Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2015, Article ID 394282, 4 pages
http://dx.doi.org/10.1155/2015/394282

Research Article

k-Smooth Points in Some Banach Spaces

A. Saleh Hamarsheh

Department of Mathematics, Al-Hussein Bin Talal University, Ma'an, Jordan

Correspondence should be addressed to A. Saleh Hamarsheh; abkmsaleh@yahoo.com

Received 22 March 2015; Accepted 20 May 2015

Academic Editor: Nawab Hussain

We characterize the k-smooth points in some Banach spaces. We will deal with injective tensor product, the Bochner space $L^\infty(\mu, X)$ of (equivalence classes of) μ-essentially bounded measurable X-valued functions, and direct sums of Banach spaces.

1. Introduction

For a unit vector x in a Banach space X, consider the state space $S_x = \{x^* \in X^* : \|x^*\| = 1 = x^*(x)\}$. The point x is a smooth point if S_x consists exactly of one point. The set of all smooth points is denoted by smooth $B(X)$. Smooth points are important tools in the study of the geometry of Banach spaces. For two Banach spaces X, Y Heinrich, [1], gave a description of smooth points of the unit ball in the space $K(X, Y)$ of compact operators from X into Y. The research then turned to the space $L(X, Y)$ of bounded operators. Kittaneh and Younis, [2], were the first to deal with this problem. They characterized smooth points in $L(l^2)$. Their result was then generalized in [3] to the space $L(l^p)$; $1 < p < \infty$. For smooth points in $L(l^p, X)$ see [4, 5]. In [6] Werner gave a description of smooth points in $L(X, Y)$ under some conditions on X and Y. Smooth points in certain vector valued function spaces were given in [7].

In [8] the authors generalize the notion of smoothness by calling a unit vector x in a Banach space X a k-smooth *point*, or *a multismooth point of order* k if S_x has exactly k linearly independent vectors, equivalently, if $\dim(\operatorname{sp} Sx) = k$. For a natural number k, the set of k-smooth points in X is denoted by k-smooth $B(X)$. Note that S_x is a weak*-compact convex set and hence it is easy to see that $x \in k$-smooth $B(X)$ if and only if $\dim(\operatorname{sp}\operatorname{ext} Sx) = k$. Multismoothness in Banach spaces was extensively studied by Lin and Rao in [9]. In paricular, they showed that, in a Banach space of finite dimension k, any k-smooth point is unitary and hence a strongly extreme point. The aim of this paper is to characterize multismoothness in some Banach spaces. Indeed, we will deal with injective tensor product, the Bochner space $L^\infty(\mu, X)$ of (equivalence classes of) μ-essentially bounded measurable X-valued functions, and direct sums of Banach spaces.

The set of all extreme points of the unit ball of a Banach space X is denoted by $\operatorname{ext} B(X)$.

2. Multismoothness in Injective Tensor Products

In [9] the authors characterized multismoothness in the completed injective tensor product $X \otimes_\epsilon Y$ when X is an L^1-predual space and Y is a smooth Banach space. We generalize their result to *any* Banach space Y. Note that $X \otimes_\epsilon Y = KW^*(Y^*, X)$ whenever either of X and Y has the approximation property. Here $KW^*(Y^*, X)$ is the space of all compact and weak* to weakly continuous operators from Y^* to X, endowed with usual operator norm.

Recall that if $x^* \in X^*$ and $y^* \in Y^*$ then $x^* \otimes y^* \in (KW^*(Y^*, X))^*$ as follows:

$$\text{For } T \in KW^*(Y^*, X), \quad \langle x^* \otimes y^*, T \rangle = \langle y^*, T^* x^* \rangle. \quad (1)$$

Theorem 1. *Let X be an L^1-predual space and Y any Banach space. Let $T \in X \otimes_\epsilon Y = KW^*(Y^*, X)$ with $\|T\| = 1$. Then T is a multismooth point of finite order in $X \otimes_\epsilon Y$ if and only if T^* attains its norm at exactly finitely many independent vectors, say at $x_1^*, x_2^*, \ldots, x_r^* \in \operatorname{ext} B(X^*)$ such that $T^* x_i^*$ is a multismooth point of finite order in Y; $i = 1, 2, \ldots, r$.*

In this case the order of smoothness of T is $k = m_1 + m_2 + \cdots + m_r$, where m_i is the order of smoothness of $T^* x_i^*$, $i = 1, 2, \ldots, r$.

