

Research Article

On the Level Set of a Function with Degenerate Minimum Point

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For $n \geq 2$, let M be an n -dimensional smooth closed manifold and $f : M \rightarrow \mathbb{R}$ a smooth function. We set $\min f(M) = m$ and assume that m is attained by unique point $p \in M$ such that p is a nondegenerate critical point. Then the Morse lemma tells us that if a is slightly bigger than m , $f^{-1}(a)$ is diffeomorphic to S^{n-1} . In this paper, we relax the condition on p from being nondegenerate to being an isolated critical point and obtain the same consequence. Some application to the topology of polygon spaces is also included.

1. Introduction and Statement of the Result

Throughout this paper, S^k denotes the standard topological sphere equipped with the standard differential structure. For $n \geq 2$, let M be an n -dimensional smooth closed manifold and $f : M \rightarrow \mathbb{R}$ a smooth function. We set $\min f(M) = m$ and assume that m is attained by unique point $p \in M$. Then the following result is a consequence of the Morse lemma (see, e.g., [1]): *If p is a nondegenerate critical point and there are no critical points in $f^{-1}((m, a])$ (where $m < a$), then there is a diffeomorphism $f^{-1}(a) \cong S^{n-1}$.*

The purpose of this paper is to study the question of whether the same result holds if we relax the condition on p from being nondegenerate to being an isolated critical point. We also give an application of our result to the topology of polygon spaces.

Now our main result is the following.

Theorem A. *For $n \geq 2$, let M be an n -dimensional smooth closed manifold and $f : M \rightarrow \mathbb{R}$ a smooth function. One sets $\min f(M) = m$ and assumes that m is attained by unique point $p \in M$. If p is an isolated critical point and there are no critical points in $f^{-1}((m, a])$ (where $m < a$), then the following results hold:*

- (i) *If $n \neq 5$, there is a diffeomorphism $f^{-1}(a) \cong S^{n-1}$.*
- (ii) *If $n = 5$, there is a homeomorphism $f^{-1}(a) \approx S^4$.*

Corollary B. *For $n \geq 2$, let M be an n -dimensional smooth closed manifold and $f : M \rightarrow \mathbb{R}$ a smooth function. One sets $\max f(M) = m$ and assumes that m is attained by unique point $p \in M$. If p is an isolated critical point and there are no critical points in $f^{-1}([a, m))$ (where $a < m$), then the following results hold:*

- (i) *If $n \neq 5$, there is a diffeomorphism $f^{-1}(a) \cong S^{n-1}$.*
- (ii) *If $n = 5$, there is a homeomorphism $f^{-1}(a) \approx S^4$.*

This paper is organized as follows. In Section 2 we prove Theorem A. In Section 3 we study an application of it. Theorem 4 is the main result in this section. Remark 7(ii) states the essential difference between the known map and ours.

2. Proof of Theorem A

We keep the notations of Theorem A. For $r \in \mathbb{R}$, we set

$$\begin{aligned} M^r &:= f^{-1}([m, r]), \\ \overset{\circ}{M}^r &:= f^{-1}((m, r)). \end{aligned} \tag{1}$$

Lemma 1. *There is a diffeomorphism $\overset{\circ}{M}^a \cong \mathbb{R}^n$.*

Proof. By [2, Lemma 3], it will suffice to prove that, for any compact set $K \subset \overset{\circ}{M}^a$, there exists an open set V of $\overset{\circ}{M}^a$ such that $K \subset V$ and $V \cong \mathbb{R}^n$.

We fix an open set U which satisfies that

$$p \in U \subset \bar{U} \subset \overset{\circ}{M}^a, \quad U \cong \mathbb{R}^n. \quad (2)$$

(Such U indeed exists because $p \in \overset{\circ}{M}^a$.) Note that $p \notin M^a \setminus U$ and $M^a \setminus U$ is compact. Note also that \bar{U} is compact. Hence there exist ε_0 and $\varepsilon_1 \in \mathbb{R}$ (where $m < \varepsilon_0 < \varepsilon_1$) such that

$$M^{\varepsilon_0} \subset U \subset \bar{U} \subset \overset{\circ}{M}^{\varepsilon_1} \subset \overset{\circ}{M}^a. \quad (3)$$

We fix such ε_0 and ε_1 .

For a compact set $K \subset \overset{\circ}{M}^a$, since $\max f(K) < a$ holds, there exists b such that $\max f(K) < b < a$. This implies that $K \subset \overset{\circ}{M}^b$.

