z, w)$ be a polynomial of elements of $\mathcal{B}_1$ and $\mathcal{B}_2$. Then, one has

- (i) $S_i^* p(z, w) S_j$ belongs to $\mathcal{B}_1$ for all $i, j = 1, \dots, M$.
- (ii) $T_k^* p(z, w) T_l$ belongs to $\mathcal{B}_2$ for all $k, l = 1, \dots, N$.
- (iii) $S_i^* p(z, w) T_l = 0$ for all $i = 1, \dots, M$, $l = 1, \dots, N$.
- (iv) $T_k^* p(z, w) S_j = 0$ for all $k = 1, \dots, N$, $j = 1, \dots, M$.

Proof. (i) By the previous lemma, we know

$$p(z, w) S_j = \sum_{h=1}^M S_h z_h \quad \text{for some } z_h \in \mathcal{B}_1 \quad (115)$$

so that

$$S_i^* p(z, w) S_j = \sum_{h=1}^M S_i^* S_h z_h = \sum_{h=1}^M \langle u_i | u_h \rangle_{\mathcal{B}_1} z_h. \quad (116)$$

As $\langle u_i | u_h \rangle_{\mathcal{B}_1} z_h$ belongs to $\mathcal{B}_1$, we see the assertion.

(ii) Is similar to (i).

(iii) As $T_k^* S_i = 0$, we have

$$T_k^* p(z, w) S_j = \sum_{i=1}^M T_k^* S_i z_i = 0. \quad (117)$$

(iv) Is similar to (i). □

We set

$$\begin{aligned} S_{1,i} &:= S_i, \quad i = 1, \dots, M, \\ S_{2,k} &:= T_k, \quad k = 1, \dots, N. \end{aligned} \quad (118)$$

Put

$$\begin{aligned} \Sigma_1 &= \{(1, i) \mid i = 1, \dots, M\}, \\ \Sigma_2 &= \{(2, k) \mid k = 1, \dots, N\}. \end{aligned} \quad (119)$$

Lemma 24. Every element of $\mathcal{P}_{\mathcal{H}}$ can be written as a linear combination of elements of the form

$$S_{g_1, i_1} S_{g_2, i_2} \cdots S_{g_m, i_m} b S_{h_n, j_n}^* \cdots S_{h_2, j_2}^* S_{h_1, j_1}^* \quad (120)$$

for some $(g_1, i_1), (g_2, i_2), \dots, (g_m, i_m), (h_1, j_1), (h_2, j_2), \dots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2$ where b is a polynomial of elements of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. The assertion follows from the preceding lemmas. $\square$

By construction, every representation of $\mathcal{B}_1$ and $\mathcal{B}_2$ on a Hilbert space H together with operators $S_i, i = 1, \dots, M, T_k, k = 1, \dots, N$ satisfying the relations $(\mathcal{H})$ extends to a representation of $\mathcal{P}_{\mathcal{H}}$ on $B(H)$. We will endow $\mathcal{P}_{\mathcal{H}}$ with the norm obtained by taking the supremum of the norms in $B(H)$ over all such representations. Note that this supremum is finite for every element of $\mathcal{P}_{\mathcal{H}}$ because of the inequalities $\|S_i\|, \|T_k\| \leq 1$, which come from (96). The completion of the algebra $\mathcal{P}_{\mathcal{H}}$ under the norm becomes a C^* -algebra denoted by $\mathcal{O}_{\mathcal{H}}$, which is called the universal C^* -algebra subject to the relations $(\mathcal{H})$.

Denote by $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ the C^* -subalgebra of $\mathcal{L}_{\mathcal{A}}(\mathcal{H})$ generated by $\phi_1(\mathcal{B}_1)$ and $\phi_2(\mathcal{B}_2)$.

Lemma 25. An element L of the C^* -algebra $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ is both a right $\mathcal{B}_1$ -module map and a right $\mathcal{B}_2$ -module map. This means that the equalities

$$[L\xi] \phi_i(b_i) = L[\xi \phi_i(b_i)] \quad \text{for } \xi \in \mathcal{H}, b_i \in \mathcal{B}_i \quad (121)$$

hold.

Proof. Since both the operators $\phi_1(z)$ for $z \in \mathcal{B}_1$ and $\phi_2(w)$ for $w \in \mathcal{B}_2$ are right $\mathcal{B}_i$ -module maps for $i = 1, 2$, any element of the $*$ -algebra algebraically generated by $\phi_1(\mathcal{B}_1)$ and $\phi_2(\mathcal{B}_2)$ is both a right $\mathcal{B}_1$ -module map and a right $\mathcal{B}_2$ -module map. Hence, it is easy to see that any element L of the C^* -algebra $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ is both a right $\mathcal{B}_1$ -module map and a right $\mathcal{B}_2$ -module map. $\square$

Denote by $\mathcal{B}_0$ the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}}$ generated by $\mathcal{B}_1$ and $\mathcal{B}_2$.

Lemma 26. The correspondence

$$z, w \in \mathcal{B}_0 \longrightarrow \phi_1(z), \phi_2(w) \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \quad (122)$$

gives rise to an isomorphism from $\mathcal{B}_0$ onto $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ as C^* -algebras.

Proof. We note that by hypothesis both the maps

$$\begin{aligned} \phi_1 : z \in \mathcal{B}_1 &\longrightarrow \phi_1(z) \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}), \\ \phi_2 : w \in \mathcal{B}_2 &\longrightarrow \phi_2(w) \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}) \end{aligned} \quad (123)$$

are injective. Denote by $P(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ the $*$ -algebra on $\mathcal{H}$ algebraically generated by $\phi_1(z), \phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. Define an operator $\pi(L) \in \mathcal{O}_{\mathcal{H}}$ for $L \in P(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ by

$$\pi(L) = \sum_{i,j=1}^M S_i \langle u_i \mid Lu_j \rangle_{\mathcal{B}_1} S_j^* + \sum_{k,l=1}^N T_k \langle v_k \mid Lv_l \rangle_{\mathcal{B}_2} T_l^*. \quad (124)$$

Let P_0 be the $*$ -subalgebra of $\mathcal{P}_{\mathcal{H}}$ algebraically generated by $\mathcal{B}_1$ and $\mathcal{B}_2$. Since $\pi(\phi_1(z)) = z$ for $z \in \mathcal{B}_1$ and $\pi(\phi_2(w)) = w$ for $w \in \mathcal{B}_2$ and by Lemma 21, the map

$$\pi : P(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \longrightarrow P_0 \subset \mathcal{B}_0 \quad (125)$$

yields a $*$ -homomorphism. As $S_i^* T_k = 0$ for $i = 1, \dots, M, k = 1, \dots, N$, we have

$$\begin{aligned} \|\pi(L)\| &= \text{Max} \left\{ \left\| \sum_{i,j=1}^M S_i \langle u_i \mid Lu_j \rangle_{\mathcal{B}_1} S_j^* \right\|, \right. \\ &\quad \left. \left\| \sum_{k,l=1}^N T_k \langle v_k \mid Lv_l \rangle_{\mathcal{B}_2} T_l^* \right\| \right\}. \end{aligned} \quad (126)$$

We then have

$$\begin{aligned} \left\| \sum_{i,j=1}^M S_i \langle u_i \mid Lu_j \rangle_{\mathcal{B}_1} S_j^* \right\| &\leq \sum_{i,j=1}^M \left\| \langle u_i \mid Lu_j \rangle_{\mathcal{B}_1} \right\| \\ &\leq \left(\sum_{i,j=1}^M \|u_i\|_{\mathcal{B}_1} \|u_j\|_{\mathcal{B}_1} \right) \|L\| \end{aligned} \quad (127)$$

and similarly

$$\left\| \sum_{k,l=1}^N T_k \langle v_k \mid Lv_l \rangle_{\mathcal{B}_2} T_l^* \right\| \leq \left(\sum_{k,l=1}^N \|v_k\|_{\mathcal{B}_2} \|v_l\|_{\mathcal{B}_2} \right) \|L\|. \quad (128)$$

By putting

$$C = \text{Max} \{ \sum_{i,j=1}^M \|u_i\|_{\mathcal{B}_1} \|u_j\|_{\mathcal{B}_1}, \sum_{k,l=1}^N \|v_k\|_{\mathcal{B}_2} \|v_l\|_{\mathcal{B}_2} \},$$

one has

$$\|\pi(L)\| \leq C \|L\|, \quad \forall L \in P(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)). \quad (129)$$

Hence, π extends to the C^* -algebra $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ such that $\pi(C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))) = \mathcal{B}_0$. The equality (124) holds for $L \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$.

