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Research Article

Periodic Solutions of Certain Differential Equations with Piecewise Constant Argument

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Existence criteria are derived for the eventually periodic solutions of a class of differential equations with piecewise constant argument whose solutions at consecutive integers satisfy nonlinear recurrence relations. The proof characterizes the initial values of periodic solutions in terms of the coefficients of the resulting difference equations. Sufficient conditions for the unboundedness, boundedness, and symmetry of general solutions also follow from the corresponding properties of the difference equations.

1. Introduction

Since the seminal works of Shah and Wiener [1] and Cooke and Wiener [2], differential equations with piecewise constant arguments of the form

$$x'(t) = f(t, x(t), x([t])), \tag{1}$$

where f is continuous and $[\cdot]$ is the greatest integer function, have been treated widely in the literature and applied to certain biomedical models (see [3–7] and references therein). Continuity of the solutions of these equations implies recurrence relations for the values of solutions at consecutive integers. Therefore, there is a natural interplay between properties of these differential equations and properties of difference equations.

In this paper, we consider a class of equations of the above form but where f is discontinuous: the chaotic and eventually periodic behavior and symmetry of solutions of initial-value problems of the form

$$x'(t) = Ax(t) + Bx([t]) + CF(x([t])), \quad x(0) = x_0 \tag{2}$$

on $[0,\infty)$ are determined, where A, B, and C are constants; $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x) \equiv \begin{cases} 0, & \text{if } x \le \lambda \\ 1, & \text{if } x > \lambda \end{cases} \tag{3}$$

for some positive number λ, and the sequence $x(n)$ ($n = 0, 1, \ldots$) satisfies a nonlinear difference equation. As in the case when f is continuous, by a solution of (2), we mean a function $x(t)$ that is defined on $[0,\infty)$ with these properties:

(1) $x(t)$ is continuous on $[0,\infty)$.

(2) The derivative $x'(t)$ exists at each point t in $[0,\infty)$, with the possible exception of the points $[t]$ in $[0,\infty)$, where one-sided derivatives exist.

(3) $x(t)$ satisfies (2) on $[n, n+1)$ for each nonnegative integer n.

Specifically, since

$$\frac{d}{dt}\left(x(t)e^{-At}\right) = Bx([t])e^{-At} + CF(x([t]))e^{-At}, \tag{4}$$

it follows that if n is a positive integer and $n-1 \le t' < n$, then

$$\begin{aligned} &x(t')e^{-At'} - x(n-1)e^{-A(n-1)} \\ &\quad = (Bx(n-1) + CF(x(n-1)))\int_{n-1}^{t'} e^{-At}dt. \end{aligned} \tag{5}$$

Letting t' increase to n, we have the difference equation

$$x(n) = a^* x(n-1) - b^* F(x(n-1)), \tag{6}$$

where $x(0) = x_0$,

$$a^* \equiv \begin{cases} e^A + B\dfrac{e^A - 1}{A}, & \text{if } A \neq 0 \\ 1 + B, & \text{if } A = 0, \end{cases}$$
$$b^* \equiv \begin{cases} -C\dfrac{e^A - 1}{A}, & \text{if } A \neq 0 \\ -C, & \text{if } A = 0. \end{cases} \tag{7}$$

Therefore, the unique solution to (2) is

$$x(t) = \left(e^{A(t-[t])} + B\int_{[t]}^{t} e^{A(t-s)}ds\right) x([t]) + C\left(\int_{[t]}^{t} e^{A(t-s)}ds\right) F(x([t])), \tag{8}$$

where

$$\int_{[t]}^{t} e^{A(t-s)}ds = \begin{cases} \dfrac{e^{A(t-[t])} - 1}{A}, & \text{if } A \neq 0 \\ t - [t], & \text{if } A = 0. \end{cases} \tag{9}$$

Moreover, $x(t)$ satisfies (1)–(3) above since $x(t)$ is continuous on $[0, \infty)$ with left derivatives

$$x'(n^-) = Ax(n) + Bx(n-1) + CF(x(n-1)) \tag{10}$$

at integers $n > 0$, with right derivatives

$$x'(n^+) = (A + B)x(n) + CF(x(n)) \tag{11}$$

at integers $n \geq 0$, and more generally with derivatives at nonintegral t given by

$$x'(t) = ((A + B)x([t]) + CF(x([t])))e^{A(t-[t])}. \tag{12}$$

We recall that the solution $x = x(t)$ to (2) is *oscillatory* if it has arbitrarily large zeros [3]. Accordingly, a sequence x_n ($n = 0, 1, \ldots$) is *oscillatory* if there exists a subsequence x_{n_k} such that $x_{n_k}x_{n_{k+1}} \leq 0$ for all k. Moreover, a *stationary state* of x_n is a term x_i such that $x_i = x_{i+1}$ [8]. The following holds for solutions of (2).

Proposition 1. *Let $x = x(t)$ be given by (8), where $x(n)$ ($n = 0, 1, \ldots$) satisfies (6).*

(1) *If $a^* \neq 1$ and $x(i)$ is a stationary state of $x(n)$, then either $x = x(t) = 0$ for all $t \geq i$ or $x = x(t) = b^*/(a^* - 1)$ for all $t \geq i$.*

(2) *Let $a^* > 1$.*

  (a) *If $x = x(t)$ is oscillatory, then $x(n)$ is oscillatory with stationary state 0.*

  (b) *If $x(n)$ diverges to ∞, then $\lim_{t\to\infty} x(t) = \infty$.*

  (c) *If $x(n)$ diverges to $-\infty$, then $\lim_{t\to\infty} x(t) = -\infty$.*

Proof. (1) Let $a^* \neq 1$ and let $x(i) = x(i+1)$. Suppose first that $x(i) \leq \lambda$. Then $x(i+1) = a^*x(i) = x(i)$ so $x(i) = x(i+1) = 0$ since $a^* \neq 1$. By induction, $x(i+j) = 0 < \lambda$ and $F(x(i+j)) = 0$ for all integers $j \geq 0$; and, by (8), $x(t) = 0$ for all $t \geq i$.

On the other hand, assume $x(i) > \lambda$. Then $x(i+1) = a^*x(i) - b^* = x(i)$ so $x(i) = x(i+1) = b^*/(a^* - 1) > \lambda$ since $a^* \neq 1$. Thus $x(i+2) = a^*x(i+1) - b^* = b^*/(a^* - 1) > \lambda$ and, by induction, $x(i+j) = b^*/(a^* - 1)$ for all integers $j \geq 0$. Since $b^*/(a^* - 1) = -C/(A + B)$ and $F(b^*/(a^* - 1)) = 1$, it follows from (8) that $x(t) = b^*/(a^* - 1)$ for all $t \geq i$.

(2) Let $a^* > 1$. Then $B > -A$ and in (8) we have

$$e^{A(t-[t])} + B\int_{[t]}^{t} e^{A(t-s)}ds > 1. \tag{13}$$

(a) Suppose by way of contradiction that $x = x(t)$ is oscillatory but 0 is not a stationary state of $x(n)$. It follows that $x(n-1) \neq 0$ for all positive integers n.

Assume first that $x(n-1) < 0$ for some integer $n \geq 1$. Then $F(x(n-1)) = 0$ and $x(n) = a^*x(n-1) < 0 < \lambda$. By induction, $x([t]) < 0$ and $F(x([t])) = 0$ for all $t \geq n - 1$.

Let $n - 1 \leq [t] < t < [t] + 1$. By (8) and (13), $x(t) < 0$ for all $t \geq n - 1$; and thus $x = x(t)$ is not oscillatory in this case, a contradiction.

Therefore, assume that $x(n-1) > 0$ for all positive integers n. Let n be a positive integer. We show that $x(t) > 0$ for all t in the interval $(n-1, n)$: note that $x'(t) = x'(n-1^+)e^{A(t-[t])}$.

Suppose that $x'(n-1^+) = 0$. Then $x'(t) = 0$ and

(i) if $A \neq 0$, then, by (2),

$$x(t) = -\frac{Bx(n-1) + CF(x(n-1))}{A} = x(n-1) > 0 \tag{14}$$

for all t in $(n-1, n)$;

(ii) if $A = 0$, then, by (2), $Bx(n-1) + CF(x(n-1)) = 0$ and, by (8),

$$x(t) = (1 + B(t - [t]))x(n-1) + C(t - [t])F(x(n-1)) = x(n-1) > 0 \tag{15}$$

for all t in $(n-1, n)$.

If $x'(n-1^+) > 0$, then $x(t)$ is strictly increasing on $(n-1, n)$ and $x(n-1) > 0$, so $x(t) > 0$ for all t in $(n-1, n)$ by continuity.

If $x'(n-1^+) < 0$, then $x(t)$ is strictly decreasing on $(n-1, n)$, but $x(t)$ is still positive on $(n-1, n)$ by continuity since $x(n-1)$ and $x(n)$ are both positive.

Therefore, $x(t) > 0$ for all t in $[n-1, n]$ (and hence for all t). Thus $x = x(t)$ is not oscillatory in this case, contrary to our assumption.

(b) Suppose that $\lim_{n\to\infty} x(n) = \infty$. There exists N such that $x(n) > \lambda$ for all integers $n \geq N$. Let $[t] \geq N$. Then $F(x([t])) = 1$ and, by (8) and (13),

$$x(t) \geq x([t]) + C\int_{[t]}^{t} e^{A(t-s)}ds. \tag{16}$$

Thus, if $C \geq 0$, then $x(t) \geq x([t])$. And if $C < 0$, then $x(t) \geq x([t]) - b^*$ since

$$0 \leq \frac{e^{A(t-[t])} - 1}{A} \leq \frac{e^A - 1}{A} \tag{17}$$

when $A \neq 0$ and $t - [t] < 1$ otherwise. Therefore, $\lim_{t\to\infty} x(t) = \infty$.

(c) Suppose that $\lim_{n\to\infty} x(n) = -\infty$. There exists N such that $x(n) \leq 0$ for all integers $n \geq N$. Let $[t] \geq N$. Then $F(x([t])) = 0$ and, by (8) and (13), $x(t) \leq x([t])$. Hence, $\lim_{t\to\infty} x(t) = -\infty$. □

Our main results characterize eventually periodic solutions $x = x(t)$ of (2). Since $x(t)$ is generally not differentiable at integers $t = n$ but is always differentiable between integers, we restrict our attention to integral periods (see [3, 6]). In this case, by (8), we have the following:

> *$x = x(t)$ is eventually periodic with positive integral period p if and only if $x(n)$ $(n = 0, 1, \ldots)$ is eventually periodic with period p.*

Equation (2) is similar to recent models related to neural networks ([8–12]). We treat a generalized version of (6) as follows.

For real numbers a, b, x_0, and $\lambda > 0$, define, for $n \geq 1$,

$$x_n = a x_{n-1} - bF(x_{n-1}), \tag{18}$$

where λ and F are given by (3).

Remark 2. Equation (18) is the difference equation (6) of a differential equation (2) with $x(0) \equiv x_0$, $\lambda \equiv \lambda$, and A being arbitrary: if $A = 0$, choose $B \equiv a - 1$ and $C \equiv -b$. And if $A \neq 0$, then let $B \equiv (a - e^A)A/(e^A - 1)$ and $C \equiv -bA/(e^A - 1)$. The resulting solution $x = x(t)$ given by (8) satisfies (2) with $a^* \equiv a$ and $b^* \equiv b$.

The next result shows that we may henceforth assume that $a > 1$ and $b > 0$ in (18).