By assumption, there are no critical points in $f^{-1}((m, a])$. Hence there is a diffeomorphism

$$\varphi : (M^{\varepsilon_1}, M^{\varepsilon_0}) \xrightarrow{\cong} (M^a, M^b). \quad (4)$$

We set $V := \varphi(U)$. Then V is an open set of $\overset{\circ}{M}^a$, $K \subset V$ and $V \cong \mathbb{R}^n$. \square

Lemma 2. *The manifold $f^{-1}(a)$ is homotopy equivalent to S^{n-1} .*

Proof. Since $f^{-1}(a)$ is a manifold, the assertion is clear for $n = 2$. We assume $n \geq 3$. We claim that

$$H_q(f^{-1}(a); \mathbb{Z}) \cong H_q(S^{n-1}; \mathbb{Z}) \quad \forall q. \quad (5)$$

To prove (5), note that M^a is a manifold with boundary such that $\partial M^a = f^{-1}(a)$. Lefschetz duality implies that

$$H_q(M^a, \partial M^a; \mathbb{Z}) \cong H^{n-q}(M^a; \mathbb{Z}) \quad \forall q. \quad (6)$$

Recall that the inclusion $i : \overset{\circ}{M}^a \rightarrow M^a$ is a homotopy equivalence. Since $\overset{\circ}{M}^a \cong \mathbb{R}^n$ by Lemma 1, M^a has the cohomology of a point. In the homology long exact sequence of the pair $(M^a, \partial M^a)$, we apply (6). Then we obtain (5).

Since the fundamental group at infinity of $\overset{\circ}{M}^a$ is trivial by Lemma 1, $f^{-1}(a)$ is simply connected. (See, e.g., [3, p. 389] or [4, Examples 5 and 6] for this kind of argument.) Hence $f^{-1}(a)$ is a homotopy sphere. \square

Proof of Theorem A. For $n = 2$ or 3, Lemma 2 immediately implies that $f^{-1}(a)$ is diffeomorphic to S^{n-1} .

For $n \geq 6$, Lemma 2 tells us that M^a is contractible with simply connected boundary. Using the h -cobordism theorem (see [5, p. 108, Proposition A]), we have $M^a \cong D^n$. Hence $f^{-1}(a) \cong S^{n-1}$.

For $n = 5$, combining Freedman's resolution of the 4-dimensional Poincaré conjecture [6] and Lemma 2, we have $f^{-1}(a) \approx S^4$.

For $n = 4$, combining Perelman's resolution of the 3-dimensional Poincaré conjecture [7], Lemma 2, and the fact [8] that the differential structure on S^3 is unique, we have $f^{-1}(a) \cong S^3$. \square

Proof of Corollary B. If f satisfies the assumption of Corollary B, the function $g := -f$ satisfies the assumption of f in Theorem A. Hence Corollary B follows. \square

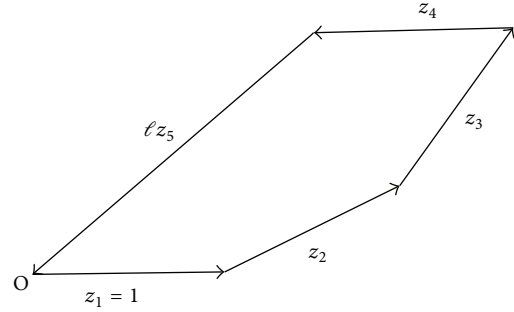


FIGURE 1: An element of $X_{5,\ell}$.

3. An Application

Starting in [9–11], the topology of the configuration space of planar polygons has been considered by many authors. We refer to [12] for an excellent exposition.

For simplicity, we consider the case that the edge lengths are $1, \dots, 1$ and ℓ . Let $SO(2)$ act on the n -dimensional torus $T^n = (S^1)^n$ diagonally. For $\ell > 0$, we set

$$X_{n,\ell} = \frac{\{(z_1, \dots, z_n) \in T^n \mid (\sum_{i=1}^{n-1} z_i) + \ell z_n = 0\}}{SO(2)}. \quad (7)$$

Here $z_i \in S^1 \subset \mathbb{C}$ denotes the unit vectors in the direction of the sides of a polygon.