Proof. One can easily prove that if T^* attains its norm at infinitely many independent vectors in $\operatorname{ext} B(X^*)$ then $T \in X\otimes_\epsilon Y$ is not a multismooth point of any finite order. The same conclusion will be obtained if T^* attains its norm at some $x^* \in \operatorname{ext} B(X^*)$ and T^*x^* is not a multismooth point of finite order in Y. So, suppose T^* attains its norm at exactly finitely many independent vectors, say at $x_1^*, x_2^*, \ldots, x_r^* \in \operatorname{ext} B(X^*)$ such that $T^*x_i^*$ is a multismooth point of finite order m_i in Y; $i = 1, 2, \ldots, r$ and let $k = m_1 + m_2 + \cdots + m_r$. We will prove that T is a multismooth point of order k in $X\otimes_\epsilon Y$.

For each i there are exactly m_i linearly independent functionals in $\operatorname{ext} B(Y^*)$ attaining their norm at $T^*x_i^*$, say $y_{i,1}^*, y_{i,2}^*, \ldots, y_{i,m_i}^*$. Since X is an L^1-predual space, then there are distinct atoms A_i with $x_i^* = \pm(1/\mu A_i)\chi_{A_i}$. Set $F_{i,t} = x_i^* \otimes y_{i,t}^*$, where $1 \le i \le r$ and $1 \le t \le m_i$. These are k extreme functionals in $(X\otimes_\epsilon Y)^*$ attaining their norms at T:

$$\left\langle x_i^* \otimes y_{i,t}^*, T\right\rangle = \left\langle y_{i,t}^*, T^*x_i^*\right\rangle = 1 = \left\|T^*x_i^*\right\|. \quad (2)$$

We claim that the $F_{i,t}$'s are linearly independent. Indeed, if $\sum_{i=1}^r\sum_{t=1}^{m_i} a_{i,t}F_{i,t} = 0$ for some scalars $a_{i,t}$ then $\sum_{i=1}^r x_i^* \otimes (\sum_{t=1}^{m_i} a_{i,t}y_{i,t}^*) = 0$. But since the x_i^* correspond to distinct atoms then $\sum_{t=1}^{m_i} a_{i,t}y_{i,t}^* = 0$ for all $i = 1, 2, \ldots, r$. Since $\{y_{i,t}^* : t = 1, 2, \ldots, m_i\}$ are linearly independent, then $a_{i,t} = 0$, $\forall 1 \le i \le r, 1 \le t \le m_i$.

Finally, Let $F \in (X\otimes_\epsilon Y)^*$ with $\|F\| = 1 = F(T)$. We will show that $F \in \operatorname{sp}\{F_{i,t} : 1 \le i \le r, 1 \le t \le m_i\}$. We can suppose that $F \in \operatorname{ext} B((X\otimes_\epsilon Y)^*)$ (see Section 1). Now, by a result of Ruess and Stegall [10], $F = x^* \otimes y^*$, where $x^* \in \operatorname{ext} B(X^*)$ and $y^* \in \operatorname{ext} B(Y^*)$. Then

$$F(T) = \left\langle x^* \otimes y^*, T\right\rangle = \left\langle x^*, Ty^*\right\rangle = \left\langle T^*x^*, y^*\right\rangle = 1. \quad (3)$$

So, $\|T^*x^*\| = 1$ and hence $x^* \in \operatorname{sp}\{x_1^*, x_2^*, \ldots, x_r^*\}$. But since the x_i^* correspond to distinct atoms then $x^* = x_i^*$ for some $i = 1, 2, \ldots, r$ and consequently $y^* \in \operatorname{sp}\{y_{i,1}^*, y_{i,2}^*, \ldots, y_{i,m_i}^*\}$. Therefore $x^* \otimes y^* = x_i^* \otimes y^* \in \operatorname{sp}\{x_i^* \otimes y_{i,1}^*, x_i^* \otimes y_{i,2}^*, \ldots, x_i^* \otimes y_{i,m_i}^*\}$. Hence $F \in \operatorname{sp}\{F_{i,t} : 1 \le i \le r, 1 \le t \le m_i\}$. This proves that the $F_{i,t}$'s form a maximal linearly independent set in S_T.

Therefore $T \in k\text{-smooth}\, B(X\otimes_\epsilon Y)$. □

As a corollary, we get the following.