Proposition 3. *Let x_n be defined by (18) for real a, b, x_0, and $\lambda > 0$, and let $x = x(t)$ satisfy (8), where a^* and b^* are given in (6). In the following, j is an arbitrary positive integer.*

(1) (a) Suppose that $0 \leq a \leq 1$. If $x_k \leq \lambda$, for some k, then $x_{k+j} = a^j x_k$. On the other hand, if $x_k > \lambda$, for all k, then

$$x_j = \begin{cases} a^j\left(x_0 - \dfrac{b}{a-1}\right) + \dfrac{b}{a-1}, & \text{if } a \neq 1 \\ x_0 - jb, & \text{if } a = 1\ (b \leq 0). \end{cases} \tag{19}$$

(b) In particular, assume that $0 \leq a^* \leq 1$. If $x(k) \leq \lambda$, for some nonnegative integer k, then, for all t' in $[k, k+1)$, one has that $x(t' + j) = a^{*j} x(t')$. On the other hand, if $x(k) > \lambda$, for all nonnegative integers k, then, for all t' in $[0, 1)$,

$$x(t'+j) = \begin{cases} a^{*j}\left(x(t') - \dfrac{b^*}{a^*-1}\right) + \dfrac{b^*}{a^*-1}, & \text{if } a^* \neq 1 \\ x(t') - jb^*, & \text{if } a^* = 1\ (b^* \leq 0). \end{cases} \tag{20}$$

(2) (a) Suppose that $a > 1$ and $b \leq 0$. If $x_0 \leq 0$, then $x_j = a^j x_0$. If $x_0 > 0$, then there exists $k \geq 1$ such that $x_{k+j} = a^j(x_k - b/(a-1)) + b/(a-1)$.

(b) In particular, assume that $a^* > 1$ and $b^* \leq 0$. If $x(0) \leq 0$, then $x(t' + j) = a^{*j} x(t')$ for all t' in $[0, 1)$. If $x(0) > 0$, then there exists a positive integer k such that $x(t' + j) = a^{*j}(x(t') - b^*/(a^* - 1)) + b^*/(a^* - 1)$ for all t' in $[k, k+1)$.

(3) (a) Suppose that $a < 0$. Either x_n is oscillatory or there exists $k \geq 0$ such that $x_{k+j} = a^j(x_k - b/(a-1)) + b/(a-1)$.

(b) In particular, assume that $a^* < 0$. Either $x = x(t)$ is oscillatory or there exists an integer $k \geq 0$ such that $x(t' + j) = a^{*j}(x(t') - b^*/(a^* - 1)) + b^*/(a^* - 1)$ for all t' in $[k, k+1)$.

Proof. (1a) Suppose that $0 \leq a \leq 1$. If $x_k \leq \lambda$, for some k, then either $0 \leq x_{k+1} = a x_k \leq x_k \leq \lambda$ or $x_{k+1} = a x_k \leq 0 < \lambda$. Hence, the desired form of x_{k+j} follows by induction in this case.

Assume that $x_k > \lambda$ for all k. Thus, if $a \neq 1$, then

$$\begin{aligned} x_j &= a x_{j-1} - b = a^2 x_{j-2} - \frac{a^2 - 1}{a - 1} b = \cdots \\ &= a^j\left(x_0 - \frac{b}{a-1}\right) + \frac{b}{a-1}. \end{aligned} \tag{21}$$

And if $a = 1$, then $x_j = x_{j-1} - b = x_{j-2} - 2b = \cdots = x_0 - jb$. (Since $x_j > \lambda > 0$, for all j, it follows that $b \leq 0$ in this case.)

(1b) Suppose that $0 \leq a^* \leq 1$. Let $x(k) \leq \lambda$ for some integer $k \geq 0$. By (a), $x(k+j) = a^{*j} x(k) \leq \lambda$ and $F(x(k+j)) = 0$. Let t' be in $[k, k+1)$. Then $k + j \leq t' + j < k + j + 1$ and, by (8),

$$\begin{aligned} &x(t'+j) \\ &= \left(e^{A(t'+j-[t'+j])} + B\int_{[t'+j]}^{t'+j} e^{A(t'+j-s)} ds\right) x(k+j) \\ &= a^{*j}\left(e^{A(t'-[t'])} + B\int_{[t']}^{t'} e^{A(t'-s)} ds\right) x(k) \\ &= a^{*j} x(t'). \end{aligned} \tag{22}$$

Next, assume that $x(k) > \lambda$ and thus $F(x(k)) = 1$ for all integers $k \geq 0$. Suppose first that $a^* \neq 1$ and $0 \leq t' < 1$. By (a), $x(j) = a^{*j}(x(0) - b^*/(a^* - 1)) + b^*/(a^* - 1)$. Since $j \leq t' + j < j + 1$ and $b^*/(a^* - 1) = -C/(A + B)$,

$$\begin{aligned} x(t'+j) = a^{*j}&\left(e^{A(t'-[t'])} + B\int_{[t']}^{t'} e^{A(t'-s)} ds\right) \\ &\cdot\left(x(0) - \frac{b^*}{a^*-1}\right) + \frac{b^*}{a^*-1} \end{aligned} \tag{23}$$

as in the proof of Proposition 1(1). Therefore, $x(t' + j) = a^{*j}(x(t') - b^*/(a^* - 1)) + b^*/(a^* - 1)$.

Finally suppose that $a^* = 1$. Thus $B = -A$ and, by (a), $x(j) = x(0) - jb^* > \lambda$ and $b^* \leq 0$. As above, for $0 \leq t' < 1$, we have that $j \leq t' + j < j + 1$ and

$$x\left(t' + j\right) = x\left(j\right) + C\int_{[t']}^{t'} e^{A(t'-s)}ds = x\left(t'\right) - jb^*. \quad (24)$$

(2a) Suppose that $a > 1$ and $b \leq 0$. If $x_0 \leq 0$ ($< \lambda$), then clearly $x_j = a^j x_0$. Assume that $x_0 > 0$. There exists a positive integer k such that $x_{k-1} = a^{k-1}x_0 \leq \lambda < x_k = a^k x_0$. Thus, $x_{k+1} = a^{k+1}x_0 - b = ax_k - b > x_k > \lambda$ since $a > 1$, $x_k > 0$, and $b \leq 0$. As in the proof of (1a), $x_{k+j} = a^j x_k - ((a^j - 1)/(a-1))b$ ($> \lambda$).

(2b) Suppose that $a^* > 1$ and $b^* \leq 0$. If $x(0) \leq 0$, then $x(j) = a^{*j}x(0) < \lambda$ by (2a); and hence the result follows as in the proof of (1b).

Assume that $x(0) > 0$. By (2a), there exists a positive integer k such that $x(k) > \lambda$ and $x(k + j) = a^{*j}(x(k) - b^*/(a^* - 1)) + b^*/(a^* - 1) > \lambda$. Let $k \leq t' < k + 1$ so that $k + j \leq t' + j < k + j + 1$. Then, by (8),

$$\begin{aligned} x\left(t' + j\right) &= a^{*j}\left(e^{A(t'-[t'])} + B\int_{[t']}^{t'} e^{A(t'-s)}ds\right) \\ &\quad \cdot \left(x\left(k\right) - \frac{b^*}{a^* - 1}\right) + \frac{b^*}{a^* - 1} \\ &= a^{*j}\left(x\left(t'\right) - \frac{b^*}{a^* - 1}\right) + \frac{b^*}{a^* - 1} \end{aligned} \quad (25)$$

as in the proof of (1b).

(3a) Suppose that $a < 0$. One of the following must be true:

(i) For every integer $k \geq 0$, there exists an integer $n_k \geq k$ such that $x_{n_k} \leq \lambda$.

(ii) There exists an integer $k \geq 0$ such that $x_n > \lambda$ for all $n \geq k$.

Assume that (i) holds. If $x_j = 0$, for some j, then x_n is oscillatory with stationary state 0; thus we further assume that x_n has no zero terms. Therefore, there exists a subsequence of x_n with alternating signs.

We may choose $x_{n_0} > 0$ as follows. If $x_0 > 0$, then let $x_{n_0} \equiv x_0$. If $x_0 < 0$ ($< \lambda$), then let $x_{n_0} \equiv x_1 = ax_0 > 0$.

Next choose $x_{n_1} < 0$, where $n_1 > n_0$: by (i), there exists $n_1' \geq n_0$ such that $x_{n_1'} \leq \lambda$. If $x_{n_1'} < 0$, then $n_1' > n_0$ so let $n_1 \equiv n_1'$. If $x_{n_1'} > 0$, then $x_{n_1'+1} = ax_{n_1'} < 0$ and $n_1' + 1 > n_0$; thus, in this case, let $n_1 \equiv n_1' + 1$.

Since $x_{n_1} < 0 < \lambda$, we have $x_{n_2} \equiv x_{n_1+1} = ax_{n_1} > 0$ and $n_2 > n_1$. By induction, a sequence x_{n_k} is constructed such that $n_{k+1} > n_k$ and $x_{n_k}x_{n_{k+1}} < 0$ for all k. Thus x_n is oscillatory.

On the other hand, if (ii) holds, then, as in the proof of (1a), there exists $k \geq 0$ such that $x_{k+j} = a^j(x_k - b/(a-1)) + b/(a-1)$.

(3b) Suppose that $a^* < 0$. By (3a), either $x(n)$ is oscillatory (and hence $x = x(t)$ is oscillatory) or $x(k + j) = a^{*j}(x(k) - b^*/(a^* - 1)) + b^*/(a^* - 1) > \lambda$ (with $x(k) > \lambda$) for some integer $k \geq 0$. The desired result follows in the latter case as in the proof of (2b). □

Thus we assume that $a^* > 1$ and $b^* > 0$; and therefore Proposition 1 applies to the resulting solution $x = x(t)$ of (2). It will follow from the third section that if λ is outside the interval $(b^*/a^{*2}, b^*/(a^* - 1) - b^*/a^{*2})$, then $x(t)$ is either eventually constant or unbounded. Moreover, if $a^* \geq 2$ and λ is in $[b^*/(a^*(a^* - 1)), b^*/a^*]$, then, by the fourth section, all initial values $x(0)$ are derived such that $x(t)$ is eventually periodic; and more generally, for any $\lambda > 0$, the eventually periodic solutions of (2) are the bounded solutions with these initializations $x(0)$.

Remark 4. For integers a, b, x_0, and $\lambda > 0$, periodic solutions of difference equation (18) were used in [13] to determine the real eigenvalues of certain arbitrarily large, sparse matrices.

2. Unbounded Solutions

Let x_n satisfy (18), where $a > 1$ and $b > 0$. If x_n is unbounded, then x_n is eventually geometric: we define

$$\begin{aligned} f_-\left(i, j\right) &\equiv a^j x_i \quad \text{whenever } x_i \leq 0, \\ f_+\left(i, j\right) &\equiv a^j\left(x_i - \frac{b}{a-1}\right) + \frac{b}{a-1} \\ &\qquad \text{when } x_i \geq \frac{b}{a-1}. \end{aligned} \quad (26)$$

The following result shows that we may assume

$$\begin{aligned} 0 &< x_0 < \frac{b}{a-1}, \\ \lambda &< \frac{b}{a-1} \end{aligned} \quad (27)$$

since otherwise there exists $i \geq 0$ such that $x_{i+j} = f_\pm(i, j)$ for all j.