It is clear that $X_{n,\ell} = \emptyset$ for $\ell > n - 1$ and $X_{n,n-1} = \{\text{one point}\}$. It is also known that there is a diffeomorphism

$$X_{n,\ell} \cong S^{n-3} \quad \text{for } n - 3 < \ell < n - 1. \quad (8)$$

Recall that (8) can be understood Morse-theoretically. The following arguments are particularly well described in [12]: Using the $SO(2)$ -action in the definition of $X_{n,\ell}$, we normalize z_1 to be 1 and write $X_{n,\ell}$ as

$$X_{n,\ell} = \left\{ (z_1, \dots, z_n) \in T^n \mid z_1 = 1, \left(\sum_{i=1}^{n-1} z_i \right) + \ell z_n = 0 \right\}. \quad (9)$$

(See Figure 1.)

We identify $T^{n-2} = \{(z_1, \dots, z_{n-1}) \in T^{n-1} \mid z_1 = 1\}$ and define a function $f_n : T^{n-2} \rightarrow \mathbb{R}$ by

$$f_n(z_1, \dots, z_{n-1}) = \left| \sum_{i=1}^{n-1} z_i \right|^2. \quad (10)$$

Then (9) gives an identification $X_{n,\ell} = f_n^{-1}(\ell^2)$. It is elementary to prove that an element $(z_1, \dots, z_{n-1}) \in T^{n-2} \setminus f_n^{-1}(0)$ is a critical point of f_n if and only if $z_i \in \{-1, 1\}$ for all $2 \leq i \leq n-1$. All these critical points are nondegenerate and their indices are known. (See, e.g., [9, 12–14].) In particular, f_n attains its maximum value at $(z_1, \dots, z_{n-1}) = (1, \dots, 1)$ and this is a nondegenerate critical point. Hence, using the Morse lemma, (8) follows.

The configuration space of spatial polygons has also been studied by many authors. (See, e.g., [12, 14, 15].) In this case, our object is defined by

$$Y_{n,\ell} = \frac{\{(a_1, \dots, a_n) \in (S^2)^n \mid (\sum_{i=1}^{n-1} a_i) + \ell a_n = 0\}}{SO(3)}. \quad (11)$$

Recently, motivated by chemistry, Crippen [16], Goto and Komatsu [17], and O'Hara [18] studied certain subspaces of $Y_{n,1}$. Namely, they studied the configuration space of *equilateral* polygons with restriction on the splay angle of each vertex.

First, we define the angle α_n to be $(7/12)\pi$ and $\arccos(-1/3)$ as $n = 5$ and $n \geq 6$, respectively. Goto and Komatsu [17] chose the angle α_n with molecular model in mind. Then they studied the space defined by

$$Z_n = \{(a_1, \dots, a_n) \in Y_{n,1} \mid -\langle a_i, a_{i+1} \rangle = \cos \alpha_n \text{ for } 1 \leq i \leq n-2\}, \quad (12)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^3 . The closed chains in Z_n are equilateral polygons in \mathbb{R}^3 with n vertices such that the interior angles are all equal to α_n except for the two angles at the successive vertices O and $\sum_{i=1}^{n-1} a_i$.

The main result in [17] states that when $n = 5, 6, 7$, Z_n is a manifold homeomorphic to S^{n-4} . Since they use Reeb's theorem, they state their result as a homeomorphism. But actually Z_n and S^{n-4} are diffeomorphic because the differential structure on S^k is unique for $k \leq 3$. But what is more important is that it is not known whether Z_n is a manifold for $n \geq 8$.

Second, for all $\beta \in \mathbb{R}$, we define a space by

$$W_n(\beta) = \{(a_1, \dots, a_n) \in Y_{n,1} \mid -\langle a_i, a_{i+1} \rangle = \cos \beta \text{ for } 1 \leq i \leq n\}. \quad (13)$$

Here we understand a_{n+1} to be a_1 . The closed chains in $W_n(\beta)$ are equilateral polygons in \mathbb{R}^3 with n vertices such that the interior angles are all equal to β . Crippen [16] studied the topological type of $W_n(\beta)$ for various β when $n = 3, 4, 5$. The result is that $W_n(\beta)$ is either \emptyset , one point, or two points depending on β . Next, O'Hara [18] studied $W_6(\beta)$ for various β . The result is that $W_6(\beta)$ is disjoint union of a certain number of S^1 's and points. It is to be noted that since $\dim Y_{n,1} = 2n - 6$, it is natural to expect that $\dim W_n(\beta) = 2n - 6 - n = n - 6$. But the above results imply that the defining equations for $W_n(\beta)$ do not intersect transversally when n is small.