We will next show that $\pi : C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \rightarrow \mathcal{B}_0$ is injective. By (124), we have for $L \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ and $h, h' = 1, \dots, M$,

$$\begin{aligned} S_h^* \pi(L) S_{h'} &= \sum_{i,j=1}^M S_h^* S_i \langle u_i | Lu_j \rangle_{\mathcal{B}_1} S_j^* S_{h'} \\ &= \sum_{i,j=1}^M \langle u_h | u_i \rangle_{\mathcal{B}_1} \langle u_i | Lu_j \rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1} \\ &= \sum_{j=1}^M \left\langle u_h | \sum_{i=1}^M u_i \langle u_i | Lu_j \rangle_{\mathcal{B}_1} \right\rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1} \\ &= \sum_{j=1}^M \langle u_h | Lu_j \rangle_{\mathcal{B}_1} \langle u_j | u_{h'} \rangle_{\mathcal{B}_1} = \langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1}. \end{aligned} \quad (130)$$

Suppose that $\pi(L) = 0$ so that $\langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1} = 0$. Since

$$Lu_{h'} = \sum_{h=1}^M u_h \langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1}, \quad (131)$$

we see that $Lu_{h'} = 0$ so that $L = 0$. We thus conclude that π is injective and hence isomorphic. $\square$

Denote by $\phi_0 : \mathcal{B}_0 \rightarrow C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ the inverse π^{-1} of the $*$ -isomorphism π given in the proof of the above lemma which satisfies

$$\begin{aligned} \phi_0(z) &= \phi_1(z) \quad \text{for } z \in \mathcal{B}_1, \\ \phi_0(w) &= \phi_2(w) \quad \text{for } w \in \mathcal{B}_2. \end{aligned} \quad (132)$$

We put $\mathcal{F}_{\mathcal{H}}^0 = \mathcal{B}_0$. For $n \in \mathbb{N}$, we denote by $\mathcal{F}_{\mathcal{H}}^n$ the closed linear span of elements of the form

$$S_{g_1, i_1} S_{g_2, i_2} \cdots S_{g_n, i_n} b S_{h_n, j_n}^* \cdots S_{h_2, j_2}^* S_{h_1, j_1}^* \quad (133)$$

for some $(g_1, i_1), (g_2, i_2), \dots, (g_n, i_n), (h_1, j_1), (h_2, j_2), \dots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2$ and $b \in \mathcal{B}_0$. Let us denote by $\mathcal{F}_{\mathcal{H}}$ the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}}$ generated by $\bigcup_{n=0}^{\infty} \mathcal{F}_{\mathcal{H}}^n$. By the relations (105) and (106), we see the following.

Lemma 27. For $x \in \mathcal{B}_0$, the following identity holds:

$$x = \sum_{i,j=1}^M S_i \langle u_i | \phi_0(x) u_j \rangle_{\mathcal{B}_1} S_j^* + \sum_{k,l=1}^N T_k \langle v_k | \phi_0(x) v_l \rangle_{\mathcal{B}_2} T_l^*. \quad (134)$$

Hence, by putting for $b \in \mathcal{B}_0$

$$\begin{aligned} b_{1,ij} &= \langle u_i | \phi_0(b) u_j \rangle_{\mathcal{B}_1}, \quad i, j = 1, \dots, M, \\ b_{2,kl} &= \langle v_k | \phi_0(b) v_l \rangle_{\mathcal{B}_2}, \quad k, l = 1, \dots, N, \end{aligned} \quad (135)$$

we have the following.

Lemma 28. For $b \in \mathcal{B}_0$, the identity

$$\begin{aligned} S_{g_1, i_1} S_{g_2, i_2} \cdots S_{g_n, i_n} b S_{h_n, j_n}^* \cdots S_{h_2, j_2}^* S_{h_1, j_1}^* \\ = \sum_{i,j=1}^M S_{g_1, i_1} S_{g_2, i_2} \cdots S_{g_n, i_n} S_{1,i} b_{1,ij} S_{1,j}^* S_{h_n, j_n}^* \cdots S_{h_2, j_2}^* S_{h_1, j_1}^* \\ + \sum_{k,l=1}^N S_{g_1, i_1} S_{g_2, i_2} \cdots S_{g_n, i_n} S_{2,k} b_{2,kl} S_{2,l}^* S_{h_n, j_n}^* \cdots S_{h_2, j_2}^* S_{h_1, j_1}^* \end{aligned} \quad (136)$$

holds and induces an embedding of $\mathcal{F}_{\mathcal{H}}^n \hookrightarrow \mathcal{F}_{\mathcal{H}}^{n+1}$ for $n \in \mathbb{Z}_+$.

Lemma 29. The C^* -algebra $\mathcal{F}_{\mathcal{H}}$ is the inductive limit $\lim_{n \rightarrow \infty} \mathcal{F}_{\mathcal{H}}^n$ of the sequence of the inclusions

$$\begin{aligned} \mathcal{F}_{\mathcal{H}}^0 \hookrightarrow \mathcal{F}_{\mathcal{H}}^1 \hookrightarrow \mathcal{F}_{\mathcal{H}}^2 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{\mathcal{H}}^n \hookrightarrow \mathcal{F}_{\mathcal{H}}^{n+1} \\ \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{\mathcal{H}}. \end{aligned} \quad (137)$$

Let $e^{2\pi\sqrt{-1}r} \in \mathbb{T}$ be a complex number of modulus one for $r \in \mathbb{R}/\mathbb{Z}$. The elements

$$\begin{aligned} e^{2\pi\sqrt{-1}r} S_i, \quad i = 1, \dots, M, \quad e^{2\pi\sqrt{-1}r} T_k, \quad k = 1, \dots, N, \\ z \in \mathcal{B}_1, \quad w \in \mathcal{B}_2 \end{aligned} \quad (138)$$

in $\mathcal{O}_{\mathcal{H}}$ instead of

$$\begin{aligned} S_i, \quad i = 1, \dots, M, \quad T_k, \quad k = 1, \dots, N, \\ z \in \mathcal{B}_1, \quad w \in \mathcal{B}_2 \end{aligned} \quad (139)$$

satisfy the relations $(\mathcal{H})$. This implies the existence of an action on $\mathcal{P}_{\mathcal{H}}$ by automorphisms of the one-dimensional torus $\mathbb{T}$ that acts on the generators by

$$\begin{aligned} h_r(S_i) &= e^{2\pi\sqrt{-1}r} S_i, \quad h_r(T_k) = e^{2\pi\sqrt{-1}r} T_k, \\ h_r(z) &= z, \quad h_r(w) = w \end{aligned} \quad (140)$$

for $i = 1, \dots, M, k = 1, \dots, N, z \in \mathcal{B}_1, w \in \mathcal{B}_2$, and $r \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$. As the C^* -algebra $\mathcal{O}_{\mathcal{H}}$ has the largest norm on $\mathcal{P}_{\mathcal{H}}$, the action $(h_r)_{r \in \mathbb{T}}$ on $\mathcal{P}_{\mathcal{H}}$ extends to an action of $\mathbb{T}$ on $\mathcal{O}_{\mathcal{H}}$, still denoted by h . The formula

$$a \in \mathcal{O}_{\mathcal{H}} \longrightarrow \int_{r \in \mathbb{T}} h_r(a) dr \in \mathcal{O}_{\mathcal{H}}, \quad (141)$$

where dr is the normalized Lebesgue measure on $\mathbb{T}$ defines a faithful conditional expectation denoted by $\mathcal{E}_{\mathcal{H}}$ from $\mathcal{O}_{\mathcal{H}}$ onto the fixed-point algebra $(\mathcal{O}_{\mathcal{H}})^h$. The following lemma is routine.

Lemma 30. $(\mathcal{O}_{\mathcal{H}})^h = \mathcal{F}_{\mathcal{H}}$.

The C^* -algebra $\mathcal{O}_{\mathcal{H}}$ satisfies the following universal property. Let $\mathcal{D}$ be a unital C^* -algebra and $\Phi_1 : \mathcal{B}_1 \rightarrow \mathcal{D}, \Phi_2 : \mathcal{B}_2 \rightarrow \mathcal{D}$ be $*$ -homomorphisms such that