Lemma 5. *Assume that x_n is defined by (18), where $a > 1$ and $b > 0$.*

(1) *If $x_n \leq \lambda$, for all $n \geq i \geq 0$, then $x_{i+j} = f_-(i, j)$ for all $j \geq 0$. In particular, if $x_i \leq 0$, for some i, then $x_{i+j} = f_-(i, j)$ for all j.*

(2) *If $x_n > \lambda$, for all $n \geq i \geq 0$, then $x_{i+j} = f_+(i, j)$ for all $j \geq 0$. In particular, if $x_i \geq b/(a-1) > \lambda$, for some i, then $x_{i+j} = f_+(i, j)$ for all j.*

(3) *If $x_i > \lambda \geq b/(a-1)$, for some i, then $x_{i+j} = f_+(i, j)$ for all $j \geq 0$. In particular, if either $0 < x_i \leq b/(a-1) \leq \lambda$ or $b/(a-1) \leq x_i \leq \lambda$, for some i, then there exists $k > i$ such that $x_{k+j} = f_+(k, j)$ for all j.*

Proof. (1) Assume that $x_n \leq \lambda$ for all $n \geq i$. Then $x_{i+j} = a^j x_i$ for all $j \geq 0$. If $x_i > 0$, then, since $a > 1$, it follows that x_n is not bounded above which contradicts our hypothesis. Thus $x_i \leq 0$ and $x_{i+j} = f_-(i, j)$.

If $x_i \le 0\ (< \lambda)$, for some i, then clearly $x_n \le \lambda$ for all $n \ge i$.

(2) Suppose that $x_n > \lambda$ for all $n \ge i$. Then

$$\begin{aligned} x_{i+1} &= ax_i - b = a\left(x_i - \frac{b}{a-1}\right) + \frac{b}{a-1}, \\ x_{i+2} &= ax_{i+1} - b = a^2\left(x_i - \frac{b}{a-1}\right) + \frac{b}{a-1} \end{aligned} \tag{28}$$

and, by induction, for all j,

$$x_{i+j} = a^j\left(x_i - \frac{b}{a-1}\right) + \frac{b}{a-1}. \tag{29}$$

If $x_i < b/(a-1)$, then, since $a > 1$, x_n is not bounded below which is contrary to our hypothesis. Thus $x_i \ge b/(a-1)$ and $x_{i+j} = f_+(i, j)$ for all j.

Assume that $x_i \ge b/(a-1) > \lambda$ for some i. Then $x_i = f_+(i, 0) > \lambda$ since $x_i \ge b/(a-1)$, and

$$x_{i+1} = ax_i - b \ge a\left(\frac{b}{a-1}\right) - b = \frac{b}{a-1} > \lambda, \tag{30}$$

where $x_{i+1} = f_+(i, 1)$ as above.

Similarly, by induction,

$$x_{i+j} = ax_{i+j-1} - b \ge \frac{b}{a-1} > \lambda \tag{31}$$

and $x_{i+j} = f_+(i, j)$ for all j.

(3) Assume that $x_i > \lambda \ge b/(a-1)$ for some i. Then $x_i = f_+(i, 0) > \lambda$ and

$$x_{i+1} = ax_i - b > a\lambda - (a-1)\lambda = \lambda, \tag{32}$$

where $x_{i+1} = f_+(i, 1)$. By induction, $x_{i+j} = ax_{i+j-1} - b > \lambda$ and $x_{i+j} = f_+(i, j)$ for all j.

Suppose that $0 < x_i \le b/(a-1) \le \lambda$ for some i. There exists $k > i$ such that

$$x_{k-1} = a^{k-1-i}x_i \le \lambda < x_k = a^{k-i}x_i. \tag{33}$$

Thus, by the general case, $x_{k+j} = f_+(k, j)$ for all j.

Similarly, if $b/(a-1) \le x_i \le \lambda$, then there is $k > i$ such that $x_{k+j} = f_+(k, j)$ for all j. □

The next result is central to our analysis.

Lemma 6. *Let x_n be given by (18), where $a > 1$, $b > 0$, and $0 < \lambda, x_0 < b/(a-1)$.*

(1) *If $\lambda \le b/a(a-1)$, then $x_n \le b/(a-1)$ for all n. In particular, if $\lambda < b/a(a-1)$, then $x_n < b/(a-1)$ for all n.*

(2) *If $b/a(a-1) \le x_0 \le \lambda$, then $x_{1+j} = f_+(1, j)$ for all $j \ge 0$. In this case, $x_0 = b/a(a-1)$ if and only if $x_1 = b/(a-1)$.*

(3) *If $\lambda \ge b/a$, then $x_n > 0$ for all n.*

(4) *If $b/a \ge x_0 > \lambda$, then $x_{1+j} = f_-(1, j)$ for all $j \ge 0$. In this case, $x_0 = b/a$ if and only if $x_1 = 0$.*

Proof. (1) Assume $\lambda \le b/a(a-1)$ and $x_0 < b/(a-1)$.

If $x_0 \le \lambda$, then $x_1 = ax_0 \le a\lambda \le b/(a-1)$. And if $x_0 > \lambda$, then $x_1 = ax_0 - b < a(b/(a-1)) - b = b/(a-1)$. Thus $x_1 \le b/(a-1)$ in both cases, and, similarly, by induction, $x_n \le b/(a-1)$ for all n.

If $\lambda < b/a(a-1)$, then $x_n < b/(a-1)$ for all n by the above argument.

(2) Suppose that $b/a(a-1) \le x_0 \le \lambda$. Then $x_1 = ax_0 \ge b/(a-1) > \lambda$ and $x_{1+j} = f_+(1, j)$ for all j by Lemma 5(2).

If $b/a(a-1) = x_0 \le \lambda$, then $x_1 = ax_0 = b/(a-1)$. Conversely, if $b/a(a-1) \le x_0 \le \lambda$ and $x_1 = b/(a-1)$, then $x_1 = ax_0$ so $x_0 = b/a(a-1)$.

(3) Assume $\lambda \ge b/a$. If $x_0 \le \lambda$, then $x_1 = ax_0 > 0$. And if $x_0 > \lambda$, then $x_1 = ax_0 - b > a\lambda - b \ge 0$. Thus $x_1 > 0$ and, by induction, $x_n > 0$ for all n.

(4) Suppose that $b/a \ge x_0 > \lambda$. Then $x_1 = ax_0 - b \le a(b/a) - b = 0$ and $x_{1+j} = f_-(1, j)$ for all j by Lemma 5(1).

Clearly, $x_0 = b/a$ if and only if $x_1 = 0$ in this case. □

Remark 7. Let x_n be defined as in Lemma 6. As in the proof of Lemma 5(3), if $x_0 \le \lambda$, then there is a unique positive integer i such that $a^{i-1}x_0 \le \lambda < a^i x_0$. In this case, $x_k = a^k x_0$ for $k = 1, \ldots, i$; and if $x_i \le b/a$, then $x_{i+1+j} = a^j x_{i+1}$ for all j by Lemma 6(4) since $x_i > \lambda$.

Similarly, if $x_0 > \lambda$, then there is a unique positive integer i such that

$$\begin{aligned} 0 < a^{i-1}\left(\frac{b}{a-1} - x_0\right) &< \frac{b}{a-1} - \lambda \\ &\le a^i\left(\frac{b}{a-1} - x_0\right). \end{aligned} \tag{34}$$

In this case, writing $x_0 = b/(a-1) - (b/(a-1) - x_0) > \lambda$, we have that

$$x_k = ax_{k-1} - b = \frac{b}{a-1} - a^k\left(\frac{b}{a-1} - x_0\right) > \lambda \tag{35}$$

for $k = 1, \ldots, i$; and if $x_i \ge b/a(a-1)$, then

$$x_{i+1+j} = \frac{b}{a-1} - a^j\left(\frac{b}{a-1} - x_{i+1}\right) \tag{36}$$

for all j by Lemma 6(2) since $x_i \le \lambda$.

Example 8. Let $a \ge 2$ and let $b > 0$ and assume that $b/a(a-1) \le \lambda \le b/a$. Let x_0 be the weighted average

$$x_0 \equiv \frac{(2a-1)\left(b/a^2\right) + b/a(a-1)}{2a}. \tag{37}$$

Then $x_0 < \lambda$ and

$$x_1 = ax_0 = \frac{((2a-1)(a-1)+a)\,b}{2a^2(a-1)} > \frac{b}{a} \ge \lambda. \tag{38}$$

Thus

$$x_2 = ax_1 - b = \frac{b}{2a(a-1)} < \frac{b}{a(a-1)} \le \lambda \tag{39}$$

and $x_3 = ax_2 = b/2(a-1)$ is the midpoint of $[b/a(a-1), b/a]$.

If $x_3 \le \lambda$, then $x_{4+j} = f_+(4, j)$ for all j by Lemma 6(2). And if $x_3 > \lambda$, then $x_{4+j} = f_-(4, j)$ for all j by Lemma 6(4).

3. Bounded Solutions

By Lemma 5, solutions x_n of (18) such that $a > 1$, $b > 0$, and $0 < \lambda, x_0 < b/(a-1)$ are bounded only when $0 \le x_n \le b/(a-1)$ for all n. In particular, by Lemma 6, if $a \le 2$ and $b/a \le \lambda \le b/a(a-1)$, then x_n is bounded. It is possible that x_n is bounded but not eventually periodic.

Example 9. Assume that $1 < a = p/q < 2$, where p and q are odd and even integers, respectively; $b > 0$ and $b/a \le \lambda \le b/a(a-1)$. If $x_0 \equiv (b/(a-1))(r/s)$, where r and s are odd and even integers, respectively, such that $r < s$, then the solution x_n is bounded but not eventually periodic: by Lemma 6, $0 < x_n \le b/(a-1)$ for all n. Note that

$$\begin{aligned} ax_0 &= \frac{b}{a-1}\left(\frac{pr}{qs}\right), \\ ax_0 - b &= \frac{b}{a-1}\left(\frac{pr-(p-q)s}{qs}\right) \end{aligned} \tag{40}$$

so that $x_1 = (b/(a-1))(p_1/qs)$, where p_1 is odd and qs is even.

Similarly, by induction, for all $n \ge 1$, $x_n = (b/(a-1))(p_n/q^n s)$, where p_n is odd and $q^n s$ is even. It follows that x_n is not eventually periodic since if $x_m = x_{m+n}$ for some $m \ge 0$ and $n \ge 1$, then $p_{m+n} = q^n p_m$ is even.

We now classify the types of solutions that may be bounded. Our results will be stated in terms of the decomposition of $(0, b/(a-1))$ into the disjoint union of the intervals $I_1 = (0, b/a^2]$, $I_2 = (b/a^2, \min\{b/a, b/a(a-1)\})$, $I_3 = [\min\{b/a, b/a(a-1)\}, \max\{b/a, b/a(a-1)\}]$, $I_4 = (\max\{b/a, b/a(a-1)\}, b/(a-1) - b/a^2)$, and $I_5 = [b/(a-1) - b/a^2, b/(a-1))$.

Definition 10. Let $a > 1$, $b > 0$, and $0 < \lambda, x_0 < b/(a-1)$. A solution x_n defined by (18) is of

(i) type A if λ and x_k are in I_2 for some k such that $x_k \le \lambda$,

(ii) type B if λ and x_k are in I_4 for some k such that $x_k > \lambda$,

(iii) type C if λ is in I_3 and either $a \ge 2$ such that x_k is in $I_2 \cup I_4$ for some k or $a < 2$.

The unbounded solution in Example 8 is of type C.

