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Research Article

Fixed Point Approximation of Generalized Nonexpansive Mappings in Hyperbolic Spaces

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We prove strong and Δ-convergence theorems for generalized nonexpansive mappings in uniformly convex hyperbolic spaces using S-iteration process due to Agarwal et al. As uniformly convex hyperbolic spaces contain Banach spaces as well as CAT(0) spaces, our results can be viewed as extension and generalization of several well-known results in Banach spaces as well as CAT(0) spaces.

1. Introduction

Let (X, d) be a metric space and let C be a nonempty subset of X. A mapping $T : C \to C$ is said to be as follows:

(i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$;

(ii) quasi-nonexpansive if $d(Tx, Tp) \leq d(x, p)$ for all $x \in C$ and $p \in F(T)$, where $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T.

We know that there exist many generalizations of nonexpansive and quasi-nonexpansive mappings. Garcia-Falset et al. [1] introduced two generalizations of nonexpansive mappings which in turn include Suzuki generalized nonexpansive mappings (see [2]).

Definition 1 (see [1]). Let T be a mapping defined on a subset C of metric space X and $\mu \geq 1$. Then T is said to satisfy the condition (E_μ), if, for all $x, y \in C$,

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y). \tag{1}$$

T is said to satisfy the condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 2 (see [1]). Let T be a mapping defined on a subset C of a metric space X and $\lambda \in (0, 1)$. Then T is said to satisfy the condition (C_λ) if, for all $x, y \in C$,

$$\lambda d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y). \tag{2}$$

In the case $0 < \lambda_1 < \lambda_2 < 1$, then the condition (C_{λ_1}) implies the condition (C_{λ_2}). Suzuki (see [2]) said that T satisfies the condition (C), when $\lambda = 1/2$.

The following example shows that the class of mappings satisfying the conditions (E) and (C_λ), for some $\lambda \in (0, 1)$, is larger than the class of mappings satisfying the condition (C).

Example 3 (see [1]). For a given $\lambda \in (0, 1)$, define a mapping T on $[0, 1]$ by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \neq 1, \\ \dfrac{1+\lambda}{2+\lambda}, & \text{if } x = 1. \end{cases} \tag{3}$$

Then the mapping T satisfies the condition (C_λ) but it fails the condition (C_{λ_1}), whenever $0 < \lambda_1 < \lambda$. Moreover, T satisfies the condition (E_μ) for $\mu = (2 + \lambda)/2$.

The basic properties and details of CAT(0) spaces can be found in the literature [3–5]. In [6], Lim introduced a concept of convergence in a general metric space which is called "Δ-convergence." In 2008, Kirk and Panyanak [7] specialized Lim's concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Since then, the existence problem and the Δ-convergence problem of iterative sequences to a fixed point for various classes of nonexpansive mappings in the frame work of CAT(0) spaces have been rapidly developed (see [1, 8–12]).

In [13], Leustean proved that CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \varepsilon^2/8$ quadratic in ε. Thus, the class of uniformly convex hyperbolic spaces are a natural generalization of both uniformly convex Banach spaces and CAT(0) spaces.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [14]. It is noted that they are different from Gromov hyperbolic spaces [15] or from other notions of hyperbolic spaces that can be found in literature (see [16–19]).

A hyperbolic space (X,d,W) is a metric space (X,d) together with a convexity mapping $W : X^2 \times [0,1] \to X$ satisfying

(W_1) $d(u,W(x,y,\alpha)) \le \alpha d(u,x) + (1-\alpha)d(u,y)$;

(W_2) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$;

(W_3) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(W_4) $d(W(x,z,\alpha),W(y,w,\alpha)) \le (1-\alpha)d(x,y) + \alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$.

A metric space is said to be a *convex metric space* in the sense of Takahashi [20], where a triple (X,d,W) satisfy only (W_1) (see [21–23]). We get the notion of the space of hyperbolic type in the sense of Goebel and Kirk [16], where a triple (X,d,W) satisfies (W_1)–(W_3). The (W_4) was already considered by Itoh [24] under the name of "condition III" and it is used by Reich and Shafrir [18] and Kirk [17] to define their notions of hyperbolic spaces.