$\Phi_1(a) = \Phi_2(a)$ for $a \in \mathcal{A}$. Assume that there exist elements $\hat{S}_1, \dots, \hat{S}_M, \hat{T}_1, \dots, \hat{T}_N$ in $\mathcal{D}$ satisfying the relations

$$\begin{aligned} \sum_{i=1}^M \hat{S}_i \hat{S}_i^* + \sum_{k=1}^N \hat{T}_k \hat{T}_k^* &= 1, \quad \hat{S}_j^* \hat{T}_l = 0, \\ \hat{S}_i^* \hat{S}_j &= \Phi_1 \left(\langle u_i | u_j \rangle_{\mathcal{B}_1} \right), \quad \hat{T}_k^* \hat{T}_l = \Phi_2 \left(\langle v_k | v_l \rangle_{\mathcal{B}_2} \right), \\ \Phi_1(z) \hat{S}_j &= \sum_{i=1}^M \hat{S}_i \Phi_1 \left(\langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \right), \\ \Phi_1(z) \hat{T}_l &= \sum_{k=1}^N \hat{T}_k \Phi_2 \left(\langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} \right), \\ \Phi_2(w) \hat{S}_j &= \sum_{i=1}^M \hat{S}_i \Phi_1 \left(\langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1} \right), \\ \Phi_2(w) \hat{T}_l &= \sum_{k=1}^N \hat{T}_k \Phi_2 \left(\langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2} \right), \end{aligned} \quad (142)$$

for $z \in \mathcal{B}_1, w \in \mathcal{B}_2, i, j = 1, \dots, M, k, l = 1, \dots, N$; then, there exists a unique $*$ -homomorphism $\Phi : \mathcal{O}_{\mathcal{H}} \rightarrow \mathcal{D}$ such that

$$\begin{aligned} \Phi(\hat{S}_i) &= \hat{S}_i, \quad \Phi(\hat{T}_k) = \hat{T}_k, \\ \Phi(z) &= \Phi_1(z), \quad \Phi(w) = \Phi_2(w) \end{aligned} \quad (143)$$

for $i = 1, \dots, M, k = 1, \dots, N$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. We further assume that both the homomorphisms $\Phi_i : \mathcal{B}_i \rightarrow \mathcal{D}$, $i = 1, 2$ are injective. We denote by $\Phi_o : \mathcal{B}_o \rightarrow \mathcal{D}$ the restriction of Φ to the subalgebra $\mathcal{B}_o$. Let us denote by $\widehat{\mathcal{O}}_{\mathcal{H}}$ the C^* -subalgebra of $\mathcal{D}$ generated by $\hat{S}_i, \hat{T}_k, i = 1, \dots, M, k = 1, \dots, N$ and $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.

Lemma 31. *Keep the above situation. The $*$ -homomorphism $\Phi_o : \mathcal{B}_o \rightarrow \mathcal{D}$ is injective.*

Proof. Since the correspondence in Lemma 26

$$\phi_o : z, w \in \mathcal{B}_o \longrightarrow \phi_1(z), \phi_2(w) \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \quad (144)$$

yields an isomorphism of C^* -algebras, it suffices to prove that the correspondence

$$\begin{aligned} \phi_1(z), \phi_2(w) &\in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \\ &\longrightarrow \Phi_1(z), \Phi_2(w) \in \mathcal{D} \end{aligned} \quad (145)$$

yields an isomorphism. Let $\widehat{\mathcal{B}}_o$ be the C^* -subalgebra of $\widehat{\mathcal{O}}_{\mathcal{H}}$ generated by elements $\Phi_1(z), \Phi_2(w) \in \mathcal{A}$ for $z \in \mathcal{B}_1,$

$w \in \mathcal{B}_2$. Define an element $\hat{\pi}(L)$ of $\mathcal{D}$ for $L \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ by setting

$$\begin{aligned} \hat{\pi}(L) &= \sum_{i,j=1}^M \hat{S}_i \Phi_1 \left(\langle u_i | Lu_j \rangle_{\mathcal{B}_1} \right) \hat{S}_j^* \\ &+ \sum_{k,l=1}^N \hat{T}_k \Phi_2 \left(\langle v_k | Lv_l \rangle_{\mathcal{B}_2} \right) \hat{T}_l^* \in \mathcal{D}. \end{aligned} \quad (146)$$

As in the proof of Lemma 26, one sees that $\hat{\pi}$ gives rise to a $*$ -homomorphism from $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ into $\mathcal{D}$. Since

$$\begin{aligned} \hat{\pi}(\phi_1(z)) &= \sum_{i,j=1}^M \hat{S}_i \Phi_1 \left(\langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1} \right) \hat{S}_j^* \\ &+ \sum_{k,l=1}^N \hat{T}_k \Phi_2 \left(\langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2} \right) \hat{T}_l^* \\ &= \Phi_1(z), \end{aligned} \quad (147)$$

and similarly $\hat{\pi}(\phi_2(w)) = \Phi_2(w)$, it is enough to show that $\hat{\pi}$ is injective. Suppose that $\hat{\pi}(L) = 0$ for some $L \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$. By following the proof of Lemma 26, one sees that $\hat{S}_h^* \hat{\pi}(L) \hat{S}_{h'} = \Phi_1(\langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1})$ for all $h, h' = 1, \dots, M$. Hence, the condition $\hat{\pi}(L) = 0$ implies that $\Phi_1(\langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1}) = 0$. Since Φ_1 is injective, we have $\langle u_h | Lu_{h'} \rangle_{\mathcal{B}_1} = 0$ for all $h, h' = 1, \dots, M$. As L is a right $\mathcal{B}_1$ -module map, we have for $\xi \in \mathcal{H}$,

$$L\xi = \sum_{h'=1}^M L(u_{h'} \langle u_{h'} | \xi \rangle_{\mathcal{B}_1}) = \sum_{h'=1}^M (Lu_{h'}) \langle u_{h'} | \xi \rangle_{\mathcal{B}_1} = 0 \quad (148)$$

so that $L = 0$. Therefore, $\hat{\pi} : C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \rightarrow \mathcal{D}$ is injective. Hence, the composition

$$\Phi_o : \hat{\pi} \circ \phi_o : \mathcal{B}_o \xrightarrow{\phi_o} C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \xrightarrow{\hat{\pi}} \mathcal{D} \quad (149)$$

is injective. $\square$

We set

$$\begin{aligned} \hat{S}_{1,i} &:= \hat{S}_i, \quad i = 1, \dots, M, \\ \hat{S}_{2,k} &:= \hat{T}_k, \quad k = 1, \dots, N. \end{aligned} \quad (150)$$

We put $\widehat{\mathcal{F}}_{\mathcal{H}}^0 = \mathcal{B}_o$. For $n \in \mathbb{N}$, let $\widehat{\mathcal{F}}_{\mathcal{H}}^n$ be the closed linear span in the C^* -algebra $\widehat{\mathcal{O}}_{\mathcal{H}}$ of elements of the form

$$\hat{S}_{g_1, i_1} \hat{S}_{g_2, i_2} \cdots \hat{S}_{g_n, i_n} \Phi_o(b) \hat{S}_{h_n, j_n}^* \cdots \hat{S}_{h_2, j_2}^* \hat{S}_{h_1, j_1}^* \quad (151)$$

for $(g_1, i_1), (g_2, i_2), \dots, (g_n, i_n), (h_1, j_1), (h_2, j_2), \dots, (h_n, j_n) \in \Sigma_1 \cup \Sigma_2$ and $b \in \mathcal{B}_o$. Similar to the subalgebras $\mathcal{F}_{\mathcal{H}}^n, n \in \mathbb{Z}_+$, of $\mathcal{O}_{\mathcal{H}}$, one knows that the closed linear span $\widehat{\mathcal{F}}_{\mathcal{H}}^n$ is a C^* -algebra and naturally regarded as a subalgebra of $\widehat{\mathcal{F}}_{\mathcal{H}}^{n+1}$ for each $n \in \mathbb{Z}_+$. Let us denote by $\widehat{\mathcal{F}}_{\mathcal{H}}$ the C^* -subalgebra of $\widehat{\mathcal{O}}_{\mathcal{H}}$ generated

by $\cup_{n=0}^{\infty} \widehat{\mathcal{F}}_{\mathcal{H}}^n$. Then, the C^* -algebra $\widehat{\mathcal{F}}_{\mathcal{H}}$ is the inductive limit $\lim_{n \rightarrow \infty} \widehat{\mathcal{F}}_{\mathcal{H}}^n$ of the sequence of the inclusions

$$\begin{aligned} \widehat{\mathcal{F}}_{\mathcal{H}}^0 &\hookrightarrow \widehat{\mathcal{F}}_{\mathcal{H}}^1 \hookrightarrow \widehat{\mathcal{F}}_{\mathcal{H}}^2 \hookrightarrow \dots \hookrightarrow \widehat{\mathcal{F}}_{\mathcal{H}}^n \hookrightarrow \widehat{\mathcal{F}}_{\mathcal{H}}^{n+1} \\ &\hookrightarrow \dots \hookrightarrow \widehat{\mathcal{F}}_{\mathcal{H}}. \end{aligned} \quad (152)$$

Lemma 32. Suppose that both the $*$ -homomorphisms $\Phi_i : \mathcal{B}_i \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}}$, $i = 1, 2$ are injective. Then, the restriction of Φ to the subalgebra $\mathcal{F}_{\mathcal{H}}$ yields a $*$ -isomorphism $\Phi|_{\mathcal{F}_{\mathcal{H}}} : \mathcal{F}_{\mathcal{H}} \rightarrow \widehat{\mathcal{F}}_{\mathcal{H}}$.