Corollary 2 (see [9]). *Let X be an L^1-predual space and Y a smooth Banach space. Let $T \in X\otimes_\epsilon Y = KW^*(Y^*, X)$ with $\|T\| = 1$. Then T is a multismooth point of finite order k in $X\otimes_\epsilon Y$ if and only if T^* attains its norm at exactly k independent vectors in* $\operatorname{ext} B(X^*)$.

Open Problem. For Banach spaces X and Y let $T \in K(X, Y)$ with $\|T\| = 1$. Is it true that T is a multismooth point of finite order k in $K(X, Y)$ if and only if T^* attains its norm at only finitely many independent vectors, say at $y_1^*, y_2^*, \ldots, y_r^* \in \operatorname{ext} B(Y^*)$ such that each $T^*y_i^*$ is a multismooth point of finite order, say m_i, in X^*, where $k = m_1 + m_2 + \cdots + m_r$?

Theorem 1 above tells us that the answer is yes when Y is an L^1-predual space, since in this case $K(X, Y) = Y\otimes_\epsilon X^* = KW^*(X^{**}, Y)$.

3. Multismoothness in Bochner Spaces

Let X be a Banach space. In this section we discuss multismoothness in the Bochner space $L^\infty(\mu, X)$ of (equivalence classes of) μ-essentially bounded measurable X-valued functions. Recall that measurable functions are constants on the atoms.

Lemma 3. *Let $f \in L^\infty(\mu, X)$ and suppose that there is no μ-atom A such that $\|f(A)\| = 1$. Then f is not a multismooth point of any finite order.*

Proof. Fix $r \in \mathbb{N}$. We will prove that f is not a multismooth point of order r. Write $\Omega = \cup_{i=1}^{r+1} E_i$, where E_i's are disjoint measurable sets of positive measure, with $\sup\{\|f(t)\| : t \in E_i\} = 1$. For $1 \le i \le r+1$, define

$$f_i(t) = \begin{cases} f(t), & \text{if } t \in E_i \\ 0, & \text{if } t \notin E_i. \end{cases} \quad (4)$$

Then $f = f_1 + f_2 + \cdots + f_{r+1}$ and $\|f_i\| = \sup\{\|f(t)\| : t \in E_i\} = 1$. Moreover, $\|f_i \pm f_j\| \le 1$ for all $i \ne j$. This shows that f is not a smooth point of any order $m \le r$; see [8]. In particular, f is not a multismooth point of order r. □

Lemma 4. *Let $f \in L^\infty(\mu, X)$ and suppose that there are exactly n μ-atoms $A_1, A_2, \ldots, A_n$ such that $\|f(A_j)\| = 1$. If $\sup\{\|f(t)\| : t \notin \cup_{j=1}^n A_j\} = 1$, then f is not a multismooth point of any finite order.*

Proof. Fix $r \in \mathbb{N}$ and write $\Omega \setminus \cup_{i=1}^n A_i$ as a disjoint union of r measurable sets E_i of positive measure and proceed as in the above proof. □

Theorem 5. *Let $f \in L^\infty(\mu, X)$ with $\|f\| = 1$. Then f is a multismooth point of finite order if and only if there are exactly finitely many distinct atoms $A_1, A_2, \ldots, A_r$ such that $\|f(A_j)\| = 1$, $j = 1, 2, \ldots, r$ and $\sup\{\|f(t)\| : t \notin \cup_{j=1}^\infty A_j\} < 1$ and each $f(A_j)$ is a multismooth point of finite order, say m_j, in X. In this case the order of smoothness of f is $k = m_1 + m_2 + \cdots + m_r$.*