Let x_n satisfy (18) as in Definition 10. Since $\lambda < b/(a-1)$, we have that c is a stationary state of x_n if and only if $x_i = c = 0$ or $x_i = c = b/(a-1)$ for some i, in which case $x_{i+j} = c$ for all $j \ge 0$: if $x_i = 0\ (< \lambda)$ or $x_i = b/(a-1)\ (> \lambda)$, then clearly $x_{i+1} = x_i = c$. Conversely, suppose that $x_{i+1} = x_i = c$ for some i. If $c \le \lambda$, then $c = x_{i+1} = ax_i = ac$ so $c = 0$ since $a > 1$. And if $c > \lambda$, then $c = x_{i+1} = ax_i - b = ac - b$ so $c = b/(a-1)$.

The solution x_n is *eventually periodic* if there are integers $m \ge 0$ and $p \ge 1$ such that $x_m = x_{m+p}$ (and thus, by (18), $x_{m+j} = x_{m+p+j}$ for all $j \ge 0$). The following eventually periodic solution is either of type A or type C.

Example 11. Let $a > 1$ and let $b > 0$ and let $p \ge 2$ be an integer such that $a^p \ge a/(a-1)$. (If $a \ge 2$, then $p \ge 2$ is arbitrary.) Choose λ such that

$$\frac{b}{a^2} \le \frac{b/a^2}{1-1/a^p} = \frac{ba^{p-2}}{a^p-1} \le \lambda \le \frac{b}{a} \tag{41}$$

and, for any integer $m \ge 0$, let

$$x_0 \equiv \frac{b}{a^m(a^p-1)}. \tag{42}$$

Then $x_m = x_{m+p}$: $x_0 \le ba^{p-2}/(a^p-1) \le \lambda$ so $x_1 = b/(a^{m-1}(a^p-1))$. Similarly, $x_m = a^m x_0 = b/(a^p-1) \le ba^{p-2}/(a^p-1) \le \lambda$ and, for $k = 1, \ldots, p-2$, $x_{m+k} = ba^k/(a^p-1) \le ba^{p-2}/(a^p-1) \le \lambda$. Thus $x_{m+(p-1)} = ba^{p-1}/(a^p-1) > b/a \ge \lambda$ and

$$x_{m+p} = ax_{m+(p-1)} - b = \frac{b}{a^p-1} = x_m. \tag{43}$$

Note that if $a \ge 2$, then, since $b/a^2 \le x_{m+p-2} \le b/a(a-1) \le b/a$, it follows that x_n is of type A whenever λ is in I_2 (and $x_{m+p-2} \le \lambda$) and is of type C when λ is in I_3. If $a < 2$, then x_n is of type A since x_{m+p-2} and λ are in I_2 and $x_{m+p-2} \le \lambda$. The following slight modification is eventually periodic of type C when $a < 2$.

Example 12. Let $1 < a < 2$ and let $b > 0$ and let $p \ge 2$ be an integer such that

$$a^p > \max\left\{\frac{a}{a-1}, \frac{1}{2-a}\right\}, \tag{44}$$

and let λ in I_3 satisfy

$$\frac{b}{a^2} < \frac{ba^{p-2}}{a^p-1} < \frac{b}{a} < \lambda < \frac{ba^{p-1}}{a^p-1} < \frac{b}{a(a-1)}. \tag{45}$$

Define

$$x_0 \equiv \frac{ba^q}{a^m(a^p-1)} \tag{46}$$

for integers $m \ge 0$ and q such that $0 \le q \le p-2$. Then $x_0 \le ba^{p-2}/(a^p-1) < \lambda$ and, as in Example 11,

$$\begin{aligned} x_m &= \frac{ba^q}{a^p-1} \le \frac{ba^{p-2}}{a^p-1} < \lambda, \\ x_{m+p-q-2} &= a^{p-q-2}x_m = \frac{ba^{p-2}}{a^p-1} < \lambda. \end{aligned} \tag{47}$$

Therefore, $x_{m+p-q-1} = ba^{p-1}/(a^p-1) > \lambda$,

$$\begin{aligned} x_{m+p-q} &= ax_{m+p-q-1} - b = \frac{b}{a^p-1} < \lambda, \\ x_{m+p} &= a^q x_{m+p-q} = \frac{ba^q}{a^p-1} = x_m. \end{aligned} \tag{48}$$

Note that, for an element x_0 in I_1, there exists an integer $k \geq 2$ such that

$$\frac{b}{a^{k+1}} < x_0 \leq \frac{b}{a^k}. \tag{49}$$

Similarly, for x_0 in I_5, there exists an integer $k \geq 2$ such that

$$\frac{b}{a-1} - \frac{b}{a^k} \leq x_0 < \frac{b}{a-1} - \frac{b}{a^{k+1}}. \tag{50}$$

Inequalities (49) and (50) will be used repeatedly in the next result.

In seeking bounded solutions without stationary states as in Examples 11 and 12, we may further assume that λ is in $I_2 \cup I_3 \cup I_4$ by the next result.

Theorem 13. *Let x_n be a solution of (18) such that $a > 1$, $b > 0$, and $0 < \lambda, x_0 < b/(a-1)$. Then x_0 fits one and only one of the following cases.*

(1) *Let λ be in I_1.*

(a) *If $x_0 \leq \lambda$ and $k \geq 2$ is given by (49), then $x_{i+j} = f_-(i, j)$ for all $j \geq 0$, where i is the first integer in $[2, k]$ such that $a^{i-1}x_0 > \lambda$; that is, $x_{i-1} > \lambda$. In this case, if $\lambda = b/a^k$, then $i = 2$; and $x_2 < 0$ when $k > 2$.*

(b) *If $\lambda < x_0 \leq b/a$, then $x_{1+j} = f_-(1, j)$ for all $j \geq 0$.*

(2) *Let λ be in I_2.*

(a) *If x_0 is in I_1 and $k \geq 2$ is given by (49), then $x_{k-1} = a^{k-1}x_0$ and one has the following.*

(i) *If $b/a^2 < x_{k-1} \leq \lambda$, then x_n is of type A.*

(ii) *If $\lambda < x_{k-1} \leq b/a$, then $x_{k+j} = f_-(k, j)$ for all j.*

(b)

(i) *If $b/a^2 < x_0 \leq \lambda$, then x_n is of type A.*

(ii) *If $\lambda < x_0 \leq b/a$, then $x_{1+j} = f_-(1, j)$ for all j.*

(3) *Let λ be in $I_1 \cup I_2$.*

(a) *Let $x_0 \neq b/a$ be in I_3 and let $\phi = (1+\sqrt{5})/2$ be the golden ratio with the property that $\phi(\phi - 1) = 1$.*

(i) *If $\phi \leq a < 2$, then either (1) or (2) holds for the sequence x_n $(n \geq 1)$ with initial term $x_1 = ax_0 - b$.*

(ii) *If $a < \phi$ and r is the least positive integer such that $a^r(a-1) \geq 1$, then there exists an integer s in $[1, r]$ such that $x_s = b/(a-1) - a^s(b/(a-1) - x_0)$ is in $I_1 \cup I_2 \cup \{b/a\}$; and therefore (1) or (2) applies to x_n $(n \geq s)$.*

(b) *Let x_0 be in I_4.*

(i) *If $a \geq 2$, then $x_1 = ax_0 - b$ is in $I_1 \cup I_2$ and x_n $(n \geq 1)$ satisfies either (1) or (2).*

(ii) *If $a < 2$, then $x_1 = ax_0 - b \neq b/a(a-1)$ is in $I_1 \cup I_2 \cup I_3$ and x_n $(n \geq 1)$ satisfies (1), (2), or (3a).*

(c) *Let x_0 be in I_5 and let $k \geq 2$ be given by (50). Then $x_k = b/(a-1) - a^k(b/(a-1) - x_0)$ and one has the following.*

(i) *If $a \geq 2$, then either $x_k \leq 0$ (and $x_{k+j} = f_-(k, j)$, for all j), or x_k is in $I_1 \cup I_2$ and x_n $(n \geq k)$ satisfies (1) or (2).*

(ii) *If $a < 2$, then $x_k \neq b/a(a-1)$ is in $I_1 \cup I_2 \cup I_3$ and x_n $(n \geq k)$ satisfies (1), (2), or (3a).*

(4) *Let λ be in I_3. By Definition 10, if either $a < 2$ or $a \geq 2$ and $I_2 \cup I_4$ contains a term of the sequence x_n, then x_n is of type C. Therefore, suppose that $a \geq 2$ and x_0 is in $I_1 \cup I_3 \cup I_5$.*

(a) *Let x_0 be in I_1 and let $k \geq 2$ be given by (49). Then $x_{k-1} = a^{k-1}x_0$ is in $I_2 \cup I_3$. If x_{k-1} is in I_2, then x_n is of type C. Let x_{k-1} be in I_3. Then*

(i) *$x_{k+j} = f_+(k, j)$ for all $j \geq 0$ whenever $b/a(a-1) \leq x_{k-1} \leq \lambda$;*

(ii) *$x_{k+j} = f_-(k, j)$ for all $j \geq 0$ when $\lambda < x_{k-1} \leq b/a$.*

(b) *Let x_0 be in I_3.*

(i) *If $x_0 \leq \lambda$, then $x_{1+j} = f_+(1, j)$ for all j.*

(ii) *If $x_0 > \lambda$, then $x_{1+j} = f_-(1, j)$ for all j.*

(c) *Let x_0 be in I_5 and let $k \geq 2$ be given by (50). Then $x_{k-1} = b/(a-1) - a^{k-1}(b/(a-1) - x_0)$ is in $I_3 \cup I_4$. Thus, if x_{k-1} is in I_4, then x_n is of type C. Let x_{k-1} be in I_3. Then*

(i) *$x_{k+j} = f_+(k, j)$ for all $j \geq 0$ whenever $b/a(a-1) \leq x_{k-1} \leq \lambda$;*

(ii) *$x_{k+j} = f_-(k, j)$ for all $j \geq 0$ when $\lambda < x_{k-1} \leq b/a$.*

(5) *Let λ be in I_4.*

(a) *If x_0 is in I_5 and $k \geq 2$ is given by (50), then $x_{k-1} = b/(a-1) - a^{k-1}(b/(a-1) - x_0)$ and one has the following.*

(i) *If $b/a(a-1) \leq x_{k-1} \leq \lambda$, then $x_{k+j} = f_+(k, j)$ for all j.*

(ii) *If $\lambda < x_{k-1} < b/(a-1) - b/a^2$, then x_n is of type B.*

(b)

(i) *If $b/a(a-1) \leq x_0 \leq \lambda$, then $x_{1+j} = f_+(1, j)$ for all j.*

(ii) *If $\lambda < x_0 < b/(a-1) - b/a^2$, then x_n is of type B.*

(6) *Let λ be in I_5.*

(a) *If $x_0 > \lambda$ and $k \geq 2$ is given by (50), then $x_{i+j} = f_+(i, j)$ for all $j \geq 0$, where i is the first integer in $[2, k]$ such that $a^{i-1}(b/(a-1)-x_0) \geq b/(a-1)-\lambda$; that is, $x_{i-1} \leq \lambda$. In this case, if $\lambda = b/(a-1) - b/a^k$, then $i = 2$ and $x_2 > b/(a-1)$.*

(b) *If $b/a(a-1) \leq x_0 \leq \lambda$, then $x_{1+j} = f_+(1, j)$ for all $j \geq 0$.*

(7) *Let λ be in $I_4 \cup I_5$.*

(a) *Let $x_0 \neq b/a(a-1)$ be in I_3.*

(i) *If $\phi \leq a < 2$, then either (5) or (6) holds for the sequence x_n $(n \geq 1)$ with initial term $x_1 = ax_0$.*

(ii) *If $a < \phi$ and r is the least positive integer such that $a^r(a-1) \geq 1$, then there exists an integer s in $[1, r]$ such that $x_s = a^s x_0$ is in $\{b/a(a-1)\} \cup I_4 \cup I_5$ and thus (5) or (6) applies to x_n $(n \geq s)$.*