Proof. By the universality of $\widehat{\mathcal{O}}_{\mathcal{H}}$, the restriction of Φ to $\mathcal{F}_{\mathcal{H}}$ yields a surjective $*$ -homomorphism $\Phi|_{\mathcal{F}_{\mathcal{H}}} : \mathcal{F}_{\mathcal{H}} \rightarrow \widehat{\mathcal{F}}_{\mathcal{H}}$. It suffices to show that $\Phi|_{\mathcal{F}_{\mathcal{H}}}$ is injective. Suppose that $\text{Ker}(\Phi|_{\mathcal{F}_{\mathcal{H}}}) \neq \{0\}$ and put $I = \text{Ker}(\Phi|_{\mathcal{F}_{\mathcal{H}}})$. Since $\Phi(\mathcal{F}_{\mathcal{H}}^n) = \widehat{\mathcal{F}}_{\mathcal{H}}^n$ and $\mathcal{F}_{\mathcal{H}} = \lim_{n \rightarrow \infty} \mathcal{F}_{\mathcal{H}}^n$, there exists $n \in \mathbb{Z}_+$ such that $I \cap \mathcal{F}_{\mathcal{H}}^n \neq \{0\}$. Let us denote by Σ^n the set of n -tuples of $\Sigma_1 \cup \Sigma_2$:

$$\Sigma^n = \{(\mu_1, \dots, \mu_n) \mid \mu_1, \dots, \mu_n \in \Sigma_1 \cup \Sigma_2\}. \quad (153)$$

For $\mu = (\mu_1, \dots, \mu_n) \in \Sigma^n$, denote by S_{μ} the operator

$$S_{\mu} = S_{\mu_1} S_{\mu_2} \dots S_{\mu_n}, \quad (154)$$

where

$$S_{\mu_m} = \begin{cases} S_{1,i} = S_i & \text{if } \mu_m = (1, i) \in \Sigma_1, \\ S_{2,k} = T_k & \text{if } \mu_m = (2, k) \in \Sigma_2, \quad m = 1, \dots, n. \end{cases} \quad (155)$$

Any element of $\mathcal{F}_{\mathcal{H}}^n$ is of the form

$$\sum_{\mu, \nu \in \Sigma^n} S_{\mu} b_{\mu, \nu} S_{\nu}^* \quad \text{for some } b_{\mu, \nu} \in \mathcal{B}_0. \quad (156)$$

Hence, one may find a nonzero element $\sum_{\mu, \nu \in \Sigma^n} S_{\mu} b_{\mu, \nu} S_{\nu}^* \in I \cap \mathcal{F}_{\mathcal{H}}^n$. Since $\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1$, the equality $\sum_{\mu \in \Sigma^n} S_{\mu} S_{\mu}^* = 1$ holds. For some $\omega, \gamma \in \Sigma^n$, one then sees

$$0 \neq S_{\omega}^* \left(\sum_{\mu, \nu \in \Sigma^n} S_{\mu} b_{\mu, \nu} S_{\nu}^* \right) S_{\gamma} \in I \cap \mathcal{F}_{\mathcal{H}}^n. \quad (157)$$

As $S_i^* T_k = 0$ and $S_i^* S_j = \langle u_i \mid u_j \rangle_{\mathcal{B}_1}$, $T_k^* T_l = \langle v_k \mid v_l \rangle_{\mathcal{B}_2}$ for $i, j = 1, \dots, M$, $k, l = 1, \dots, N$, the element $S_{\omega}^* (\sum_{\mu, \nu \in \Sigma^n} S_{\mu} b_{\mu, \nu} S_{\nu}^*) S_{\gamma}$ belongs to $I \cap \mathcal{B}_0$. By the preceding lemma, the homomorphism $\Phi_0 : \mathcal{B}_0 \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}}$ is injective, so that we have $\Phi_0(S_{\omega}^* (\sum_{\mu, \nu \in \Sigma^n} S_{\mu} b_{\mu, \nu} S_{\nu}^*) S_{\gamma}) \neq 0$ a contradiction. Therefore we conclude that $\Phi|_{\mathcal{F}_{\mathcal{H}}} : \mathcal{F}_{\mathcal{H}} \rightarrow \widehat{\mathcal{F}}_{\mathcal{H}}$ is injective and hence isomorphic. $\square$

The following theorem is one of the main results of the paper.

Theorem 33. Let $\mathcal{D}$ be a unital C^* -algebra. Suppose that there exist $*$ -homomorphisms $\Phi_1 : \mathcal{B}_1 \rightarrow \mathcal{D}$, $\Phi_2 : \mathcal{B}_2 \rightarrow \mathcal{D}$

such that $\Phi_1(a) = \Phi_2(a)$ for $a \in \mathcal{A}$ and there exist elements $\widehat{S}_1, \dots, \widehat{S}_M, \widehat{T}_1, \dots, \widehat{T}_N$ in $\mathcal{D}$ satisfying the relations

$$\sum_{i=1}^M \widehat{S}_i \widehat{S}_i^* + \sum_{k=1}^N \widehat{T}_k \widehat{T}_k^* = 1, \quad S_j^* T_l = 0,$$

$$\widehat{S}_i^* \widehat{S}_j = \Phi_1(\langle u_i \mid u_j \rangle_{\mathcal{B}_1}),$$

$$\widehat{T}_k^* \widehat{T}_l = \Phi_2(\langle v_k \mid v_l \rangle_{\mathcal{B}_2}),$$

$$\Phi_1(z) \widehat{S}_j = \sum_{i=1}^M \widehat{S}_i \Phi_1(\langle u_i \mid \phi_1(z) u_j \rangle_{\mathcal{B}_1}), \quad (158)$$

$$\Phi_1(z) \widehat{T}_l = \sum_{k=1}^N \widehat{T}_k \Phi_2(\langle v_k \mid \phi_1(z) v_l \rangle_{\mathcal{B}_2}),$$

$$\Phi_2(w) \widehat{S}_j = \sum_{i=1}^M \widehat{S}_i \Phi_1(\langle u_i \mid \phi_2(w) u_j \rangle_{\mathcal{B}_1}),$$

$$\Phi_2(w) \widehat{T}_l = \sum_{k=1}^N \widehat{T}_k \Phi_2(\langle v_k \mid \phi_2(w) v_l \rangle_{\mathcal{B}_2})$$

for $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$, $i, j = 1, \dots, M$, $k, l = 1, \dots, N$. Let us denote by $\widehat{\mathcal{O}}_{\mathcal{H}}$ the C^* -subalgebra of $\mathcal{D}$ generated by $\widehat{S}_i, \widehat{T}_k$, $i = 1, \dots, M$, $k = 1, \dots, N$ and $\Phi_1(z), \Phi_2(w)$, for $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$. One further assumes that the algebra $\widehat{\mathcal{O}}_{\mathcal{H}}$ admits a gauge action. If both the $*$ -homomorphisms $\Phi_i : \mathcal{B}_i \rightarrow \mathcal{A}$, $i = 1, 2$ are injective, then there exists a $*$ -isomorphism $\Phi : \widehat{\mathcal{O}}_{\mathcal{H}} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}}$ satisfying

$$\begin{aligned} \Phi(S_i) &= \widehat{S}_i, & \Phi(T_k) &= \widehat{T}_k, \\ \Phi(z) &= \Phi_1(z), & \Phi(w) &= \Phi_2(w) \end{aligned} \quad (159)$$

for $i = 1, \dots, M$, $k = 1, \dots, N$ and $z \in \mathcal{B}_1$, $w \in \mathcal{B}_2$.

Proof. By assumption, $\widehat{\mathcal{O}}_{\mathcal{H}}$ admits a gauge action, which we denote by $\hat{h}$. Let us denote by $(\widehat{\mathcal{O}}_{\mathcal{H}})^{\hat{h}}$ the fixed-point algebra of $\widehat{\mathcal{O}}_{\mathcal{H}}$ under the gauge action $\hat{h}$ and by $\widehat{\mathcal{F}}_{\mathcal{H}}$ the C^* -subalgebra of $\widehat{\mathcal{O}}_{\mathcal{H}}$ defined by the inductive limit (152). Then, it is routine to check that $(\widehat{\mathcal{O}}_{\mathcal{H}})^{\hat{h}}$ is canonically $*$ -isomorphic to $\widehat{\mathcal{F}}_{\mathcal{H}}$. There exists a conditional expectation

$$\widehat{\mathcal{E}}_{\mathcal{H}} : \widehat{\mathcal{O}}_{\mathcal{H}} \longrightarrow \widehat{\mathcal{F}}_{\mathcal{H}} \quad (160)$$

defined by

$$\widehat{\mathcal{E}}_{\mathcal{H}}(x) = \int_{r \in \mathbb{T}} \hat{h}_r(x) dr \quad \text{for } x \in \widehat{\mathcal{O}}_{\mathcal{H}}. \quad (161)$$