The class of hyperbolic spaces include normed spaces and convex subsets thereof, the Hilbert space ball equipped with the hyperbolic metric [25], Hadrmard manifold, and the CAT(0) spaces in the sense of Gromov (see [15]).

If $x,y \in X$ and $\lambda \in [0,1]$, then we use the notation $(1-\lambda)x \oplus \lambda y$ for $W(x,y,\lambda)$. The following holds even for the more general setting of convex metric space [20]: for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y),$$
$$d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda) d(x,y). \tag{4}$$

A hyperbolic space (X,d,W) is uniformly convex [13] if, for any $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that, for all $a,x,y \in X$,

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1-\delta) r, \tag{5}$$

provided $d(x,a) \le r$, $d(y,a) \le r$, and $d(x,y) \ge \varepsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$, providing such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called a modulus of uniform convexity. We called that η is *monotone* if it decreases with r for fix ε.

Recently, Agarwal et al. [26] introduced S-iteration process as follows (see [27]).

Let C be a convex subset of a linear space X and let T be a mapping from C into itself. Then the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by

$$x_{n+1} = (1-\alpha_n) Tx_n + \alpha_n Ty_n,$$
$$y_n = (1-\beta_n) x_n + \beta_n Tx_n, \quad n \in \mathbb{N}, \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the certain condition. It is observed that rate of convergence of S-iteration process is similar to the Picard iteration process but faster than the Mann iteration process for contraction mapping (see [26, 27]).

The purpose of this paper is to prove the strong and Δ-convergence theorems for generalized nonexpansive mappings in uniformly convex hyperbolic spaces by using S-iteration process. Our results can be viewed as extension and generalization of several well-known results in Banach spaces as well as CAT(0) spaces [10–12, 28, 29].

2. Preliminaries

Let C be a nonempty subset of metric space X and let $\{x_n\}$ be any bounded sequence in C. Consider a continuous functional $r_a(\cdot,\{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x), \quad x \in X. \tag{7}$$

Then, the infimum of $r_a(\cdot,\{x_n\})$ over C is said to be the *asymptotic radius* of $\{x_n\}$ with respect to C and is denoted by $r_a(C,\{x_n\})$.

A point $z \in C$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r_a(z,\{x_n\}) = \inf\{r_a(x,\{x_n\}) : x \in C\}; \tag{8}$$

the set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $AC(C,\{x_n\})$. This set may be empty or a singleton or contain infinitely many points.

If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X,\{x_n\}) = r_a(\{x_n\})$ and $AC(X,\{x_n\}) = AC(\{x_n\})$, respectively. We know that, for $x \in X$, $r_a(x,\{x_n\}) = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

It is known that every bounded sequence has a unique asymptotic center with respect to each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces.

The following lemma is due to Leuştean [30] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 4 (see [30]). *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has* *a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Recall that a sequence $\{x_n\}$ in X is said to be Δ-*convergent* to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 5 (see [31]). *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le c, \qquad \limsup_{n\to\infty} d(y_n, x) \le c,$$
$$\lim_{n\to\infty} d(W(x_n, y_n, t_n), x) = c, \tag{9}$$

for some $c \ge 0$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$.

3. Main Results

We begin with the definition of *Fejér* monotone sequences.

Definition 6. Let C be a nonempty subset of hyperbolic space X and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *Fejér* monotone with respect to C if for all $x \in C$ and $n \in \mathbb{N}$

$$d(x_{n+1}, x) \le d(x_n, x). \tag{10}$$

Example 7. Let C be a nonempty subset of hyperbolic space X and let $T : C \to C$ be a quasi-nonexpansive (in particular, nonexpansive) mapping such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ of Picard iteration is *Fejér* monotone with respect to $F(T)$.

We can easily prove the following proposition.

Proposition 8. *Let $\{x_n\}$ be a sequence in X and let C be a nonempty subset of X. Suppose that $\{x_n\}$ is Fejér monotone with respect to C. Then we have the following:*

(1) *$\{x_n\}$ is bounded;*

(2) *the sequence $\{d(x_n, p)\}$ is decreasing and convergent for all $p \in F(T)$.*

We now define S-iteration process in hyperbolic spaces:

Let C be a nonempty closed convex subset of a hyperbolic space X and let T be a mapping of C into itself. For any $x_1 \in C$ the sequence $\{x_n\}$ of S-iteration process is defined as

$$\begin{aligned} x_{n+1} &= W(Tx_n, Ty_n, \alpha_n) \\ y_n &= W(x_n, Tx_n, \beta_n), \quad n \in \mathbb{N}, \end{aligned} \tag{11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences with $0 < a \le \alpha_n, \beta_n \le b < 1$.