Proof. The above two lemmas prove the "only if" part. For the converse, we choose, for any $j = 1, 2, \ldots, r$, linearly independent set $\{x_{j,1}^*, x_{j,2}^*, \ldots, x_{j,m_j}^*\} \subseteq S_{f(A_j)}$. So, $\|x_{j,i}^*\| = 1 = \langle x_{j,i}^*, f(A_j)\rangle$. For $1 \le j \le r$ and $1 \le i \le m_j$ define $F_{j,i} \in (L^\infty(\mu, X))^*$ by $F_{j,i}(g) = \langle x_{j,i}^*, g(A_j)\rangle$. These are $m_1 + m_2 + \cdots + m_r = k$ linear functionals attaining their norm at f. Suppose that $\sum_{j=1}^r\sum_{i=1}^{m_j} a_{j,i}F_{j,i} = 0$ for some scalars $a_{j,i}$. Then $\sum_{j=1}^r\sum_{i=1}^{m_j} a_{j,i}\langle x_{j,i}^*, g(A_j)\rangle = 0$, $\forall g \in L^\infty(\mu, X)$. Choosing $g_j(t) = x$; if $t \in A_j$ and $g_j(t) = 0$; if $t \notin A_j$, where $x \in X$, we get $(\sum_{i=1}^{m_j} a_{j,i}x_{j,i}^*)(x) = 0$ for all $x \in X$ and $j = 1, 2, \ldots, r$. Consequently, $\sum_{i=1}^{m_j} a_{j,i}x_{j,i}^* = 0$ for all $j = 1, 2, \ldots, r$. Since $\{x_{j,1}^*, x_{j,2}^*, \ldots, x_{j,m_j}^*\} \subseteq S_{f(A_j)}$ is linearly independent, then $a_{j,i} = 0$ for all $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, m_j$. Therefore, the $F_{j,i}$'s are linearly independent.

For an atom A and $g \in L^\infty(\mu, X)$ let $g_A = g(A)\chi_A \in L^\infty(\mu, X)$. We will prove that if $F \in (L^\infty(\mu, X))^*$ with $\|F\| = 1 = F(f)$ then $F(g) = \sum_{j=1}^r F(g_{A_j})$ for all $g \in L^\infty(\mu, X)$. Without loss of generality, say $\|g\| = 1$. We claim that there is $\epsilon > 0$ such that $\|f \pm \epsilon g\| \le 1$. Indeed, if such ϵ does not exist, we would have a sequence (t_k) outside $\cup_{j=1}^r A_j$ such that $\|f(t_k) \pm (1/k)g(t_k)\| > 1$. But then

$$1 < \left\| f(t_k) \pm \frac{1}{k} g(t_k) \right\| \le \|f(t_k)\| + \frac{1}{k}\|g(t_k)\| \le \|f(t_k)\| + \frac{1}{k}. \tag{5}$$

Hence, $\|f(t_k)\| > 1 - 1/k$ for all k, a contradiction to our assumption. Thus, $|F(f \pm \epsilon g)| \le 1$ and therefore $F(g) = 0$ for all $g \in L^\infty(\mu, X)$ such that $g(A_j) = 0$, $\forall 1 \le j \le r$. This proves that $F(g) = \sum_{j=1}^r F(g_{A_j})$ for all $g \in L^\infty(\mu, X)$. Let $Z = \oplus_{j=1}^r X(l^\infty\text{-sum})$ and let $E = \{h \in Z^* : \|h\| = 1 = h(f(A_1), f(A_2), \ldots, f(A_r))\}$. Then by Krein-Millman Theorem we have $E = \overline{\text{co}\,\text{ext}\,E}$ with $\text{ext}\,E \subseteq \text{ext}\,B(Z^*)$. So, any $h \in \text{ext}\,E$ has the form $h = (0, 0, \ldots, 0, x^*, 0, \ldots, 0)$ for some $x^* \in \text{ext}\,B(X^*)$. Note that if, for example, $h_1, h_2 \in \text{ext}\,E$ has the form $h_1 = (x^*, 0, 0, \ldots, 0)$, $h_2 = (y^*, 0, 0, \ldots, 0)$, then $\langle x^*, f(A_1)\rangle = \langle y^*, f(A_1)\rangle = 1$. Since $f(A_1) \in m_1\text{-smooth}\,B(X)$, then any $m_1 + 1h$'s of the above form must be linearly dependent. Consequently, any $m_1 + m_2 + \cdots + m_r + 1 = k + 1$ elements $h \in E$ must be linearly dependent.