By the universality of the algebra $\widehat{\mathcal{O}}_{\mathcal{H}}$, there exists a surjective $*$ -homomorphism Φ from $\widehat{\mathcal{O}}_{\mathcal{H}}$ to $\widehat{\mathcal{O}}_{\mathcal{H}}$ such that

$$\begin{aligned} \Phi(S_i) &= \widehat{S}_i, & \Phi(T_k) &= \widehat{T}_k, \\ \Phi(z) &= \Phi_1(z), & \Phi(w) &= \Phi_2(w) \end{aligned} \quad (162)$$

for $i, j = 1, \dots, M, k, l = 1, \dots, N, z \in \mathcal{B}_1, w \in \mathcal{B}_2$. Then, $\Phi(\mathcal{F}_{\mathcal{H}}) = \widehat{\mathcal{F}}_{\mathcal{H}}$ and the following diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{H}} & \xrightarrow{\Phi} & \widehat{\mathcal{O}}_{\mathcal{H}} \\ \varepsilon_{\mathcal{H}} \downarrow & & \downarrow \widehat{\varepsilon}_{\mathcal{H}} \\ \mathcal{F}_{\mathcal{H}} & \xrightarrow{\Phi|_{\mathcal{F}_{\mathcal{H}}}} & \widehat{\mathcal{F}}_{\mathcal{H}} \end{array} \quad (163)$$

is commutative. Denote by Φ_0 the restriction of Φ to the C^* -subalgebra $\mathcal{B}_0$ of $\mathcal{O}_{\mathcal{H}}$ generated by $z \in \mathcal{B}_1, w \in \mathcal{B}_2$. By assumption, both the maps $\Phi_i : \mathcal{B}_i \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}}, i = 1, 2$ are injective, so that $\Phi_0 : \mathcal{B}_0 \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}}$ is injective by Lemma 31. By the preceding lemma, $\Phi|_{\mathcal{F}_{\mathcal{H}}} : \mathcal{F}_{\mathcal{H}} \rightarrow \widehat{\mathcal{F}}_{\mathcal{H}}$ is an isomorphism. Since the conditional expectation $\mathcal{E}_{\mathcal{H}} : \mathcal{O}_{\mathcal{H}} \rightarrow \mathcal{F}_{\mathcal{H}}$ is faithful, a routine argument shows that Φ is injective and hence isomorphic. $\square$

Therefore, we have the following.

Theorem 34. *For a C^* -quad module $\mathcal{H}$ of finite type, the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ generated by the quotients $[s_{\xi}], [t_{\xi}]$ of the creation operators s_{ξ}, t_{ξ} for $\xi \in \mathcal{H}$ on the Fock spaces $F(\mathcal{H})$ is canonically isomorphic to the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by operators $S_1, \dots, S_M, T_1, \dots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations*

$$\begin{aligned} \sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* &= 1, & S_j^* T_l &= 0, \\ S_i^* S_j &= \langle u_i | u_j \rangle_{\mathcal{B}_1}, & T_k^* T_l &= \langle v_k | v_l \rangle_{\mathcal{B}_2}, \\ z S_j &= \sum_{i=1}^M S_i \langle u_i | \phi_1(z) u_j \rangle_{\mathcal{B}_1}, \\ z T_l &= \sum_{k=1}^N T_k \langle v_k | \phi_1(z) v_l \rangle_{\mathcal{B}_2}, \\ w S_j &= \sum_{i=1}^M S_i \langle u_i | \phi_2(w) u_j \rangle_{\mathcal{B}_1}, \\ w T_l &= \sum_{k=1}^N T_k \langle v_k | \phi_2(w) v_l \rangle_{\mathcal{B}_2} \end{aligned} \quad (164)$$

for $i, j = 1, \dots, M, k, l = 1, \dots, N$ and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$.

Proof. Theorem 17 implies that the operators $S_1, \dots, S_M, T_1, \dots, T_N$ and the elements $\Phi_1(z), \Phi_2(w)$ for $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ in $\mathcal{O}_{F(\mathcal{H})}$ satisfy the eight relations of Theorem 33. By Theorem 33, we see that the correspondences

$$\begin{aligned} S_i &\longrightarrow S_i, & T_k &\longrightarrow T_k, \\ z \in \mathcal{B}_1 &\longrightarrow \Phi_1(z), & w \in \mathcal{B}_2 &\longrightarrow \Phi_2(w) \end{aligned} \quad (165)$$

for $i = 1, \dots, M, k = 1, \dots, N$, and $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ give rise to an isomorphism from $\mathcal{O}_{\mathcal{H}}$ to $\mathcal{O}_{F(\mathcal{H})}$. $\square$

The eight relations of the operators above are called the relations $(\mathcal{H})$. The above generating operators $S_1, \dots, S_M$ and $T_1, \dots, T_N$ of the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ correspond to two finite bases $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$ of the Hilbert C^* -quad module $\mathcal{H}$, respectively. On the other hand, the other C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ is generated by the quotients of the creation operators s_{ξ}, t_{ξ} for $\xi \in \mathcal{H}$ on the Fock spaces $F(\mathcal{H})$, which do not depend on the choice of the two finite bases. Hence, we have the following.

Corollary 35. *For a C^* -quad module $\mathcal{H}$ of finite type, the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by operators $S_1, \dots, S_M, T_1, \dots, T_N$ and elements $z \in \mathcal{B}_1, w \in \mathcal{B}_2$ subject to the relations $(\mathcal{H})$ does not depend on the choice of the finite bases $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$.*

6. K -Theory Formulae

Let $\mathcal{H}$ be a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ of finite type as in the preceding section. In this section, we will state K -theory formulae for the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$. By the previous section, the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ is regarded as the universal C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by the operators $S_1, \dots, S_M$ and $T_1, \dots, T_N$ and the elements $z \in \mathcal{B}_1$ and $w \in \mathcal{B}_2$ subject to the relations $(\mathcal{H})$. Let us denote by $\mathcal{B}_0$ the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}}$ generated by elements $z \in \mathcal{B}_1$ and $w \in \mathcal{B}_2$. By Lemma 26, the correspondence

$$z, w \in \mathcal{B}_0 \longrightarrow \phi_1(z), \quad (166)$$

$$\phi_2(w) \in C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2)) \subset \mathcal{L}_{\mathcal{A}}(\mathcal{H})$$

gives rise to a $*$ -isomorphism from $\mathcal{B}_0$ onto $C^*(\phi_1(\mathcal{B}_1), \phi_2(\mathcal{B}_2))$ as C^* -algebras, which is denoted by ϕ_0 . We will restrict our interest to the case when

- (i) $S_1, \dots, S_M$ and $T_1, \dots, T_N$ are partial isometries, and
- (ii) $S_1 S_1^*, \dots, S_M S_M^*, T_1 T_1^*, \dots, T_N T_N^*$ commute with all elements of $\mathcal{B}_0$.

If the bases $\{u_1, \dots, u_M\}$ and $\{v_1, \dots, v_N\}$ satisfy the conditions

$$\begin{aligned} \langle u_i | u_j \rangle_{\mathcal{B}_1} &= 0 \quad \text{for } i \neq j, \\ \langle v_k | v_l \rangle_{\mathcal{B}_2} &= 0 \quad \text{for } k \neq l, \end{aligned} \quad (167)$$

the condition (i) holds. Furthermore, if $\phi_1(z)$ acts diagonally on $\{u_1, \dots, u_M\}$ for $z \in \mathcal{B}_1$ and $\phi_2(w)$ acts diagonally on $\{v_1, \dots, v_N\}$ for $w \in \mathcal{B}_2$, the condition (ii) holds. Recall that the gauge action is denoted by h which is an action of $\mathbb{T}$ on $\mathcal{O}_{\mathcal{H}}$ such that the fixed-point algebra $(\mathcal{O}_{\mathcal{H}})^h$ under h is canonically isomorphic to the C^* -algebra $\mathcal{F}_{\mathcal{H}}$. Denote by $\hat{h}$ the dual action of h which is an action of $\mathbb{Z} = \widehat{\mathbb{T}}$ on the C^* -crossed product $\mathcal{O}_{\mathcal{H}} \rtimes_h \mathbb{T}$ by the gauge action h of $\mathbb{T}$. As in the argument of [16], $\mathcal{O}_{\mathcal{H}} \rtimes_h \mathbb{T}$ is stably isomorphic to $\mathcal{F}_{\mathcal{H}}$. Hence, we have that $K_*(\mathcal{O}_{\mathcal{H}} \rtimes_h \mathbb{T})$ is isomorphic to $K_*(\mathcal{F}_{\mathcal{H}})$. The dual action $\hat{h}$ induces an automorphism on the group $K_*(\mathcal{O}_{\mathcal{H}} \rtimes_h \mathbb{T})$ and hence on $K_*(\mathcal{F}_{\mathcal{H}})$, which is denoted by σ_* . Then, by [16] (cf. [10, 17], etc.), we have the following.