Note that the above results in [16–18] concentrate on the case for small n . This is understandable because imposing some conditions on the interior angles causes difficulties in computations. Nevertheless, we would like to prove some assertion which holds for general n . Modifying the definition of Z_n , we define a space as follows:

$$A_{n,\ell} = \{(a_1, \dots, a_n) \in Y_{n,\ell} \mid \langle a_i, a_{i+1} \rangle = 0 \text{ for } 1 \leq i \leq n-2\}. \quad (14)$$

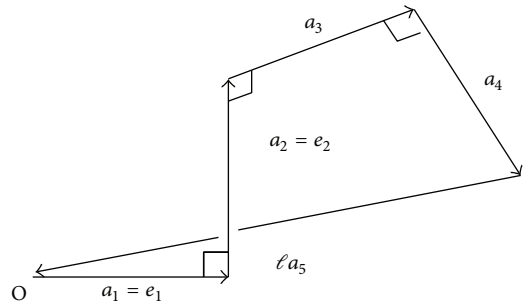


FIGURE 2: An element of $A_{5,\ell}$.

The closed chains in $A_{n,\ell}$ are polygons in \mathbb{R}^3 with edge lengths $1, \dots, 1$ and ℓ such that the interior angles are all equal to $\pi/2$ except for the two angles at the endpoints of the edge of length ℓ .

Let us obtain a similar description to (9). We set $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. By the $SO(3)$ -action in the definition of $Y_{n,\ell}$, we can normalize a_1 and a_2 to be e_1 and e_2 , respectively. Thus we have the following description of $A_{n,\ell}$:

$$\begin{aligned} A_{n,\ell} &= \left\{ (a_1, \dots, a_n) \in (S^2)^n \mid a_1 = e_1, a_2 = e_2, \left(\sum_{i=1}^{n-1} a_i \right) + \ell a_n = 0, \langle a_i, a_{i+1} \rangle = 0 \ (2 \leq i \leq n-2) \right\}. \end{aligned} \quad (15)$$

Hereafter we use description (15). (See Figure 2.)

We recall the following.

Theorem 3 (see [19, Theorem 5.6], [20, Theorem 4]). *One sets*

$$r_n := \begin{cases} \sqrt{2}m & n = 2m + 1, \\ \sqrt{2m^2 - 2m + 1} & n = 2m. \end{cases} \quad (16)$$

Then the following results hold:

- (i) $A_{n,\ell} = \emptyset$ if $\ell > r_n$.
- (ii) $A_{n,r_n} = \{\text{one point}\}$. We write the element of A_{n,r_n} by $P_n = (a_1, \dots, a_n)$ in the notation of (15). Then P_n is given as follows: For $1 \leq i \leq n-1$, a_i is given by

$$a_i = \begin{cases} e_1 & i \text{ is odd,} \\ e_2 & i \text{ is even.} \end{cases} \quad (17)$$

As a consequence, a_n is given by

$$a_n = \begin{cases} -\frac{m}{r_n} (e_1 + e_2) & n = 2m + 1, \\ -\frac{1}{r_n} (me_1 + (m-1)e_2) & n = 2m. \end{cases} \quad (18)$$

Note that P_n is a planar n -gon. (See Figure 3.)

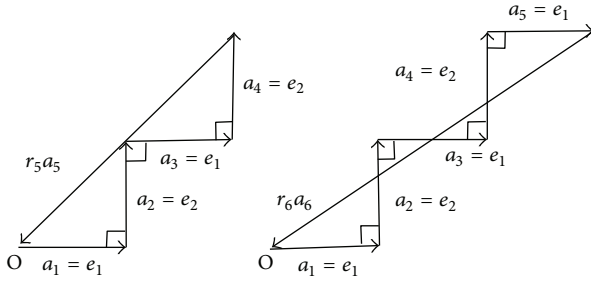


FIGURE 3: P_5 and P_6 .

Now we state the main result in this section (cf. (8)).

Theorem 4. *Let ℓ be slightly smaller than r_n . Then, for all $n \geq 5$, there is a diffeomorphism $A_{n,\ell} \cong S^{n-4}$.*

For the rest of this paper, we prove Theorem 4. For that purpose, we define

$$B_n = \left\{ (a_1, \dots, a_{n-1}) \in (S^2)^{n-1} \mid a_1 = e_1, a_2 = e_2, \langle a_i, a_{i+1} \rangle = 0 \ (2 \leq i \leq n-2) \right\}. \quad (19)$$

Moreover, similarly to f_n in (10), we define a map $g_n : B_n \rightarrow \mathbb{R}$ by

$$g_n(a_1, \dots, a_{n-1}) = \left\| \sum_{i=1}^{n-1} a_i \right\|^2. \quad (20)$$

Then (15) gives an identification $A_{n,\ell} = g_n^{-1}(\ell^2)$ and Theorem 3 tells us that $\max g_n(B_n)$ is attained by P_n . In order to compute the Hessian matrix of g_n at P_n , we construct the commutative diagram shown in Figure 4.