Now, let $F_1, F_2, \ldots, F_{k+1} \in L^\infty(\mu, X)^*$ such that $\|F_i\| = 1 = F_i(f)$. We will prove that the F_i's are linearly dependent. By the argument above we see that $F_i(g) = \sum_{j=1}^r F_i(g_{A_j})$ for all $g \in L^\infty(\mu, X)$. For $1 \le i \le k+1$ define $h_i \in Z^* = \oplus_{j=1}^r X^*(l^1\text{-sum})$ by $h_i(x_1, x_2, \ldots, x_r) = F_i(\sum_{j=1}^r h_{x_j})$, where $h_{x_j} = x_j \chi_{A_j}$. Then $h_i \in Z^*$ with $\|h_i\| = 1 = h_i(f(A_1), f(A_2), \ldots, f(A_r)) = \sum_{j=1}^r F_i(f_{A_j}) = F_i(f)$ so that $h_i \in E$ for all $i = 1, 2, \ldots, k+1$ and therefore $\{h_1, h_2, \ldots, h_{k+1}\}$ is linearly dependent. Hence, there are scalars $a_1, a_2, \ldots, a_{k+1}$, not all zeros, with $\sum_{i=1}^{k+1} a_i h_i = 0$. The proof will be complete if we show that $\sum_{i=1}^{k+1} a_i F_i = 0$. Indeed, if $g \in L^\infty(\mu, X)$ then $\sum_{i=1}^{k+1} a_i h_i(g(A_1), g(A_2), \ldots, g(A_r)) = \sum_{i=1}^{k+1} a_i F_i(\sum_{j=1}^r g_{A_j}) = \sum_{i=1}^{k+1} a_i F_i(g) = 0$. This shows that $f \in k\text{-smooth}\,B(L^\infty(\mu, X))$ and completes the proof of the "if" part. □

4. Multismoothness in Direct Sums of Banach Spaces

Lin and Rao characterized in [9] multisoothness in l^∞-direct sums and proved the following theorem.

Theorem 6 (see [9]). *Let $\{X_i : i \in I\}$ be an infinite family of nonzero Banach spaces. Let $X = \oplus_\infty X_i$ and let $x = (x_i)$ be a unit vector in X. Let $I_1 = \{i \in I : \|x_i\| < 1\}$, let $I_2 = \{i \in I : \|x_i\| = 1$ and x_i is a multismooth point of finite order$\}$, and let $I_3 = I \setminus (I_1 \cup I_2)$. Then, x is a multismooth point of finite order if and only if $I_3 = \emptyset$, I_2 is finite, and $\sup_{i \in I_1} \|x_i\| < 1$. In this case the order of smoothness of x is $k = \sum_{i \in I_2} m_i$, where m_i is the order of smoothness of x_i in X_i, $i \in I_2$.*

In this section we deal with l^1-direct sums. Indeed we prove the following result.

Theorem 7. *Let $\{X_i : i \in I\}$ be any family of nonzero Banach spaces. Let $X = \oplus_1 X_i$ and let $x = (x_i)$ be a unit vector in X. Let $I_1 = \{i \in I : x_i = 0$ and $\dim X_i < \infty\}$, $I_2 = \{i \in I : x_i = 0$ and $\dim X_i = \infty\}$, $I_3 = \{i \in I : x_i \ne 0$ and $x_i/\|x_i\| \in \text{smooth}\,B(X_i)\}$, $I_4 = \{i \in I : x_i \ne 0$ and $x_i/\|x_i\| \in r\text{-smooth}\,B(X_i)$ for some natural number $r \ge 2\}$, and $I_5 = I \setminus \cup_{i=1}^4 I_i$. Then x is a multismooth point of finite order if and only if $I_2 = I_5 = \emptyset$ and $I_1 \cup I_4$ is finite. In this case the order of smoothness of x is $k = \sum_{i \in I_1} \dim X_i + \sum_{i \in I_4} m_i - |I_4| + 1$, where m_i is the order of smoothness of x_i in X_i, $i \in I_4$ and $|I_4|$ is the number of elements in I_4.*

For the sake of completeness, let us first state and prove the characterization of smoothness.

Theorem 8. *Let $\{X_i : i \in I\}$ be any family of nonzero Banach spaces. Let $X = \oplus_1 X_i$ and let $x = (x_i)$ be a unit vector in X. Then x is a smooth point if and only if for any $i \in I$, $x_i \ne 0$ and $x_i/\|x_i\| \in \text{smooth}\,B(X_i)$.*

Proof. Suppose $j \in I$ and $x_j = 0$, or $x_j \ne 0$ but $x_j/\|x_j\|$ is not a smooth point in X_j. Then there are distinct unit functionals $y_0^*, z_0^* \in X_j^*$ with $y_0^*(x_j) = z_0^*(x_j) = \|x_j\|$. For $i \ne j$, choose any unit functional $x_i^* \in X_i^*$ with $x_i^*(x_i) = \|x_i\|$. Let $y^* = (y_i^*)$ and $z^* = (z_i^*)$ where $y_i^* = z_i^* = x_i^*$, $i \ne j$, $y_j^* = y_0^*$ and $z_j^* = z_0^*$. Clearly, y^* and z^* are distinct elements in S_x. So x is not a smooth point.