(b) *Let x_0 be in I_2.*

(i) *If $a \geq 2$, then $x_1 = ax_0$ is in $I_4 \cup I_5$ and x_n $(n \geq 1)$ satisfies either (5) or (6).*

(ii) *If $a < 2$, then $x_1 = ax_0 \neq b/a$ is in $I_3 \cup I_4 \cup I_5$ and x_n $(n \geq 1)$ satisfies (5), (6), or (7a).*

(c) *Let x_0 be in I_1 and let $k \geq 2$ be given by (49). Then $x_k = a^k x_0$ and one has the following.*

(i) *If $a \geq 2$, then either $x_k \geq b/(a-1)$ (hence $x_{k+j} = f_+(k, j)$ for all j) or x_k is in $I_4 \cup I_5$ and x_n $(n \geq k)$ satisfies (5) or (6).*

(ii) *If $a < 2$, then $x_k \neq b/a$ is in $I_3 \cup I_4 \cup I_5$ and x_n $(n \geq k)$ satisfies (5), (6), or (7a).*

Proof. (1a) Assume $0 < x_0 \leq \lambda \leq b/a^2$. By (49), there exists an integer $k \geq 2$ such that

$$\frac{b}{a^{k+1}} < x_0 \leq \frac{b}{a^k} \leq \frac{b}{a^2}, \qquad \lambda \leq \frac{b}{a^2} < a^{k-1} x_0. \tag{51}$$

Therefore, as in Remark 7, there is a smallest integer i in $[2, k]$ such that

$$x_{i-2} = a^{i-2} x_0 \leq \lambda < x_{i-1} = a^{i-1} x_0. \tag{52}$$

It follows that

$$x_i = ax_{i-1} - b = a^i x_0 - b \leq a^k x_0 - b \leq b - b = 0 \tag{53}$$

and $x_{i+j} = f_-(i, j)$ for all j by Lemma 5(1).

Suppose that $\lambda = b/a^k$. Then $\lambda/a < x_0 \leq \lambda$ and $\lambda < x_1 = ax_0$ so $i = 2$. And if $k > 2$, then

$$x_2 = ax_1 - b = a^2 x_0 - b < 0 \quad \text{since } x_0 \leq \frac{b}{a^k} < \frac{b}{a^2}. \tag{54}$$

(1b) Lemma 6(4).

(2) Let λ be in I_2. By (49), if x_0 is in I_1, then

$$\frac{b}{a^{k+1}} < x_0 \leq \frac{b}{a^k} \leq \frac{b}{a^2} < \lambda. \tag{55}$$

Therefore, $b/a^2 < x_{k-1} = a^{k-1} x_0$ and (2) follows from Definition 10 and Lemma 6(4).

Note that, for any λ in $I_1 \cup I_2$, (1) and (2) cover the following cases:

(i) x_0 is in $I_1 \cup I_2 \cup I_3$ whenever $a \geq 2$.

(ii) x_0 is in $I_1 \cup I_2 \cup \{b/a\}$ when $a < 2$.

(3) Let λ be in $I_1 \cup I_2$.

(a) Suppose that $x_0 \neq b/a$ is in I_3 and $a < 2$. Then $\lambda < b/a < x_0 \leq b/a(a-1)$ and

$$0 = a\left(\frac{b}{a}\right) - b < x_1 = ax_0 - b \leq \frac{b}{a-1} - b < \frac{b}{a(a-1)}. \tag{56}$$

(i) Suppose that $\phi \leq a < 2$. Then $a(a-1) \geq 1$ so $b/(a-1) - b \leq b/a$ since $b/a(a-1) \leq b$. Thus, by (56), x_1 is in $I_1 \cup I_2 \cup \{b/a\}$ and (1) or (2) applies to x_n $(n \geq 1)$.

(ii) On the other hand, assume that $a < \phi$ so that $a(a-1) < 1$. By (56), x_1 is in $I_1 \cup I_2 \cup I_3$. If x_1 is in $I_1 \cup I_2 \cup \{b/a\}$, then (1) or (2) applies to x_n $(n \geq 1)$. And if $x_1 \neq b/a$ is in I_3, then, since $x_1 \leq b/(a-1) - b$ by (56), the above argument shows

$$0 < x_2 = ax_1 - b \leq \frac{b}{a-1} - ab < \frac{b}{a(a-1)} \tag{57}$$

and x_2 is in $I_1 \cup I_2 \cup I_3$.

Note that, in the latter case, if $x_2 \neq b/a$ is in I_3, then $\lambda < b/a < x_2 \leq b/(a-1) - ab$ so $a^2(a-1) < 1$.

It follows by induction that, for every $j \geq 1$, either x_j is in $I_1 \cup I_2 \cup \{b/a\}$ or $x_j \neq b/a$ is in I_3 and $a^j(a-1) < 1$. Since $a > 1$, there are least positive integers r and $s \leq r$ such that $a^r(a-1) \geq 1$ and x_s is in $I_1 \cup I_2 \cup \{b/a\}$. Since $x_i > \lambda$, for $i = 0, \ldots, s-1$, the form of x_s follows from Remark 7.

(b) Let x_0 be in I_4.

(i) Let $a \geq 2$. Then $\lambda < b/a < x_0 < b/(a-1) - b/a^2$,

$$0 = a\left(\frac{b}{a}\right) - b < x_1 = ax_0 - b < a\left(\frac{b}{a-1} - \frac{b}{a^2}\right) - b = \frac{b}{a(a-1)}, \tag{58}$$

and x_1 is in $I_1 \cup I_2$.

(ii) Let $a < 2$. Then $\lambda < b/a(a-1) < x_0 < b/(a-1) - b/a^2$,

$$0 < a\left(\frac{b}{a(a-1)}\right) - b < x_1 = ax_0 - b < a\left(\frac{b}{a-1} - \frac{b}{a^2}\right) - b = \frac{b}{a(a-1)}, \tag{59}$$

and $x_1 \neq b/a(a-1)$ is in $I_1 \cup I_2 \cup I_3$.

(c) Let x_0 be in I_5 and let $k \geq 2$ be given by (50). Then

$$\lambda < \frac{b}{a-1} - \frac{b}{a^2} \leq \frac{b}{a-1} - \frac{b}{a^k} \leq x_0 < \frac{b}{a-1} - \frac{b}{a^{k+1}} \quad (60)$$

so that, for $j = 1, \ldots, k-2$,

$$\begin{aligned} \lambda &< \frac{b}{a-1} - \frac{b}{a^2} \leq \frac{b}{a-1} - \frac{b}{a^{k-j}} \leq x_j = ax_{j-1} - b \\ &< \frac{b}{a-1} - \frac{b}{a^{k-j+1}}. \end{aligned} \quad (61)$$

Hence,

$$\begin{aligned} \lambda &< \frac{b}{a(a-1)} = \frac{b}{a-1} - \frac{b}{a} \leq x_{k-1} \\ &= ax_{k-2} - b < \frac{b}{a-1} - \frac{b}{a^2}, \end{aligned} \quad (62)$$

$$\frac{b}{a-1} - b \leq x_k = ax_{k-1} - b < \frac{b}{a-1} - \frac{b}{a} = \frac{b}{a(a-1)},$$

where $b/(a-1) - b > 0$ when $a < 2$. Therefore, (3c) follows as above.

Moreover, by Remark 7, $x_k = b/(a-1) - a^k(b/(a-1) - x_0)$.

(4) Assume that λ is in I_3.

(a) Let x_0 be in I_1 and $a \geq 2$. Then $b/a(a-1) \leq \lambda \leq b/a$ and, as in the proof of (2), if k is given by (49), then $b/a^2 < x_{k-1} = a^{k-1}x_0 \leq b/a$. Thus (4a) follows from Definition 10 and Lemma 6((2), (4)).

(b) Let x_0 be in I_3 and $a \geq 2$ so that $b/a(a-1) \leq x_0 \leq b/a$. Hence, (i) and (ii) follow from Lemma 6((2), (4)).

(c) Let x_0 be in I_5 and $a \geq 2$. As in the proof of (3c),

$$\begin{aligned} \frac{b}{a(a-1)} &\leq x_{k-1} = \frac{b}{a-1} - a^{k-1}\left(\frac{b}{a-1} - x_0\right) \\ &< \frac{b}{a-1} - \frac{b}{a^2}. \end{aligned} \quad (63)$$

Therefore, (4c) follows from Definition 10 and Lemma 6((2), (4)).

(5) Assume that λ is in I_4.

(a) Let x_0 be in I_5. As above, (63) holds for x_{k-1}. Moreover,

$$\max\left\{\frac{b}{a}, \frac{b}{a(a-1)}\right\} < \lambda < \frac{b}{a-1} - \frac{b}{a^2}. \quad (64)$$

Thus (5a) follows from Lemma 6(2) and Definition 10.

(b) A direct consequence of Lemma 6(2) and Definition 10.

(6) Let λ be in I_5.

(a) Assume that $b/(a-1) - b/a^2 \leq \lambda < x_0 < b/(a-1)$ and let $k \geq 2$ be the integer satisfying (50):

$$\frac{b}{a-1} - \frac{b}{a^k} \leq x_0 < \frac{b}{a-1} - \frac{b}{a^{k+1}}. \quad (65)$$

Thus

$$\frac{b}{a-1} - a^{k-1}\left(\frac{b}{a-1} - x_0\right) < \frac{b}{a-1} - \frac{b}{a^2} \leq \lambda \quad (66)$$

and, by Remark 7, there is a smallest integer i in $[2, k]$ such that $x_{i-1} \leq \lambda$ and

$$\begin{aligned} x_{i-1} &= \frac{b}{a-1} - a^{i-1}\left(\frac{b}{a-1} - x_0\right) \leq \lambda < x_{i-2} \\ &= \frac{b}{a-1} - a^{i-2}\left(\frac{b}{a-1} - x_0\right). \end{aligned} \quad (67)$$

It follows that

$$x_i = ax_{i-1} = \frac{ab}{a-1} - a^i\left(\frac{b}{a-1} - x_0\right) \geq \frac{b}{a-1} \quad (68)$$

if and only if $a^i(b/(a-1) - x_0) \leq b$. But, by (50), $a^i(b/(a-1) - x_0) \leq b/a^{k-i} \leq b$. Therefore, $x_i \geq b/(a-1) > \lambda$ and $x_{i+j} = f_+(i, j)$ for all j by Lemma 5(2).

Suppose that $\lambda = b/(a-1) - b/a^k$. Then

$$\begin{aligned} \frac{b}{a-1} - \frac{b}{a^2} &\leq \frac{b}{a-1} - \frac{b}{a^k} = \lambda < x_0 < \frac{b}{a-1} - \frac{b}{a^{k+1}}, \\ \frac{b}{a(a-1)} &\leq \frac{b}{a-1} - \frac{b}{a^{k-1}} < x_1 = ax_0 - b \\ &< \frac{b}{a-1} - \frac{b}{a^k} = \lambda. \end{aligned} \quad (69)$$

Thus $i = 2$ and $x_2 = ax_1 > a(b/a(a-1)) = b/(a-1)$.

(b) Follows immediately from Lemma 6(2).

Note that, for any λ in $I_4 \cup I_5$, (5) and (6) cover the following cases:

(i) x_0 is in $I_3 \cup I_4 \cup I_5$ whenever $a \geq 2$.

(ii) x_0 is in $\{b/a(a-1)\} \cup I_4 \cup I_5$ when $a < 2$.