First, we set

$$\pi_n(\theta_1, \dots, \theta_{n-3}) = (e^{i\theta_1}, \dots, e^{i\theta_{n-3}}). \quad (21)$$

Namely, π_n is the universal cover.

Second, we construct ψ_n by induction on n . We define ψ_3 to be the unique map between one-point spaces.

Assuming that ψ_n is constructed, we construct ψ_{n+1} . We write

$$\psi_n(\theta_1, \dots, \theta_{n-3}) = (a_1, \dots, a_{n-1}). \quad (22)$$

Since $\langle a_{n-2}, a_{n-1} \rangle = 0$, there are two choices for $\xi \in S^2$ which satisfies that $\langle \xi, a_{n-2} \rangle = \langle \xi, a_{n-1} \rangle = 0$. Among these ξ 's, we choose the one which satisfies the condition that $\det(a_{n-2}, a_{n-1}, \xi) = (-1)^{n+1}$. Using this, we define

$$\eta := (\cos \theta_{n-2}) a_{n-2} + (\sin \theta_{n-2}) \xi. \quad (23)$$

And we set

$$\psi_{n+1}(\theta_1, \dots, \theta_{n-3}, \theta_{n-2}) := (a_1, \dots, a_{n-1}, \eta). \quad (24)$$

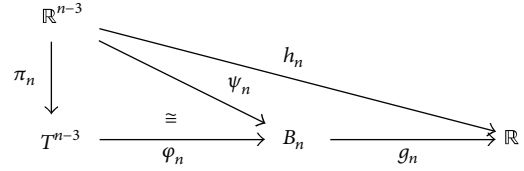


FIGURE 4

From the construction, ψ_n induces a map $\varphi_n : T^{n-3} \rightarrow B_n$ such that $\psi_n = \varphi_n \circ \pi_n$. It is easy to see that φ_n is a diffeomorphism. (See Remark 7(ii).)

Third, we set $h_n := g_n \circ \psi_n$, where g_n is defined in (20). Thus we have completed the construction of the diagram in Figure 4.

Note that $\psi_n(0, \dots, 0) = P_n$, where P_n is defined in Theorem 3.

Lemma 5. *Let $H(h_n)(0, \dots, 0)$ be the Hessian matrix of h_n at $(\theta_1, \dots, \theta_{n-3}) = (0, \dots, 0)$. Note that this is an $(n-3) \times (n-3)$ matrix. Let $\gamma(n, i, j)$ be the (i, j) th entry of $H(h_n)(0, \dots, 0)$. Then, for all $n \geq 5$, the following result holds:*

$$\gamma(n, i, j) = \begin{cases} -2 \left\lfloor \frac{n-1-i}{2} \right\rfloor \left\lfloor \frac{j+1}{2} \right\rfloor & \text{if } i \geq j, \\ -2 \left\lfloor \frac{n-1-j}{2} \right\rfloor \left\lfloor \frac{i+1}{2} \right\rfloor & \text{if } i < j. \end{cases} \quad (25)$$

Proof. The lemma is proved by direct computations. \square

Proof of Theorem 4 for Even n. We set

$$\Gamma(n, i) := \gamma(n, i, i+1) - \frac{\gamma(n, i, 1)}{\gamma(n, n-3, 1)} \gamma(n, n-3, i+1). \quad (26)$$

It is easy to see that when n is even,

$$\det(H(h_n)(0, \dots, 0)) = -2 \prod_{i=1}^{n-4} \Gamma(n, i). \quad (27)$$

Thus we have $\det(H(h_n)(0, \dots, 0)) \neq 0$; hence $(0, \dots, 0)$ is a nondegenerate critical point of h_n . Using the Morse lemma, we complete the proof of Theorem 4 for even n . \square

Proof of Theorem 4 for Odd n. Lemma 5 tells us that when n is odd, $\det(H(h_n)(0, \dots, 0)) = 0$. Hence we need to use Corollary B. For that purpose, it will suffice to prove that $(0, \dots, 0)$ is an isolated critical point of h_n .