Conversely, suppose $x_i^* \in X_i^*$ is the unique element in $S_{x_i/\|x_i\|}$, $i \in I$. Let $x^* = (x_i^*)$. Then $x^* \in X^*$ with $\|x^*\| = \sup_{i \in I} \|x_i^*\| = 1$ and $x^*(x) = \sum_{i \in I} x_i^*(x_i) = \sum_{i \in I} \|x_i\| = \|x\| = 1$. Now if $y^* = (y_i^*) \in S_x$ then $\sum_{i \in I} y_i^*(x_i) = 1 = \sum_{i \in I} \|x_i\|$. Since $y_i^*(x_i) \le \|x_i\|$ then $y_i^*(x_i/\|x_i\|) = 1$ for all $i \in I$ and hence $y^* = x^*$. □

Lemma 9. *Let Y and Z be nonzero Banach spaces and $X = Y \oplus_1 Z$. Let $x = (0, z) \in X$ where $z \in k\text{-smooth}\,B(Z)$. If $\dim Y = n < \infty$ then $x \in (k+n)\text{-smooth}\,B(X)$.*

Proof. Let $z_1^*, z_2^*, \ldots, z_k^* \in S_z$ be linearly independent and choose n linearly independent unit functionals $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$. Let

$$x_i^* = \begin{cases} (0, z_i^*), & \text{if } 1 \le i \le k, \\ (y_{i-k}^*, z_1^*), & \text{if } k+1 \le i \le k+n. \end{cases} \tag{6}$$

Clearly $x_i^* \in S_x$; $1 \le i \le k+n$. They are linearly independent. Indeed if $\sum_{i=1}^{k+n} a_i x_i^* = 0$ then $\sum_{i=1}^n a_{k+i} y_i^* = 0$ and $\sum_{i=1}^k a_i z_i^* + (\sum_{i=1}^n a_{k+i}) z_1^* = 0$. Since $\{y_1^*, y_2^*, \ldots, y_n^*\}$ is lineraly independent then $a_{k+i} = 0$; $1 \le i \le n$ and thus $\sum_{i=1}^k a_i z_i^* = 0$. But $\{z_1^*, z_2^*, \ldots, z_k^*\}$ is also lineraly independent. So $a_i = 0$; $1 \le i \le k$. Hence $\{x_1^*, x_2^*, \ldots, x_{k+n}^*\}$ is lineraly independent.

Now suppose $g_i^* = (h_i^*, k_i^*) \in X^* = Y^* \oplus_\infty Z^*$; $1 \le i \le k+n+1$, where $h_i^* \in Y^*$ and $k_i^* \in Z^*$ with $\|g_i^*\| = 1 = g_i^*(x) = k_i^*(z)$. Thus $g_i^* = (h_i^*, k_i^*) \in$

$Y^*\oplus_\infty \operatorname{sp} S_z$. Since $\dim(Y^*\oplus_\infty \operatorname{sp} S_z) = k+n$ then we see that $g_1^*, g_2^*, \ldots, g_{k+n+1}^*$ must be linearly dependent. Therfore $x \in (k+n)$-smooth $B(X)$. □

Lemma 10. *Let Y and Z be nonzero Banach spaces and $X = Y\oplus_1 Z$. Let $x = (y,z) \in X$ where $y, z \neq 0$, $y/\|y\| \in$ m-smooth $B(Y)$ and $z/\|z\| \in k$-smooth $B(Z)$. Then $x \in (m+k-1)$-smooth $B(X)$.*

Proof. Say $m \geq k$. Let $\{y_1^*, y_2^*, \ldots, y_m^*\} \subseteq S_{y/\|y\|}$ and let $\{z_1^*, z_2^*, \ldots, z_k^*\} \subseteq S_{z/\|z\|}$ be linearly independent sets. Let

$$x_i^* = \begin{cases} (y_i^*, z_1^*), & \text{if } 1 \leq i \leq m, \\ (y_{i-m+1}^*, z_{i-m+1}^*), & \text{if } m+1 \leq i \leq m+k-1. \end{cases} \tag{7}$$