(7) Let λ be in $I_4 \cup I_5$.

(a) Let $a < 2$ and let $x_0 \neq b/a(a-1)$ be in I_3. Then $b/a \leq x_0 < b/a(a-1) < \lambda$ and

$$\begin{aligned} \frac{b}{a} &< b = a\left(\frac{b}{a}\right) \leq x_1 = ax_0 < a\left(\frac{b}{a(a-1)}\right) \\ &= \frac{b}{a-1}. \end{aligned} \quad (70)$$

(i) Assume that $\phi \leq a < 2$. Then $a(a-1) \geq 1$ so $b/a(a-1) \leq b$ and therefore, by (70), x_1 is in $\{b/a(a-1)\} \cup I_4 \cup I_5$. Hence, (5) or (6) applies to x_n $(n \geq 1)$.

(ii) Suppose that $a < \phi$ so that $a(a-1) < 1$. By (70), x_1 is in $I_3 \cup I_4 \cup I_5$. If x_1 is in $\{b/a(a-1)\} \cup I_4 \cup I_5$, then (5) or (6) applies to x_n $(n \geq 1)$.

Assume $x_1 \neq b/a(a-1)$ is in I_3. Then, by (70), $b \leq x_1 < b/a(a-1) < \lambda$ and

$$\frac{b}{a} < b < ab \leq x_2 = ax_1 < \frac{b}{a-1}. \quad (71)$$

Hence, x_2 is in $I_3 \cup I_4 \cup I_5$. Note that if $x_2 \neq b/a(a-1)$ is in I_3, then $ab \leq x_2 < b/a(a-1) < \lambda$ and $a^2(a-1) < 1$.

Continuing by induction, for every $j \geq 1$, either x_j is in $\{b/a(a-1)\} \cup I_4 \cup I_5$ or $x_j \neq b/a(a-1)$ is in I_3 and $a^j(a-1) < 1$. Therefore, since $a > 1$, if r is the least positive integer such that $a^r(a-1) \geq 1$, then there exists an integer s in $[1, r]$ such that $x_s = a^s x_0$ is in $\{b/a(a-1)\} \cup I_4 \cup I_5$.

(b) Let $a > 1$ and let x_0 be in I_2. Then $b/a^2 < x_0 < b/a(a-1) < \lambda$ and $b/a < x_1 = ax_0 < b/(a-1)$. Therefore, x_1 is in $I_4 \cup I_5$ when $a \geq 2$; and $x_1 \neq b/a$ is in $I_3 \cup I_4 \cup I_5$ otherwise.

(c) Let x_0 be in I_1 and let $k \geq 2$ be given by (49). Then, for $j = 0, 1, \ldots, k-2$,

$$\frac{b}{a^{k+1-j}} < x_j = a^j x_0 \leq \frac{b}{a^{k-j}} \leq \frac{b}{a^2} < \lambda. \tag{72}$$

Moreover, $b/a^2 < x_{k-1} = a^{k-1}\ x_0 \leq b/a < \lambda$ and

$$\frac{b}{a} < x_k = a^k x_0 \leq b. \tag{73}$$

(i) If $a \geq 2$, then, by (73), either $x_k \geq b/(a-1)$ or x_k is in $I_4 \cup I_5$.

(ii) If $a < 2$, then $b < b/(a-1)$ and therefore, by (73), $x_k \neq b/a$ is in $I_3 \cup I_4 \cup I_5$. □

4. Periodic Solutions

The following sets are basic components of any eventually periodic solution of the difference equation (18).

Definition 14. Let $\mathbb{P}_{-1}(a) \equiv \{0\}$ and, for $k \geq 0$, let $\mathbb{P}_k(a)$ be the set of all polynomials in a with degree at most k and with all coefficients either 0 or 1.

A consequence of the next result is that if $x_m = x_{m+p}$, then there are 2^{m+p} possibilities for x_0.

Lemma 15. *Let x_n be defined by (18) with $a > 1$, $b > 0$, and $0 < \lambda, x_0 < b/(a-1)$. If $x_m = x_{m+p}$, for some integers $m \geq 0$ and $p \geq 1$, then there exist polynomials $\mathbf{p}_{p-1}(a)$ in $\mathbb{P}_{p-1}(a)$ and $\mathbf{q}_{m-1}(a)$ in $\mathbb{P}_{m-1}(a)$ such that*

$$x_0 = \frac{b\left(\mathbf{p}_{p-1}(a) + (a^p - 1)\,\mathbf{q}_{m-1}(a)\right)}{a^m(a^p - 1)}. \tag{74}$$

Proof. Note that $\mathbb{P}_0(a) = \{0, 1\}$ and, for $k \geq 1$,

$$\mathbb{P}_k(a) = \left\{ \sum_{j=0}^{k} \mathbf{p}_0^{(k-j)}(a)\, a^j : \mathbf{p}_0^{(k-j)}(a) \in \mathbb{P}_0(a)\ \forall j \right\}. \tag{75}$$

We solve the equation $x_m = x_{m+p}$ for general initial value x_0.

The possible expressions for x_{m+1} may be written as follows:

$$x_{m+1} = ax_m - \mathbf{p}_0^{(0)}(a)\, b, \quad \mathbf{p}_0^{(0)}(a) \in \mathbb{P}_0(a). \tag{76}$$

Similarly,

$$\begin{aligned} x_{m+2} &= ax_{m+1} - \mathbf{p}_0^{(1)}(a)\, b \\ &= a^2 x_m - \left(\mathbf{p}_0^{(1)}(a) + a\mathbf{p}_0^{(0)}(a)\right) b, \end{aligned} \tag{77}$$

where $\mathbf{p}_0^{(1)}(a) \in \mathbb{P}_0(a)$.

Continuing in this manner, we have

$$x_{m+p} = a^p x_m - \mathbf{p}_{p-1}(a)\, b, \tag{78}$$

where $\mathbf{p}_{p-1}(a) = \sum_{j=0}^{p-1} \mathbf{p}_0^{(p-1-j)}(a) a^j$ is an arbitrary element of $\mathbb{P}_{p-1}(a)$.

In particular,

$$x_m = a^m x_0 - \mathbf{q}_{m-1}(a)\, b, \quad \mathbf{q}_{m-1}(a) \in \mathbb{P}_{m-1}(a). \tag{79}$$

Therefore, $x_m = x_{m+p}$ is equivalent to

$$x_m = \frac{\mathbf{p}_{p-1}(a)\, b}{a^p - 1} = a^m x_0 - \mathbf{q}_{m-1}(a)\, b \tag{80}$$

and thus the desired form (74) for x_0 follows. □

Note that the initial value of the unbounded solution of Example 8 is not of form (74).

Our main results are converses of Lemma 15. If x_0 is defined by (74), then

$$\begin{aligned} 0 &\leq x_0 \\ &\leq \frac{b\left((a^p - 1)/(a-1) + (a^p - 1)\left((a^m - 1)/(a-1)\right)\right)}{a^m(a^p - 1)} \\ &= \frac{b}{a-1}. \end{aligned} \tag{81}$$

Furthermore, we have the following refinement.

Lemma 16. *Let $a > 1$ and let $b > 0$ and suppose that x_0 satisfies (74) for integers $m \geq 0$ and $p \geq 1$ and polynomials $\mathbf{p}_{p-1}(a)$ in $\mathbb{P}_{p-1}(a)$ and $\mathbf{q}_{m-1}(a)$ in $\mathbb{P}_{m-1}(a)$.*

(1) If

$$\begin{aligned} &\deg \mathbf{q}_{m-1}(a) < m-1 \quad \text{when } m > 1, \\ &\mathbf{q}_{m-1}(a) = 0 \quad \text{when } m = 1, \\ &\deg \mathbf{p}_{p-1}(a) < p-1 \quad \text{when } m = 0,\ p > 1, \\ &\mathbf{p}_{p-1}(a) = 0 \quad \text{when } m = 0,\ p = 1, \end{aligned} \tag{82}$$

then $0 \leq x_0 \leq b/a(a-1)$.

(2) If

$$\begin{aligned} &\deg \mathbf{q}_{m-1}(a) = m-1 \quad \text{when } m > 1, \\ &\mathbf{q}_{m-1}(a) = 1 \quad \text{when } m = 1, \\ &\deg \mathbf{p}_{p-1}(a) = p-1 \quad \text{when } m = 0,\ p > 1, \\ &\mathbf{p}_{p-1}(a) = 1 \quad \text{when } m = 0,\ p = 1, \end{aligned} \tag{83}$$

then $b/a \leq x_0 \leq b/(a-1)$.

Moreover, the converses of (1) and (2) hold whenever x_0 is not in I_3.

Proof. Suppose that $m > 1$ and $\deg \mathbf{q}_{m-1}(a) < m-1$. Then $0 \le \mathbf{q}_{m-1}(a) \le (a^{m-1}-1)/(a-1)$ and $0 \le \mathbf{p}_{p-1}(a) \le (a^p-1)/(a-1)$ and thus, by (74),

$$\begin{aligned} 0 &\le x_0 \\ &\le \frac{b\left((a^p-1)/(a-1)+(a^p-1)\left(\left(a^{m-1}-1\right)/(a-1)\right)\right)}{a^m(a^p-1)} \\ &= \frac{b}{a-1}. \end{aligned} \quad (84)$$

Similarly, if $m > 1$ and $\deg \mathbf{q}_{m-1}(a) = m-1$, then $\mathbf{q}_{m-1}(a) \ge a^{m-1}$ and, by (74) and (81),

$$\frac{b}{a-1} \ge x_0 \ge \frac{b\left(\mathbf{p}_{p-1}(a)+(a^p-1)a^{m-1}\right)}{a^m(a^p-1)} \ge \frac{b}{a}. \quad (85)$$

Suppose that $m = 1$. Then $\mathbb{P}_{m-1}(a) = \{0,1\}$. If $\mathbf{q}_{m-1}(a) = 0$, then, by (74),

$$0 \le x_0 = \frac{b}{a(a-1)}\frac{\mathbf{p}_{p-1}(a)}{((a^p-1)/(a-1))} \le \frac{b}{a(a-1)}. \quad (86)$$

And if $\mathbf{q}_{m-1}(a) = 1$, then

$$\frac{b}{a-1} \ge x_0 = \frac{b\left(\mathbf{p}_{p-1}(a)+(a^p-1)\right)}{a(a^p-1)} \ge \frac{b}{a}. \quad (87)$$

Assume next that $m = 0$ and $p > 1$. Then $\mathbb{P}_{m-1}(a) = \{0\}$. If $\deg \mathbf{p}_{p-1}(a) < p-1$, then $\mathbf{p}_{p-1}(a) \le (a^{p-1}-1)/(a-1)$ and

$$0 \le x_0 \le b\frac{\mathbf{p}_{p-1}(a)}{a^p-1} \le \frac{b}{a-1}\left(\frac{a^{p-1}-1}{a^p-1}\right) < \frac{b}{a(a-1)} \quad (88)$$

since $a > 1$. And if $\deg \mathbf{p}_{p-1}(a) = p-1$, then $\mathbf{p}_{p-1}(a) \ge a^{p-1}$ and

$$\frac{b}{a-1} \ge x_0 = b\frac{\mathbf{p}_{p-1}(a)}{a^p-1} \ge \frac{ba^{p-1}}{a^p-1} > \frac{b}{a}. \quad (89)$$

Finally assume that $m = 0$ and $p = 1$. Then $\mathbb{P}_{m-1}(a) = \{0\}$. If $\mathbf{p}_{p-1}(a) = 0$, then $x_0 = 0$. And if $\mathbf{p}_{p-1}(a) = 1$, then $x_0 = b/(a-1)$.