Lemma 9. *Let C be a nonempty closed convex subset of a hyperbolic space X and let $T : C \to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda \in (0, 1)$. If $\{x_n\}$ is a sequence defined by (11), then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.*

Proof. Let $p \in F(T)$. Then we have

$$\begin{aligned} \lambda d(p, Tp) &= 0 \le d(p, y_n), \\ \lambda d(p, Tp) &= 0 \le d(p, x_n), \end{aligned} \tag{12}$$

for all $n \in \mathbb{N}$. Since T satisfies the condition (C_λ), for some $\lambda \in (0, 1)$, we have

$$\begin{aligned} d(Tp, Ty_n) &\le d(p, y_n), \\ d(Tp, Tx_n) &\le d(p, x_n). \end{aligned} \tag{13}$$

Using (11), we have

$$\begin{aligned} d(y_n, p) &= d(W(x_n, Tx_n, \beta_n), p) \\ &\le (1 - \beta_n) d(x_n, p) + \beta_n d(Tx_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{14}$$

Again, using (11) and (14), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(Tx_n, Ty_n, \alpha_n), p) \\ &\le (1 - \alpha_n) d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &\le (1 - \alpha_n) d(x_n, p) + \alpha_n d(y_n, p) \\ &= d(x_n, p); \end{aligned} \tag{15}$$

that is, $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in F(T)$. Thus, $\{x_n\}$ is *Fejér* monotone with respect to $F(T)$. □

Lemma 10. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $T : C \to C$ be a mapping which satisfies the conditions (C_λ) and (E) on C. If $\{x_n\}$ is a sequence defined by (11), then $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. Suppose that the fixed point set $F(T)$ is nonempty and $p \in F(T)$. From Lemma 9, we know that $\{x_n\}$ is *Fejér* monotone with respect to $F(T)$. Hence, by Proposition 8, $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = c$. Since

$$\begin{aligned} \lambda d(p, Tp) &= 0 \le d(p, y_n), \\ \lambda d(p, Tp) &= 0 \le d(p, x_n), \end{aligned} \tag{16}$$

for all $n \in \mathbb{N}$, from the condition (C_λ), we have

$$\begin{aligned} d(Tp, Ty_n) &\le d(p, y_n), \\ d(Tp, Tx_n) &\le d(p, x_n). \end{aligned} \tag{17}$$

Therefore,

$$d(Tx_n, p) \le d(Tx_n, Tp) \le d(x_n, p), \tag{18}$$

for all $n \in \mathbb{N}$. Taking the limit supremum on both sides, we get

$$\limsup_{n\to\infty} d(Tx_n, p) \le c. \tag{19}$$

Similarly, we have

$$\limsup_{n\to\infty} d\left(Ty_n, p\right) \le c. \tag{20}$$

Taking the limit supremum on both sides of (14), we have

$$\limsup_{n\to\infty} d\left(y_n, p\right) \le c. \tag{21}$$

Since

$$c = \limsup_{n\to\infty} d\left(x_{n+1}, p\right) = \lim_{n\to\infty} d\left(W\left(Tx_n, Ty_n, \alpha_n\right), p\right), \tag{22}$$

by using (19), (20), and Lemma 5, we get

$$\lim_{n\to\infty} d\left(Tx_n, Ty_n\right) = 0. \tag{23}$$

Next, we know that

$$\begin{aligned} d\left(x_{n+1}, Tx_n\right) &= d\left(W\left(Tx_n, Ty_n, \alpha_n\right), Tx_n\right) \\ &\le bd\left(Ty_n, Tx_n\right) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{24}$$

Hence, from (23) and (24), we have

$$\begin{aligned} d\left(x_{n+1}, Ty_n\right) &\le d\left(x_{n+1}, Tx_n\right) + d\left(Tx_n, Ty_n\right) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{25}$$