Clearly $x_i^* \in S_x$; $1 \leq i \leq m+k-1$. They are linearly independent. Indeed if $\sum_{i=1}^{m+k-1} a_i x_i^* = 0$ then $\sum_{i=1}^{m} a_i y_i^* + \sum_{i=m+1}^{m+k-1} a_i y_{i-m+1}^* = 0$ and $\sum_{i=m+1}^{m+k-1} a_i z_{i-m+1}^* + (\sum_{i=1}^{m} a_i) z_1^* = 0$. Since $\{y_1^*, y_2^*, \ldots, y_m^*\}$ and $\{z_1^*, z_2^*, \ldots, z_k^*\}$ are linearly independent sets then $a_1 = 0$, $a_i = 0$; $k+1 \leq i \leq m$, $a_i + a_{m+i-1} = 0$; $2 \leq i \leq n$ and $a_i = 0$; $m+1 \leq i \leq m+k-1$. This makes $a_i = 0$ for all $1 \leq i \leq m+k-1$. Hence $\{x_1^*, x_2^*, \ldots, x_{m+k-1}^*\}$ is linearly independent.

Now to prove that $x \in (m+k-1)$-smooth $B(X)$ it is clearly enough to show that any $x^* \in S_x$ must be in $\operatorname{sp}\{x_1^*, x_2^*, \ldots, x_{m+k-1}^*\}$. So suppose $x^* = (h^*, k^*) \in S_x$, where $h^* \in Y^*$ and $k^* \in Z^*$. Then

$$\begin{aligned} 1 = x^*(x) = h^*(y) + k^*(z) &\leq \|h^*\| \|y\| + \|k^*\| \|z\| \\ &\leq \|y\| + \|z\| = \|x\| = 1. \end{aligned} \tag{8}$$

So $h^* \in S_{y/\|y\|}$ and $k^* \in S_{z/\|z\|}$ and hence there are scalars α_i; $1 \leq i \leq m$ and β_j; $1 \leq j \leq k$ such that $h^* = \sum_{i=1}^{m} \alpha_i y_i^*$ and $k^* = \sum_{j=1}^{k} \beta_j z_j^*$. Thus $1 = h^*(y/\|y\|) = \sum_{i=1}^{m} \alpha_i y_i^*(y/\|y\|) = \sum_{i=1}^{m} \alpha_i$. Similarly $\sum_{j=1}^{k} \beta_j = 1$.

let $G = (y^{**}, z^{**}) \in X^{**} = Y^{**}\oplus_1 Z^{**}$ with $G(x_i^*) = 0$ for all $1 \leq i \leq m+k-1$. Then $y^{**}(y_i^*) + z^{**}(z_1^*) = 0$; $1 \leq i \leq m$ and $y^{**}(y_{i-m+1}^*) + z^{**}(z_{i-m+1}^*) = 0$; $m+1 \leq i \leq m+k-1$. Thus $G(x^*) = y^{**}(h^*) + z^{**}(k^*) = y^{**}(\sum_{i=1}^{m} \alpha_i y_i^*) + z^{**}(\sum_{j=1}^{k} \beta_j z_j^*) = -\sum_{i=1}^{m} \alpha_i z^{**}(z_1^*) + \beta_1 z^{**}(z_1^*) - \sum_{j=2}^{k} \beta_j z^{**}(z_1^*) = 0$ since $\sum_{i=1}^{m} \alpha_i = 1 = \sum_{j=1}^{k} \beta_j$. This proves that $x^* \in \operatorname{sp}\{x_1^*, x_2^*, \ldots, x_{m+k-1}^*\}$. Therefore $x \in (m+k-1)$-smooth $B(X)$. □

We now can easily prove Theorem 7.

Proof of Theorem 7. First note that if $I_2 \neq \emptyset$, $I_5 \neq \emptyset$, I_1 is infinite, or I_4 is infinite, then one can easily construct an infinite linearly independent subset of S_x. On the other hand, if $I_2 = I_5 = \emptyset$ and $I_1 \cup I_4$ is finite, then writing $X = \oplus_1 X_i$ as $X = W_1\oplus_1 W_2\oplus_1 W_3$, where $W_1 = \oplus_1\{X_i : i \in I_1\}$, $W_2 = \oplus_1\{X_i : i \in I_3\}$, $W_3 = \oplus_1\{X_i : i \in I_4\}$ and applying Theorem 8 and Lemmas 9 and 10 we get the result. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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