Conversely, suppose that $0 \le x_0 \le b/a(a-1)$ and $m > 1$ but x_0 is not in I_3. Either $\deg \mathbf{q}_{m-1}(a) < m-1$ or $\deg \mathbf{q}_{m-1}(a) = m-1$. If $\deg \mathbf{q}_{m-1}(a) = m-1$, then $b/a \le x_0 \le b/(a-1)$ by (2). Thus, in this case, if $a < 2$, then x_0 is in I_3; and if $a \ge 2$, then $a = 2$ and x_0 is again in I_3. Hence, $\deg \mathbf{q}_{m-1}(a) < m-1$.

The other converses follow similarly. □

The converses in Lemma 16 may fail when x_0 is in I_3: in Example 12, we have $a < 2$, $x_{m+p-q-1} = ba^{p-1}/(a^p-1)$ is in I_3, $0 \le x_{m+p-q-1} \le b/a(a-1)$, and $x_{m+p-q-1}$ satisfies (74) with $m = 0$ and $p > 1$ but $\deg \mathbf{p}_{p-1}(a) = p-1$.

If λ is in $I_1 \cup I_5$, then, by Theorem 13, x_n either has a stationary state or is unbounded. Using Lemma 16, we may extend this result to the other cases of λ.

Theorem 17. *Let x_n be a solution of (18) such that $a \ge 2$, $b > 0$, and x_0 satisfies (74) for some integers $m \ge 0$ and $p \ge 1$ and some polynomials $\mathbf{p}_{p-1}(a)$ in $\mathbb{P}_{p-1}(a)$ and $\mathbf{q}_{m-1}(a)$ in $\mathbb{P}_{m-1}(a)$.*

(1) *If λ is in I_2, then either $x_m = x_{m+p}$ or there exists a positive integer $i \le m+p$ such that $x_{i+j} = f_-(i,j)$ for all $j \ge 0$.*

(2) *Let λ be in I_3. If $\lambda \ne b/a$ or $\mathbf{p}_{p-1}(a) \ne 0$, then $x_m = x_{m+p}$. On the other hand, if $\lambda = b/a$ and $\mathbf{p}_{p-1}(a) = 0$, then either $x_0 = 0$ or there exists a positive integer $i \le m$ such that $x_{i+j} = f_+(i,j)$ for all $j \ge 0$, where $x_i = b$.*

(3) *If λ is in I_4, then either $x_m = x_{m+p}$ or there exists a positive integer $i \le m+p$ such that $x_{i+j} = f_+(i,j)$ for all $j \ge 0$.*

Proof. (2) Let λ be in I_3 and suppose that either $\lambda \ne b/a$ or $\mathbf{p}_{p-1}(a) \ne 0$. We first verify that, for $m \ge 1$,

$$x_m = \frac{b\mathbf{p}_{p-1}(a)}{a^p-1}. \quad (90)$$

Suppose that $\mathbf{q}_{m-1}(a) = 0$. Then

$$\begin{aligned} x_0 &= \frac{b}{a-1}\frac{\mathbf{p}_{p-1}(a)}{a^m\left(1+a+\cdots+a^{p-1}\right)} \le \frac{b}{a-1}\frac{1}{a^m} \\ &\le \frac{b}{a-1}\frac{1}{a} \le \lambda \end{aligned} \quad (91)$$

since $a \ge 2$ and λ is in I_3. Thus $x_1 = ax_0$ and similarly $x_j = ax_{j-1} \le \lambda$ for $j = 1,\ldots,m-1$. Hence, $x_m = a^m x_0$ satisfies (90).

Next assume $\mathbf{q}_{m-1}(a) \ne 0$ and $m = 1$. Since $a \ge 2$ and λ is in I_3, it follows that

$$x_0 = \frac{b\left(\mathbf{p}_{p-1}(a)+(a^p-1)\right)}{a(a^p-1)} \ge \frac{b}{a} \ge \lambda. \quad (92)$$

Additionally, since $\lambda < b/a$ or $\mathbf{p}_{p-1}(a) > 0$, at least one of the inequalities is strict and $x_0 > \lambda$. Thus $x_1 = ax_0 - b$ satisfies (90).

Hence, assume that $\mathbf{q}_{m-1}(a) \ne 0$ and $m > 1$. There exists a nonempty subset Λ_{m-1} of $\{0,1,\ldots,m-1\}$ such that

$$x_0 = b\left(\frac{\mathbf{p}_{p-1}(a)}{a^m(a^p-1)} + \frac{\mathbf{q}_{m-1}(a)}{a^m}\right), \quad (93)$$

where $\mathbf{q}_{m-1}(a) = \sum_{k\in\Lambda_{m-1}} a^k$. Let $i \equiv \min\Lambda_{m-1}$ and $j \equiv \max\Lambda_{m-1} = \deg \mathbf{q}_{m-1}(a)$. We will show that

$$x_{m-i} = \frac{b\mathbf{p}_{p-1}(a)}{a^i(a^p-1)} \quad (94)$$

and therefore (90) will hold as in the initial case $\mathbf{q}_{m-1}(a) = 0$.

We begin by showing that, for $m > 1$,

$$x_0 > \lambda \text{ iff } j = m-1. \quad (95)$$

Let $x_0 > \lambda$. If $a = 2$, then $b/a(a-1) = \lambda = b/a < x_0$ and therefore x_0 is not in I_3. On the other hand, if $a > 2$, then $b/a(a-1) < b/a$ and, by Lemma 16, x_0 is again not in I_3.

Since $b/a(a-1) \le \lambda \le b/a$, $x_0 > \lambda$, and x_0 is not in I_3, it follows that $b/a < x_0 \le b/(a-1)$ and, by Lemma 16, $j = m-1$.

Conversely, let $j = m-1$. Then $\mathbf{q}_{m-1}(a) \ge a^{m-1}$ and

$$x_0 \ge b\left(\frac{\mathbf{p}_{p-1}(a)}{a^m(a^p-1)} + \frac{a^{m-1}}{a^m}\right) \ge \frac{b}{a} \ge \lambda. \tag{96}$$

Since $\lambda < b/a$ or $\mathbf{p}_{p-1}(a) > 0$, at least one of the latter inequalities is strict and thus $x_0 > \lambda$. Hence, (95) holds.

Next consider x_1, where $\mathbf{q}_{m-1}(a) \ne 0$ and x_0 is given by (93). By (95),

$$x_1 = b\left(\frac{\mathbf{p}_{p-1}(a)}{a^{m-1}(a^p-1)} + \frac{\mathbf{q}_{m-2}(a)}{a^{m-1}}\right), \tag{97}$$

where $\mathbf{q}_{m-2}(a) = \sum_{k\in\Lambda_{m-2}} a^k$ such that

$$\Lambda_{m-2} \equiv \begin{cases} \Lambda_{m-1}, & \text{if } j < m-1 \\ \Lambda_{m-1} \setminus \{j\}, & \text{if } j = m-1, \end{cases} \tag{98}$$

which is a subset of $\Lambda_{m-1} \cap \{0,\ldots,m-2\}$. In addition, since $i \le j \le m-1$, it follows that

$$\mathbf{q}_{m-2}(a) = 0 \text{ iff } 1 = m-i, \tag{99}$$

in which case (94) holds for x_1.

Suppose that $1 < m-i$. Then $\mathbf{q}_{m-2}(a) \ne 0$, i is in Λ_{m-2}, and x_1 is of form (93) with m replaced by $m-1$. By (95),

$$x_1 > \lambda \text{ iff } \max\Lambda_{m-2} = m-2. \tag{100}$$

Moreover, as above, (94) holds for x_2 if $2 = m-i$.

By induction, for $r = 0, 1, \ldots, m-i-1$, we may assume

$$x_r = b\left(\frac{\mathbf{p}_{p-1}(a)}{a^{m-r}(a^p-1)} + \frac{\mathbf{q}_{m-r-1}(a)}{a^{m-r}}\right), \tag{101}$$

where $\mathbf{q}_{m-r-1}(a) = \sum_{k\in\Lambda_{m-r-1}} a^k \ne 0$ such that

$$\Lambda_{m-r-1} \equiv \begin{cases} \Lambda_{m-r}, & \text{if } \max\Lambda_{m-r} < m-r \\ \Lambda_{m-r} \setminus \{\max\Lambda_{m-r}\}, & \text{if } \max\Lambda_{m-r} = m-r, \end{cases} \tag{102}$$

which is a subset of $\Lambda_{m-r} \cap \{0,\ldots,m-r-1\}$, and, by (95),

$$x_r > \lambda \text{ iff } \max\Lambda_{m-r-1} = m-r-1. \tag{103}$$

Moreover, after $r \equiv m-i$ steps, we conclude that $\mathbf{q}_{m-r-1}(a) = 0$ and x_{m-i} satisfies (94). Thus (90) follows as in the case $\mathbf{q}_{m-1}(a) = 0$.

Finally, we verify that $x_m = x_{m+p}$, where x_m is given by (90). If $p = 1$, then x_m is 0 or $b/(a-1)$; so $x_m = x_{m+p}$. Thus assume $p > 1$. In this case, we prove that

$$x_m \le \lambda \text{ iff } \deg\mathbf{p}_{p-1}(a) < p-1. \tag{104}$$

Let $x_m \le \lambda$. Suppose first that x_m is in I_3. Since $a \ge 2$, we have $b/a(a-1) \le b/a$; and, by Lemma 16, $b/a(a-1) = x_m = \lambda = b/a$ and $a = 2$. Hence, by (90), $\mathbf{p}_{p-1}(a) = (a^p-1)/a$, which is impossible since $a = 2$ and $\mathbf{p}_{p-1}(a)$ is a positive integer but a^p-1 and a are relatively prime.

Thus assume x_m is not in I_3. Then x_m is in $I_1 \cup I_2$ since λ is in I_3 and $x_m \le \lambda$. Hence, by Lemma 16, $\deg\mathbf{p}_{p-1}(a) < p-1$ since $p > 1$.

Conversely, let $\deg\mathbf{p}_{p-1}(a) < p-1$ (and $p > 1$). Then $0 \le x_m \le b/a(a-1) \le \lambda$ by Lemma 16 and therefore $x_m \le \lambda$. Thus (104) follows.

As in (93), by (90),

$$x_m = \frac{b\sum_{k\in\Lambda_{p-1}} a^k}{a^p-1} \tag{105}$$

for some nonempty subset Λ_{p-1} of $\{0,\ldots,p-1\}$. By (104),

$$x_{m+1} = \frac{b\sum_{k\in\Lambda_{p-1}} T_{p-1}(a^k)}{a^p-1}, \tag{106}$$

where T_{p-1} is the mapping on the set $\{a^0, a^1, \ldots, a^{p-1}\}$ defined by

$$T_{p-1}(a^k) \equiv \begin{cases} a^{k+1}, & \text{if } k < p-1 \\ a^0, & \text{if } k = p-1. \end{cases} \tag{107}$$

Similarly, for $r = 1, 2, \ldots$,

$$x_{m+r} = \frac{b\sum_{k\in\Lambda_{p-1}} T^r_{p-1}(a^k)}{a^p-1} \tag{108}$$

and since $T^p_{p-1}(a^k) = a^k$, for all k, it follows that $x_m = x_{m+p}$.