Proposition 36. *The following six-term exact sequence of K -theory holds:*

$$\begin{array}{ccccc} K_0(\mathcal{F}\mathcal{H}) & \xrightarrow{\text{id}-\sigma_*} & K_0(\mathcal{F}\mathcal{H}) & \xrightarrow{\iota_*} & K_0(\mathcal{O}\mathcal{H}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}\mathcal{H}) & \xleftarrow{\iota_*} & K_1(\mathcal{F}\mathcal{H}) & \xleftarrow{\text{id}-\sigma_*} & K_1(\mathcal{F}\mathcal{H}) \end{array} \quad (168)$$

We put for $x \in \mathcal{B}_0$,

$$\begin{aligned} \lambda_{1,i}(x) &= S_i^* x S_i, \quad i = 1, \dots, M, \\ \lambda_{2,k}(x) &= T_k^* x T_k, \quad k = 1, \dots, N. \end{aligned} \quad (169)$$

Both the families $\lambda_{1,i}, \lambda_{2,k}$ yield endomorphisms on $\mathcal{B}_0$ which give rise to endomorphisms on the K -groups:

$$\begin{aligned} \lambda_{1,i*} : K_0(\mathcal{B}_0) &\longrightarrow K_0(\mathcal{B}_0), \quad i = 1, \dots, M, \\ \lambda_{2,k*} : K_0(\mathcal{B}_0) &\longrightarrow K_0(\mathcal{B}_0), \quad k = 1, \dots, N. \end{aligned} \quad (170)$$

We put $\lambda_0 = \sum_{i=1}^M \lambda_{1,i*} + \sum_{k=1}^N \lambda_{2,k*}$ which is an endomorphism on $K_0(\mathcal{B}_0)$. Now, we further assume that $K_1(\mathcal{F}\mathcal{H}) = \{0\}$. It is routine to show that the groups $\text{Coker}(\text{id} - \sigma_*)$ in $K_0(\mathcal{F}\mathcal{H})$ and $\text{Ker}(\text{id} - \sigma_*)$ in $K_0(\mathcal{F}\mathcal{H})$ are isomorphic to the groups $\text{Coker}(\text{id} - \lambda_0)$ in $K_0(\mathcal{B}_0)$, and $\text{Ker}(\text{id} - \lambda_0)$ in $K_0(\mathcal{B}_0)$, respectively, by an argument of [17]. Therefore, we have

Proposition 37. *The following formulae hold:*

$$\begin{aligned} K_0(\mathcal{O}\mathcal{H}) &= \text{Coker}(\text{id} - \lambda_0) \text{ in } K_0(\mathcal{B}_0), \\ K_1(\mathcal{O}\mathcal{H}) &= \text{Ker}(\text{id} - \lambda_0) \text{ in } K_0(\mathcal{B}_0). \end{aligned} \quad (171)$$

7. Examples

In this section, we will study the C^* -algebras $\mathcal{O}\mathcal{H}$ for the Hilbert C^* -quad modules presented in Examples in Section 2.

(1) Let α, β be automorphisms of a unital C^* -algebra $\mathcal{A}$ satisfying $\alpha \circ \beta = \beta \circ \alpha$. Let $\mathcal{H}_{\alpha, \beta}$ be the associated Hilbert C^* -quad module of finite type as in (1) in Section 2. It is easy to see the following proposition.

Proposition 38. *The C^* -algebra $\mathcal{O}\mathcal{H}_{\alpha, \beta}$ associated with the Hilbert C^* -quad module $\mathcal{H}_{\alpha, \beta}$ coming from commuting automorphisms α, β of a unital C^* -algebra $\mathcal{A}$ is isomorphic to the universal C^* -algebra generated by two isometries U, V and elements x of $\mathcal{A}$ subject to the following relations:*

$$\begin{aligned} UU^* + VV^* &= 1, \\ UU^*x &= xUU^*, \quad VV^*x = xVV^*, \\ \alpha(x) &= U^*xU, \quad \beta(x) = V^*xV \end{aligned} \quad (172)$$

for $x \in \mathcal{A}$.

(2) We fix natural numbers $1 < N, M \in \mathbb{N}$. Consider finite-dimensional commutative C^* -algebras $\mathcal{A} = \mathbb{C}$,

$\mathcal{B}_1 = \mathbb{C}^N$, and $\mathcal{B}_2 = \mathbb{C}^M$. The algebras $\mathcal{B}_1, \mathcal{B}_2$ have the ordinary product structure and the inner product structure which we denote by $\langle \cdot | \cdot \rangle_N$ and $\langle \cdot | \cdot \rangle_M$, respectively. Let us denote by $\mathcal{H}_{M,N}$ the Hilbert C^* -quad module $\mathbb{C}^M \otimes \mathbb{C}^N$ over $(\mathbb{C}; \mathbb{C}^N, \mathbb{C}^M)$ defined in (2) in Section 2. Put the finite bases

$$u_i = e_i \otimes 1 \in \mathcal{H}_{M,N}, \quad i = 1, \dots, M$$

as a right $\mathcal{B}_1$ -module,

$$v_k = 1 \otimes f_k \in \mathcal{H}_{M,N}, \quad k = 1, \dots, N$$

as a right $\mathcal{B}_2$ -module.

We set $\Sigma^\circ = \{(i, k) \mid 1 \leq i \leq M, 1 \leq k \leq N\}$ and put $e_{(i,k)} = e_i \otimes f_k$, $(i, k) \in \Sigma^\circ$ the standard basis of $\mathcal{H}_{M,N}$. Then, the C^* -algebra $\mathcal{B}_0$ on $\mathcal{H}_{M,N}$ generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ is regarded as $\mathbb{C}^M \otimes \mathbb{C}^N = \mathcal{B}_2 \otimes \mathcal{B}_1$. Hence,

$$\mathcal{B}_0 = \sum_{(i,k) \in \Sigma^\circ} \mathbb{C} e_{(i,k)}. \quad (174)$$

Lemma 39. *The C^* -algebra $\mathcal{O}\mathcal{H}_{M,N}$ is generated by operators $S_i, T_k, e_{(i,k)}$, $i = 1, \dots, M$, $k = 1, \dots, N$ satisfying*

$$\sum_{i=1}^M S_i S_i^* + \sum_{k=1}^N T_k T_k^* = 1, \quad (175)$$

$$S_i^* S_j = \delta_{i,j}, \quad T_k^* T_l = \delta_{k,l}, \quad (176)$$

$$e_{(i,k)} S_j = \delta_{i,j} \sum_{h=1}^M S_j e_{(h,k)}, \quad e_{(i,k)} T_l = \delta_{k,l} \sum_{m=1}^N T_l e_{(i,m)} \quad (177)$$

for $i, j = 1, \dots, M$, $k, l = 1, \dots, N$.

Proof. It suffices to show the equalities (177). We have

$$\begin{aligned} e_{(i,k)} S_j &= S_j \langle u_j \mid \phi_0(e_{(i,k)}) u_j \rangle_{\mathcal{B}_1} \\ &= S_j \langle e_j \otimes 1 \mid (e_i \otimes f_k)(e_j \otimes 1) \rangle_{\mathcal{B}_1} \\ &= \delta_{i,j} S_j (1 \otimes f_k) = \delta_{i,j} S_j \sum_{h=1}^M e_{(h,k)}. \end{aligned} \quad (178)$$

The other equality of (177) is similarly shown. $\square$

Put

$$S_{(i,k)} = e_{(i,k)} S_i, \quad T_{(i,k)} = e_{(i,k)} T_k \quad \text{for } (i, k) \in \Sigma^\circ. \quad (179)$$

Then we have the following.

Lemma 40. *The following equalities hold:*

$$e_{(i,k)} = S_{(i,k)} S_{(i,k)}^* + T_{(i,k)} T_{(i,k)}^*, \quad (180)$$

$$S_i = \sum_{k=1}^N S_{(i,k)}, \quad T_k = \sum_{i=1}^M T_{(i,k)}, \quad (181)$$

$$\sum_{(i,k) \in \Sigma^\circ} S_{(i,k)} S_{(i,k)}^* + \sum_{(i,k) \in \Sigma^\circ} T_{(i,k)} T_{(i,k)}^* = 1, \quad (182)$$

$$S_{(i,k)}^* S_{(i,k)} = \sum_{j=1}^M (S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*), \quad (183)$$

$$T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^N (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*) \quad (184)$$

for $i = 1, \dots, M$, $k = 1, \dots, N$ and $(i, k) \in \Sigma^\circ$.