Now we observe that

$$\begin{aligned} d\left(x_{n+1}, p\right) &\le d\left(x_{n+1}, Ty_n\right) + d\left(Ty_n, p\right) \\ &\le d\left(x_{n+1}, Ty_n\right) + d\left(y_n, p\right), \end{aligned} \tag{26}$$

which yields that

$$c \le \liminf_{n\to\infty} d\left(y_n, p\right). \tag{27}$$

From the estimates of (21) and (27), we have that

$$\lim_{n\to\infty} d\left(y_n, p\right) = c. \tag{28}$$

Thus, from (11), we have

$$\lim_{n\to\infty} d\left(W\left(x_n, Tx_n, \beta_n\right), p\right) = c, \tag{29}$$

which gives

$$\lim_{n\to\infty} d\left(x_n, Tx_n\right) = 0. \tag{30}$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $AC(C, \{x_n\}) = \{x\}$. Then, by Lemma 4, $x \in C$. As T satisfies the condition (E_μ) on C, there exists $\mu > 1$ such that

$$d\left(x_n, Tx\right) \le \mu d\left(x_n, Tx_n\right) + d\left(x_n, x\right), \tag{31}$$

which implies that

$$\begin{aligned} \limsup_{n\to\infty} d\left(x_n, Tx\right) &\le \limsup_{n\to\infty} \left\{\mu d\left(x_n, Tx_n\right) + d\left(x_n, x\right)\right\} \\ &= \limsup_{n\to\infty} d\left(x_n, x\right). \end{aligned} \tag{32}$$

By using the uniqueness of asymptotic center, $Tx = x$, so x is a fixed point of T. □

Theorem 11. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \to C$ be a mapping which satisfies conditions (C_λ) and (E), for some $\lambda \in (0,1)$ on C with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ which is defined by (11), is Δ-convergent to a fixed point of T.*

Proof. From Lemma 10, we know that $\{x_n\}$ is a bounded sequence; therefore, $\{x_n\}$ has a Δ-convergent subsequence. We now prove that every Δ-convergent subsequence of $\{x_n\}$ has unique Δ-limt in $F(T)$. For this, let u and v be Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$, respectively. By Lemma 4, $AC(C, \{u_n\}) = \{u\}$ and $AC(C, \{v_n\}) = \{v\}$. Since $\{u_n\}$ is bounded sequence, by Lemma 10, $\lim_{n\to\infty} d(u_n, Tu_n) = 0$. We claim that u is a fixed point of T. Since T satisfies the condition (E), there exists a $\mu \ge 1$ such that

$$d\left(u_n, Tu\right) \le \mu d\left(u_n, Tu_n\right) + d\left(u_n, u\right). \tag{33}$$

Taking the limit supremum on both sides, we have

$$\begin{aligned} r_a\left(\{u_n\}, Tu\right) &= \limsup_{n\to\infty} d\left(u_n, Tu\right) \\ &\le \limsup_{n\to\infty} \left\{\mu d\left(u_n, Tu_n\right) + d\left(u_n, u\right)\right\} \\ &\le \limsup_{n\to\infty} d\left(u_n, u\right) \\ &= r_a\left(\{u_n\}, u\right). \end{aligned} \tag{34}$$

Hence, we obtain

$$r_a\left(\{u_n\}, Tu\right) \le r_a\left(\{u_n\}, u\right). \tag{35}$$

By uniqueness of the asymptotic center, $Tu = u$.

Similarly, we can prove that $Tv = v$. Thus, u and v are fixed points of T. Now we show that $u = v$. If not, then by the uniqueness of asymptotic center,

$$\begin{aligned} \limsup_{n\to\infty} d\left(x_n, u\right) &= \limsup_{n\to\infty} d\left(u_n, u\right) \\ &< \limsup_{n\to\infty} d\left(u_n, v\right) \\ &= \limsup_{n\to\infty} d\left(x_n, v\right) \\ &= \limsup_{n\to\infty} d\left(v_n, v\right) \\ &< \limsup_{n\to\infty} d\left(v_n, u\right) \\ &= \limsup_{n\to\infty} d\left(x_n, u\right), \end{aligned} \tag{36}$$

which is a contradiction. Hence $u = v$. □

Theorem 12. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \to C$ be a mapping which satisfies conditions (C_λ) and (E), for some $\lambda \in (0,1)$ on C with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ which* *is defined by (11) converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D\left(x_n, F(T)\right) = 0, \tag{37}$$

where $D(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.