We define a map $F_n : S^{n-4} \times [0, \infty) \rightarrow \mathbb{R}^{n-3}$ by

$$F_n(u_1, \dots, u_{n-3}, t) = (\text{grad } h_n)(tu_1, \dots, tu_{n-3}, t), \quad (28)$$

where the right-hand side denotes the Jacobian matrix of h_n at

$$(\theta_1, \dots, \theta_{n-3}) = (tu_1, \dots, tu_{n-3}). \quad (29)$$

First we prove the following.

Lemma 6. For each $(u_1, \dots, u_{n-3}) \in S^{n-4}$, there exists $\varepsilon > 0$ such that $F_n(u_1, \dots, u_{n-3}, t) \neq 0$ for all $t \in (0, \varepsilon)$.

Proof. We fix $(u_1, \dots, u_{n-3}) \in S^{n-4}$. In order to prove the lemma by contradiction, assume that $t = 0$ were an accumulation point of $F_n(u_1, \dots, u_{n-3}, t)$. For $1 \leq k \leq n - 3$, let $F_{n,k}(u_1, \dots, u_{n-3}, t)$ be the k th element of $F_n(u_1, \dots, u_{n-3}, t)$. Since $F_{n,k}(u_1, \dots, u_{n-3}, t)$ is a polynomial in $\{\sin(tu_i), \cos(tu_i) \mid 1 \leq i \leq n - 3\}$, we can define $F_{n,k}(u_1, \dots, u_{n-3}, t)$ for $t \in \mathbb{C}$ and this is a holomorphic function. If $t = 0$ were an accumulation point of $F_n(u_1, \dots, u_{n-3}, t)$, then the identity theorem would tell us that $F_{n,k}(u_1, \dots, u_{n-3}, t)$ is identically 0 for all $t \in \mathbb{C}$ and for all $1 \leq k \leq n - 3$.

We write the Maclaurin expansion of $F_{n,k}(u_1, \dots, u_{n-3}, t)$ as

$$\begin{aligned}
 F_{n,k}(u_1, \dots, u_{n-3}, t) &= \sigma_k(u_1, \dots, u_{n-3})t \\
 &\quad + \tau_k(u_1, \dots, u_{n-3})t^3 \\
 &\quad + \text{Higher Order Terms in } t,
 \end{aligned}
 \tag{30}$$

where $\sigma_k(u_1, \dots, u_{n-3})$ and $\tau_k(u_1, \dots, u_{n-3})$ are polynomials in u_1, \dots, u_{n-3} of degrees 1 and 3, respectively. Since $F_{n,k}(u_1, \dots, u_{n-3}, t) \equiv 0$, we have

$$\begin{aligned}
 \sigma_k(u_1, \dots, u_{n-3}) &= \tau_k(u_1, \dots, u_{n-3}) = 0 \\
 \forall 1 \leq k \leq n - 3.
 \end{aligned}
 \tag{31}$$

Then standard computations show that $u_1 = \dots = u_{n-3} = 0$. This contradicts the assumption that $(u_1, \dots, u_{n-3}) \in S^{n-4}$.

This completes the proof of Lemma 6. \square

Now using Lemma 6 and the continuity of the function F_n , there exists an open neighborhood U of O in \mathbb{R}^{n-3} such that if $(a_1, \dots, a_{n-3}) \in U \setminus O$, then

$$(\text{grad } h_n)(a_1, \dots, a_{n-3}) \neq (0, \dots, 0).
 \tag{32}$$

Thus $(0, \dots, 0)$ is an isolated critical point of h_n .

This completes the proof of Theorem 4 for odd n .

Remark 7. (i) Recall that Corollary B for $\dim M = 5$ just gives a topological assertion. In order to prove Theorem 4, we have studied the map $g_n : B_n \rightarrow \mathbb{R}$. Since $\dim B_n = n - 3$, $\dim B_n = 5$ implies that $n = 8$. But we have seen in Lemma 5 that when n is even, P_n is a nondegenerate critical point and deduced Theorem 4 directly from the Morse lemma, which gives a differential assertion.

(ii) Although we have not used the diffeomorphism φ_n in the above arguments, it is to be noted that the map $g_n \circ \varphi_n : T^{n-3} \rightarrow \mathbb{R}$ is much more difficult than the known map $f_n : T^{n-2} \rightarrow \mathbb{R}$ in (10).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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