Proof. Since $e_{(i,k)} S_j = \delta_{i,j} e_{(i,k)} S_i$, we have

$$S_{(i,k)} S_{(i,k)}^* = e_{(i,k)} S_i S_i^* e_{(i,k)} = e_{(i,k)} \left(\sum_{j=1}^M S_j S_j^* \right) e_{(i,k)} \quad (185)$$

and similarly $T_{(i,k)} T_{(i,k)}^* = e_{(i,k)} (\sum_{l=1}^N T_l T_l^*) e_{(i,k)}$. Hence, we have

$$\begin{aligned} e_{(i,k)} &= e_{(i,k)} \left(\sum_{j=1}^M S_j S_j^* + \sum_{l=1}^N T_l T_l^* \right) e_{(i,k)} \\ &= S_{(i,k)} S_{(i,k)}^* + T_{(i,k)} T_{(i,k)}^*, \end{aligned} \quad (186)$$

so that (180) holds. As $1 = \sum_{(j,k) \in \Sigma^\circ} e_{(j,k)}$, the equality (182) holds. Since $e_{(j,k)} S_i = 0$ for $j \neq i$, we have

$$S_i = \left(\sum_{(j,k) \in \Sigma^\circ} e_{(j,k)} \right) S_i = \sum_{k=1}^N S_{(i,k)} \quad (187)$$

and similarly $T_k = \sum_{i=1}^M T_{(i,k)}$, so that (181) holds. By (177), it follows that

$$\begin{aligned} S_{(i,k)}^* S_{(i,k)} &= S_i^* \sum_{j=1}^M S_i e_{(j,k)} = \sum_{j=1}^M e_{(j,k)} \\ &= \sum_{j=1}^M (S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*), \end{aligned} \quad (188)$$

and similarly, we have

$$T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^N (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*). \quad (189)$$

□

Theorem 41. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_{M,N}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_{M,N} = \mathbb{C}^M \otimes \mathbb{C}^N$ is generated by*

partial isometries $S_{(i,k)}, T_{(i,k)}$ for $(i, k) \in \Sigma^\circ = \{(i, k) \mid i = 1, \dots, M, k = 1, \dots, N\}$ satisfying the relations

$$\sum_{(i,k) \in \Sigma^\circ} S_{(i,k)} S_{(i,k)}^* + \sum_{(i,k) \in \Sigma^\circ} T_{(i,k)} T_{(i,k)}^* = 1,$$

$$S_{(i,k)}^* S_{(i,k)} = \sum_{j=1}^M (S_{(j,k)} S_{(j,k)}^* + T_{(j,k)} T_{(j,k)}^*), \quad (190)$$

$$T_{(i,k)}^* T_{(i,k)} = \sum_{l=1}^N (S_{(i,l)} S_{(i,l)}^* + T_{(i,l)} T_{(i,l)}^*)$$

for $(i, k) \in \Sigma^\circ$.

Proof. By the preceding lemma, one knows that $e_{(i,k)}, S_i, T_k$ are generated by the operators $S_{(i,k)}, T_{(i,k)}$ so that the algebra $\mathcal{O}_{\mathcal{H}_{M,N}}$ is generated by the partial isometries $S_{(i,k)}, T_{(i,k)}$, $(i, k) \in \Sigma^\circ$. □

Let I_n be the $n \times n$ identity matrix and E_n the $n \times n$ matrix whose entries are all 1s. For an $M \times M$ -matrix $C = [c_{i,j}]_{i,j=1}^M$ and an $N \times N$ -matrix $D = [d_{k,l}]_{k,l=1}^N$, denote by $C \otimes D$ the $MN \times MN$ matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1M}D \\ c_{21}D & c_{22}D & \cdots & c_{2M}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{M1}D & c_{M2}D & \cdots & c_{MM}D \end{bmatrix} \quad (191)$$

so that

$$\begin{aligned} E_M \otimes I_N &= \begin{bmatrix} I_N & I_N & \cdots & I_N \\ I_N & I_N & \cdots & I_N \\ \vdots & \vdots & \ddots & \vdots \\ I_N & I_N & \cdots & I_N \end{bmatrix}, \\ I_M \otimes E_N &= \begin{bmatrix} E_N & 0 & \cdots & 0 \\ 0 & E_N & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & E_N \end{bmatrix}. \end{aligned} \quad (192)$$

The index set $\{(i, k) \mid i = 1, \dots, M, k = 1, \dots, N\}$ of the standard basis of $\mathbb{C}^M \otimes \mathbb{C}^N$ is ordered lexicographically from left as in the following way:

$$(1, 1), \dots, (1, N), (2, 1), \dots, (2, N), \dots, (M, 1), \dots, (M, N). \quad (193)$$

Put the $MN \times MN$ matrices

$$A_{M,N} = E_M \otimes I_N, \quad B_{M,N} = I_M \otimes E_N \quad (194)$$

and the $2MN \times 2MN$ matrix

$$H_{M,N} = \begin{bmatrix} A_{M,N} & A_{M,N} \\ B_{M,N} & B_{M,N} \end{bmatrix}. \quad (195)$$

Then, we have the following.

Theorem 42. The C^* -algebra $\mathcal{O}_{\mathcal{H}_{M,N}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_{M,N}}$ for the matrix $H_{M,N}$. The algebra $\mathcal{O}_{H_{M,N}}$ is simple and purely infinite and is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_{M,N}+B_{M,N}}$ for the matrix $A_{M,N}+B_{M,N}$.

Proof. By the preceding proposition, the C^* -algebra $\mathcal{O}_{\mathcal{H}_{M,N}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_{M,N}}$ for the matrix $H_{M,N}$. Since the matrix $H_{M,N}$ is aperiodic, the algebra is simple and purely infinite. The n th column of the matrix $H_{M,N}$ coincides with the $(n+N)$ th column for every $n = 1, \dots, M$. One sees that the matrix $A_{M,N} + B_{M,N}$ is obtained from $H_{M,N}$ by amalgamating them. The procedure is called the column amalgamation and induces an isomorphism on their Cuntz-Krieger algebras (see [15]). $\square$

In [15], the abelian groups $\mathbb{Z}^{MN}/(A_{M,N} + B_{M,N} - I_{MN})\mathbb{Z}^{MN}$, $\text{Ker}(A_{M,N} + B_{M,N} - I_{MN})$ in $\mathbb{Z}^{MN}$ have been computed by using Euclidean algorithms. For the case $M = 2$, they are $\mathbb{Z}/(N^2 - 1)\mathbb{Z}, \{0\}$, respectively, so that we see $K_0(\mathcal{O}_{\mathcal{H}_{2,N}}) = \mathbb{Z}/(N^2 - 1)\mathbb{Z}, K_1(\mathcal{O}_{\mathcal{H}_{2,N}}) = 0$ (see [15] for details).

(3) For a C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, let $\mathcal{H}_\kappa^{\rho, \eta}$ be the C^* -quad module over $(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ as in (3) in Section 2. The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ has been studied in [12].

8. Higher-Dimensional Analogue

In this final section, we will state a generalization of Hilbert C^* -quad modules to Hilbert modules with multi actions of C^* -algebras.

Let $\mathcal{A}$ be a unital C^* -algebra and let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be n -family of unital C^* -algebras. Suppose that there exists a unital embedding

$$\iota_i : \mathcal{A} \hookrightarrow \mathcal{B}_i \quad (196)$$

for each $i = 1, \dots, n$. Suppose that there exists a right action ψ_i of $\mathcal{A}$ on $\mathcal{B}_i$ such that

$$b_i \psi_i(a) \in \mathcal{B}_i \quad \text{for } b_i \in \mathcal{B}_i, a \in \mathcal{A}, i = 1, \dots, n. \quad (197)$$

Hence, $\mathcal{B}_i$ is a right $\mathcal{A}$ -module through ψ_i for $i = 1, \dots, n$. Let $\mathcal{H}$ be a Hilbert C^* -bimodule over $\mathcal{A}$ with a right action of $\mathcal{A}$, an $\mathcal{A}$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$, and a $*$ -homomorphism $\phi_{\mathcal{A}}$ from $\mathcal{A}$ to $\mathcal{L}_{\mathcal{A}}(\mathcal{H})$. It is called a Hilbert C^* -multimodule over $(\mathcal{A}; \mathcal{B}_i, i = 1, \dots, n)$ if $\mathcal{H}$ has a multistructure of Hilbert C^* -bimodules over $\mathcal{B}_i$ for $i = 1, \dots, n$ such that for each $i = 1, \dots, n$ there exist a right action φ_i of $\mathcal{B}_i$ on $\mathcal{H}$ and a left action ϕ_i of $\mathcal{B}_i$ on $\mathcal{H}$ and a $\mathcal{B}_i$ -valued right inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ such that $\phi_i(z_i) \in \mathcal{L}_{\mathcal{A}}(\mathcal{H})$ and

$$[\phi_i(z_i) \xi] \varphi_j(w_j) = \phi_i(z_i) [\xi \varphi_j(w_j)], \quad (198)$$

$$\xi \varphi_j(z_j \psi_j(a)) = [\xi \varphi_j(z_j)] a$$

for $\xi \in \mathcal{H}, z_i \in \mathcal{B}_i, w_j \in \mathcal{B}_j, a \in \mathcal{A}, i, j = 1, \dots, n$ and

$$\phi_{\mathcal{A}}(a) = \phi_i(\iota_i(a)), \quad a \in \mathcal{A}, i = 1, \dots, n. \quad (199)$$