Proof. Necessity is obvious; we have to prove only sufficient part. First, we show that the fixed point set $F(T)$ is closed; let $\{x_n\}$ be a sequence in $F(T)$ which converges to some point $z \in C$. Since

$$\lambda d\left(x_n, Tx_n\right) = 0 \le d\left(x_n, z\right), \tag{38}$$

from the condition (C_λ), we have

$$d\left(x_n, Tz\right) = d\left(Tx_n, Tz\right) \le d\left(x_n, z\right). \tag{39}$$

By taking the limit on both sides, we obtain

$$\lim_{n\to\infty} d\left(x_n, Tz\right) \le \lim_{n\to\infty} d\left(x_n, z\right) = 0. \tag{40}$$

In view of the uniqueness of the limit, we have $z = Tz$, so that $F(T)$ is closed. Suppose that

$$\liminf_{n\to\infty} D\left(x_n, F(T)\right) = 0. \tag{41}$$

Then, from (15)

$$D\left(x_{n+1}, F(T)\right) \le D\left(x_n, F(T)\right); \tag{42}$$

it follows from Lemma 9 and Proposition 8 that $\lim_{n\to\infty} D(x_n, F(T))$ exists. Hence we know that $\lim_{n\to\infty} D(x_n, F(T)) = 0$.

Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d\left(x_{n_k}, p_k\right) < \frac{1}{2^k}, \quad \forall k \ge 1, \tag{43}$$

where $\{p_k\}$ is in $F(T)$. By Lemma 9, we have

$$d\left(x_{n_{k+1}}, p_k\right) \le d\left(x_{n_k}, p_k\right) < \frac{1}{2^k}, \tag{44}$$

which implies that

$$\begin{aligned} d\left(p_{k+1}, p_k\right) &\le d\left(p_{k+1}, x_{n_{k+1}}\right) + d\left(x_{n_{k+1}}, p_k\right) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}. \end{aligned} \tag{45}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then, we know that $\{x_n\}$ converges to p. In fact, since

$$d\left(x_{n_k}, p\right) \le d\left(x_{n_k}, p_k\right) + d\left(p_k, p\right) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{46}$$

we have

$$\lim_{k\to\infty} d\left(x_{n_k}, p\right) = 0. \tag{47}$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, the sequence $\{x_n\}$ is convergent to p. □

We recall the definition of condition (I) due to Senter and Doston [32].

Let C be a nonempty subset of a metric space X. A mapping $T : C \to C$ is said to satisfy condition (I), if there is a nondecreasing function $f[0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$d\left(x, Tx\right) \ge f\left(D\left(x, F(T)\right)\right), \tag{48}$$

for all $x \in C$, where $D(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

Theorem 13. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $T : C \to C$ be a mapping which satisfies conditions (C_λ) and (E), for some $\lambda \in (0,1)$ on C. Moreover, T satisfies condition (I) with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ which is defined by (11) converges strongly to some fixed point of T.*

Proof. As in the proof of Theorem 12, it can be shown that $F(T)$ is closed. Observe that, by Lemma 9, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from condition (I) that

$$\lim_{n\to\infty} f\left(D\left(x_n, F(T)\right)\right) \le \lim_{n\to\infty} d\left(x_n, Tx_n\right) = 0. \tag{49}$$

Therefore, we have

$$\lim_{n\to\infty} f\left(D\left(x_n, F(T)\right)\right) = 0. \tag{50}$$

Since $f : [0,\infty] \to [0,\infty)$ is a nondecreasing mapping satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, we have

$$\lim_{n\to\infty} D\left(x_n, F(T)\right) = 0. \tag{51}$$

Rest of the proof follows in lines of Theorem 12. □

Remark 14. Our Theorems 11, 12, and 13 improve and extend the previous well-known results from Banach spaces and CAT(0) spaces to more general class of uniformly convex hyperbolic spaces (see [10, 28, 29], in particular, Theorems 3.4 and 3.6 of [12]). In our results, we considered the faster iteration process to approximate the fixed point of underlying mapping T in the framework of uniformly convex hyperbolic spaces.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved final paper.

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