The operator $\phi_i(z_i)$ on $\mathcal{H}$ is adjointable with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ whose adjoint $\phi_i(z_i)^*$ coincides with the adjoint of $\phi_i(z_i)$ with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ so that $\phi_i(z_i)^* = \phi_i(z_i^*)$. We assume that the left actions ϕ_i of $\mathcal{B}_i$ on $\mathcal{H}$ for $i = 1, 2$ are faithful. We require the following compatibility conditions between the right $\mathcal{A}$ -module structure of $\mathcal{H}$ and the right $\mathcal{A}$ -module structure of $\mathcal{B}_i$ through ψ_i :

$$\langle \xi | \eta a \rangle_{\mathcal{B}_i} = \langle \xi | \eta \rangle_{\mathcal{B}_i} \psi_i(a), \quad (200)$$

$$\xi, \eta \in \mathcal{H}, a \in \mathcal{A}, i = 1, \dots, n.$$

We further assume that $\mathcal{H}$ is a full Hilbert C^* -bimodule with respect to the inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}, \langle \cdot | \cdot \rangle_{\mathcal{B}_i}$ for each. A Hilbert C^* -multimodule $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_i, i = 1, \dots, n)$ is said to be of *general type* if there exists a faithful completely positive map $\lambda_i : \mathcal{B}_i \rightarrow \mathcal{A}$ for $i = 1, \dots, n$ such that

$$\lambda_i(b_i \psi_i(a)) = \lambda_i(b_i) a, \quad b_i \in \mathcal{B}_i, a \in \mathcal{A}, \quad (201)$$

$$\lambda_i(\langle \xi | \eta \rangle_{\mathcal{B}_i}) = \langle \xi | \eta \rangle_{\mathcal{A}}, \quad \xi, \eta \in \mathcal{H}, i = 1, \dots, n.$$

A Hilbert C^* -multimodule $\mathcal{H}$ over $(\mathcal{A}; \mathcal{B}_i, i = 1, \dots, n)$ is said to be of *finite type* if there exists a family $\{u_1^{(i)}, \dots, u_{M^{(i)}}^{(i)}\}$, $i = 1, \dots, n$ of finite bases of $\mathcal{H}$ as a right Hilbert $\mathcal{B}_i$ -module for each $i = 1, \dots, n$ such that

$$\sum_{j=1}^{M^{(i)}} u_j^{(i)} \varphi_i(\langle u_j^{(i)} | \xi \rangle_{\mathcal{B}_i}) = \xi, \quad \xi \in \mathcal{H}, i = 1, \dots, n$$

$$\langle u_j^{(i)} | \phi_k(w_k) u_h^{(i)} \rangle_{\mathcal{B}_i} \in \mathcal{A}, \quad w_k \in \mathcal{B}_k, j, h = 1, \dots, M^{(i)},$$

$$\sum_{j=1}^{M^{(i)}} \langle u_j^{(i)} | \phi_k(\langle \xi | \eta \rangle_{\mathcal{B}_k}) u_j^{(i)} \rangle_{\mathcal{B}_i} = \langle \xi | \eta \rangle_{\mathcal{A}} \quad (202)$$

for all $\xi, \eta \in \mathcal{H}, i, k = 1, \dots, n$ with $i \neq k$.

By a generalizing argument to the preceding sections, we may construct a C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ associated with the Hilbert C^* -multimodule $\mathcal{H}$ by a similar manner to the preceding sections; that is, the C^* -algebra is generated by the quotients of the n -kinds of creation operators $s_\xi^{(i)}$, $\xi \in \mathcal{H}, i = 1, \dots, n$ on the generalized Fock space $F(\mathcal{H})$ by the ideal generated by the finite-rank operators. One may show the following generalization.

Proposition 43. Let $\mathcal{H}$ be a Hilbert C^* -multimodule over $(\mathcal{A}; \mathcal{B}_i, i = 1, \dots, n)$ of finite type with a finite basis $\{u_1^{(i)}, \dots, u_{M^{(i)}}^{(i)}\}$ of $\mathcal{H}$ as a Hilbert C^* -right module over $\mathcal{B}_i$ for each $i = 1, \dots, n$. Then, the C^* -algebra $\mathcal{O}_{F(\mathcal{H})}$ generated by the quotients of the n -kinds of creation operators on the generalized Fock spaces $F(\mathcal{H})$ is canonically isomorphic to the universal

C^* -algebra $\mathcal{O}_{\mathcal{H}}$ generated by the operators $S_1^{(i)}, \dots, S_{M^{(i)}}^{(i)}$ and elements $z_i \in \mathcal{B}_i$ for $i = 1, \dots, n$ subject to the relations

$$\sum_{i=1}^n \sum_{k=1}^{M^{(i)}} S_k^{(i)} S_k^{(i)*} = 1, \quad S_k^{(i)*} S_m^{(j)} = 0, \quad i \neq j,$$

$$S_k^{(i)*} S_l^{(i)} = \langle u_k^{(i)} | u_l^{(i)} \rangle_{\mathcal{B}_i}, \quad (203)$$

$$z_j S_k^{(i)} = \sum_{l=1}^{M^{(i)}} S_l^{(i)} \langle u_l^{(i)} | \phi_j(z_j) u_k^{(i)} \rangle_{\mathcal{B}_i}$$

for $z_j \in \mathcal{B}_j$, $i, j = 1, \dots, n$, $k, l = 1, \dots, M^{(i)}$, $m = 1, \dots, M^{(j)}$.

The proof of the above proposition is similar to the proof of Theorem 1.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was supported by JSPS KAKENHI Grant Number 23540237.

References

- [1] G. Robertson and T. Steger, "Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras," *Journal für die Reine und Angewandte Mathematik*, vol. 513, pp. 115–144, 1999.
- [2] A. Kumjian and D. Pask, "Higher rank graph C^* -algebras," *New York Journal of Mathematics*, vol. 6, pp. 1–20, 2000.
- [3] V. Deaconu, " C^* -algebras and Fell bundles associated to a textile system," *Journal of Mathematical Analysis and Applications*, vol. 372, no. 2, pp. 515–524, 2010.
- [4] R. Excel, D. Gonçalves, and C. Starling, "The tiling C^* -algebra viewed as a tight inverse semigroup algebra," *Semigroup Forum*, vol. 84, no. 2, pp. 229–240, 2012.
- [5] G. A. Pino, J. Clark, A. an Huff, and I. Raeburn, "Kumjian-Pask algebras of higher rank graphs," *Transactions of the American Mathematical Society*, vol. 365, no. 7, pp. 3613–3641, 2013.
- [6] D. Pask, I. Raeburn, and N. A. Weaver, "A family of 2-graphs arising from two-dimensional subshifts," *Ergodic Theory and Dynamical Systems*, vol. 29, no. 5, pp. 1613–1639, 2009.
- [7] K. Matsumoto, "Actions of symbolic dynamical systems on C^* -algebras," *Journal für die Reine und Angewandte Mathematik*, vol. 605, pp. 23–49, 2007.
- [8] T. Kajiwara, C. Pinzari, and Y. Watatani, "Ideal structure and simplicity of the C^* -algebras generated by Hilbert bimodules," *Journal of Functional Analysis*, vol. 159, no. 2, pp. 295–322, 1998.
- [9] P. S. Muhly and B. Solel, "On the simplicity of some Cuntz-Pimsner algebras," *Mathematica Scandinavica*, vol. 83, no. 1, pp. 53–73, 1998.
- [10] M. V. Pimsner, "A class of C^* -algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$," in *Free Probability Theory*, vol. 12, pp. 189–212, American Mathematical Society, Providence, RI, USA, 1997.
- [11] K. Matsumoto, " C^* -algebras associated with textile dynamical systems," <http://arxiv.org/pdf/1201.1056.pdf>.
- [12] K. Matsumoto, " C^* -algebras associated with Hilbert C^* -quad modules of C^* -textile dynamical systems," <http://arxiv.org/pdf/1111.3091.pdf>.
- [13] Y. Watatani, "Index for C^* -subalgebras," *Memoirs of the American Mathematical Society*, vol. 83, no. 424, 1990.
- [14] Y. Katayama, "A generalized Cuntz algebra," *Sūrikaiseikiken-kyūsho Kōkyūroku*, no. 858, pp. 87–90, 1994.
- [15] K. Matsumoto, "Cuntz-Krieger algebras associated with Hilbert C^* -quad modules of commuting matrices," *Mathematica Scandinavica*. In press.
- [16] W. L. Paschke, "K for actions of the circle group on C^* -algebras," *Journal of Operator Theory*, vol. 6, no. 1, pp. 125–133, 1981.
- [17] J. Cuntz, "A class of C^* -algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^* -algebras," *Inventiones Mathematicae*, vol. 63, no. 1, pp. 25–40, 1981.