We now turn to the second part of (2). Assume that $a \ge 2$, $\lambda = b/a$, and $x_0 = b\mathbf{q}_{m-1}(a)/a^m > 0$ and thus $m \ge 1$. We prove the following:

$$\begin{aligned} &\text{If } x_0 = \frac{b\mathbf{Q}_j(a)}{a^k} \ne 0 \quad \text{where } \mathbf{Q}_j(a) \text{ is in } \mathbb{P}_j(a) \\ &\text{such that } 0 \le j = \deg\mathbf{Q}_j(a) < k \le m, \\ &\text{then either } x_0 = \frac{b}{a} = \lambda \\ &\text{or } x_1 = \frac{b\mathbf{Q}_{j'}(a)}{a^{k-1}} \ne 0 \\ &\text{where } \mathbf{Q}_{j'}(a) \text{ is in } \mathbb{P}_{j'}(a) \text{ and } 0 \le j' = \deg\mathbf{Q}_{j'}(a) < k-1 \le m. \end{aligned} \tag{109}$$

Suppose that $x_0 = b\mathbf{Q}_j(a)/a^k \ne 0$, where $\mathbf{Q}_j(a)$ satisfies the above hypotheses. Consider two cases for j.

(i) Assume $j \le k-2$. Then $x_0 = b\mathbf{Q}_j(a)/a^k < \lambda = b/a$: $\mathbf{Q}_j(a) \le 1 + a + \cdots + a^j = (a^{j+1}-1)/(a-1) < a^{k-1}$ since

$$\begin{aligned} a^{k-1} + a^{j+1} - 1 &\le a^{k-1} + a^{k-1} - 1 = 2a^{k-1} - 1 \\ &\le aa^{k-1} - 1 < a^k. \end{aligned} \tag{110}$$

Hence, (109) holds since $x_1 = ax_0 = b\mathbf{Q}_j(a)/a^{k-1} \ne 0$, where $0 \le j = \deg\mathbf{Q}_j(a) \le k-2 < k-1 \le m$ so let $j' = j$.

(ii) Suppose that $j = k-1$. If $\mathbf{Q}_j(a) = a^j$, then $x_0 = b/a = \lambda$ so assume that $\mathbf{Q}_j(a) > a^j$ since $j = \deg \mathbf{Q}_j(a)$. In this case, (109) also holds since $x_0 = b\mathbf{Q}_j(a)/a^k > \lambda = b/a$ and hence $x_1 = ax_0 - b = b\mathbf{Q}_{j'}(a)/a^{k-1} \neq 0$, where $\mathbf{Q}_{j'}(a) \equiv \mathbf{Q}_j(a) - a^j$ and $0 \leq j' \equiv \deg \mathbf{Q}_{j'}(a) \leq k-2 < k-1 \leq m$.

Thus, starting with $x_0 = b\mathbf{q}_{m-1}(a)/a^m = b\mathbf{Q}_j(a)/a^k \neq 0$, $\mathbf{Q}_j(a) \in \mathbb{P}_j(a)$, $0 \leq j = \deg \mathbf{Q}_j(a) < k = m$, and, applying (109) to $x_1, x_2, \ldots$, we conclude that $x_{i-1} = b/a = \lambda$ for some i in $[1, m]$. Therefore,

$$x_i = ax_{i-1} = b \geq \frac{b}{a-1} > \frac{b}{a} = \lambda \tag{111}$$

and $x_{i+j} = f_+(i, j)$ for all $j \geq 0$ by Lemma 5(2).

(1) Let λ be in I_2. The condition "$x_{i+j} = f_-(i, j)$ for all $j \geq 0$" is equivalent to "$x_i \leq 0$." Thus we further assume that $x_i > 0$ for $i = 1, \ldots, m+p$ and show that $x_m = x_{m+p}$ following the outline of the proof of (2).

We first verify that x_m satisfies (90) for $m \geq 1$. Suppose that $\mathbf{q}_{m-1}(a) = 0$. Then, since $a \geq 2$ and $m \geq 1$,

$$\begin{aligned} x_0 &= \frac{b}{a-1}\frac{\mathbf{p}_{p-1}(a)}{a^m((a^p-1)/(a-1))} \leq \frac{b}{a-1}\frac{1}{a^m} \\ &\leq \frac{b}{a-1}\frac{1}{a} \leq \frac{b}{a}. \end{aligned} \tag{112}$$

If $x_0 > \lambda$, then $x_1 \leq 0$ by Lemma 6(4). Therefore, $x_0 \leq \lambda$ and

$$x_1 = ax_0 = \frac{b\mathbf{p}_{p-1}(a)}{a^{m-1}(a^p-1)}. \tag{113}$$

Similarly, $x_j \leq \lambda$ for $j = 1, \ldots, m-1$ and hence $x_m = a^m x_0$ satisfies (90).

Suppose next that $\mathbf{q}_{m-1}(a) \neq 0$ and $m = 1$. Since λ is in I_2,

$$x_0 = \frac{b\left(\mathbf{p}_{p-1}(a) + (a^p-1)\right)}{a(a^p-1)} \geq \frac{b}{a} > \lambda \tag{114}$$

and $x_1 = ax_0 - b$ satisfies (90).

Thus assume that $\mathbf{q}_{m-1}(a) \neq 0$ and $m > 1$. Then x_0 is given by (93), where $\mathbf{q}_{m-1}(a) = \sum_{k\in\Lambda_{m-1}} a^k$ for some nonempty subset Λ_{m-1} of $\{0, 1, \ldots, m-1\}$. We will show that (94) holds, where $i \equiv \min \Lambda_{m-1}$, and therefore (90) will hold as in the case $\mathbf{q}_{m-1}(a) = 0$.

Let $j \equiv \max \Lambda_{m-1} = \deg \mathbf{q}_{m-1}(a)$. We begin by showing (95) for $m > 1$.

Let $x_0 > \lambda$. If x_0 is in $(\lambda, b/a(a-1)) \cup I_3$, then $x_1 \leq 0$ by Lemma 6(4). Since $x_1 > 0$ by assumption and $x_0 > \lambda$, it follows that x_0 is in $I_4 \cup I_5$; and therefore $j = m-1$ by Lemma 16 since $m > 1$.

Conversely, let $j = m-1$. By Lemma 16,

$$\lambda < \frac{b}{a(a-1)} \leq \frac{b}{a} \leq x_0 \leq \frac{b}{a-1} \tag{115}$$

and $x_0 > \lambda$. Thus (95) holds.

Equation (94) and consequently (90) follow from (95) as in the proof of (2).

Finally, we verify that $x_m = x_{m+p}$, where x_m is given by (90). We may assume $p > 1$ since otherwise x_m is 0 or $b/(a-1)$, and therefore $x_m = x_{m+p}$. We start by proving (104): since λ is in I_2 and $x_{m+1} > 0$ by assumption, it follows by Lemma 6(4) that x_m is not in $(\lambda, b/a(a-1)) \cup I_3$.

Let $x_m \leq \lambda$. Then x_m is in $I_1 \cup I_2$ so $\deg \mathbf{p}_{p-1}(a) < p-1$ by Lemma 16 since x_m is not in I_3 and $p > 1$.

Conversely, assume that $\deg \mathbf{p}_{p-1}(a) < p-1$. By Lemma 16, $0 \leq x_m \leq b/a(a-1)$, but x_m is not in $(\lambda, b/a(a-1)]$. Therefore $x_m \leq \lambda$ and (104) holds.

By (90) and (104), the desired result $x_m = x_{m+p}$ follows as in the proof of (2).

(3) Let λ be in I_4. The condition "$x_{i+j} = f_+(i, j)$ for all $j \geq 0$" is equivalent to "$x_i \geq b/(a-1)$." Thus we assume that $x_i < b/(a-1)$ for $i = 1, \ldots, m+p$ and show that $x_m = x_{m+p}$ following the outline of the proof of (2).

We first verify that x_m satisfies (90) for $m \geq 1$. Suppose that $\mathbf{q}_{m-1}(a) = 0$. Then, since $a \geq 2$, $m \geq 1$, and λ is in I_4,

$$\begin{aligned} x_0 &= \frac{b}{a-1}\frac{\mathbf{p}_{p-1}(a)}{a^m((a^p-1)/(a-1))} \leq \frac{b}{a-1}\frac{1}{a^m} \\ &\leq \frac{b}{a-1}\frac{1}{a} \leq \frac{b}{a} < \lambda. \end{aligned} \tag{116}$$

Thus $x_1 = ax_0$ and similarly $x_j \leq \lambda$ for $j = 1, \ldots, m-1$. Hence, $x_m = a^m x_0$ satisfies (90).

Next, we assume $\mathbf{q}_{m-1}(a) \neq 0$ and $m = 1$. Then

$$x_0 = \frac{b\left(\mathbf{p}_{p-1}(a) + (a^p-1)\right)}{a(a^p-1)} \geq \frac{b}{a} \geq \frac{b}{a(a-1)}. \tag{117}$$

If $x_0 \leq \lambda$, then $x_1 \geq b/(a-1)$ by Lemma 6(2), contrary to our hypothesis. Thus $x_0 > \lambda$ and $x_1 = ax_0 - b$ satisfies (90).

Therefore assume that $\mathbf{q}_{m-1}(a) \neq 0$ and $m > 1$. Then x_0 is given by (93), where $\mathbf{q}_{m-1}(a) = \sum_{k\in\Lambda_{m-1}} a^k$ for some nonempty subset Λ_{m-1} of $\{0, 1, \ldots, m-1\}$. We will show that (94) holds, where $i \equiv \min \Lambda_{m-1}$, and hence (90) will follow as in the case $\mathbf{q}_{m-1}(a) = 0$.

Let $j \equiv \max\Lambda_{m-1} = \deg \mathbf{q}_{m-1}(a)$. We begin by showing (95) for $m > 1$: if x_0 is in $I_3 \cup (b/a, \lambda)$, then $x_1 \geq b/(a-1)$ by Lemma 6(2). Thus x_0 is not in $I_3 \cup (b/a, \lambda)$ by our hypothesis.

Let $x_0 > \lambda$. Then x_0 is in $I_4 \cup I_5$, $m > 1$, and hence $j = m-1$ by Lemma 16 since x_0 is not in I_3.

Conversely, let $j = m-1$ (and $m > 1$). By Lemma 16, $b/a(a-1) \leq b/a \leq x_0 \leq b/(a-1)$. Since x_0 is not in $(b/a, \lambda)$, we have that $x_0 > \lambda$; and (95) holds.

Equation (94) and consequently (90) follow from (95) as in the proof of (2).

Finally, we verify that $x_m = x_{m+p}$, where x_m is given by (90). We may assume $p > 1$ since otherwise x_m is 0 or $b/(a-1)$, and therefore $x_m = x_{m+p}$. We start by proving (104).

Let $x_m \leq \lambda$. Then x_m is not in $I_3 \cup (b/a, \lambda)$ by Lemma 6(2) since $x_{m+1} < b/(a-1)$. Thus x_m is in $I_1 \cup I_2$ and therefore $\deg \mathbf{p}_{p-1}(a) < p-1$ by Lemma 16 since x_m is not in I_3 and $p > 1$.

Conversely, if $\deg \mathbf{p}_{p-1}(a) < p-1$ (and $p > 1$), then $0 \leq x_m \leq b/a(a-1) < \lambda$ by Lemma 16, and (104) follows.

Finally, $x_m = x_{m+p}$ by (90) and (104) as in the proof of (2). $\square$

The following two examples illustrate symmetry between type C solutions about the point $b/2(a-1)$ that will be generalized in the next section. In particular, type B solutions will be shown to be reflections of type A solutions.

Example 18. Let $a \geq 2$ and let $b > 0$ and let λ be in I_3. For positive integer k, consider the weighted average

$$x_0 \equiv \frac{b/a^2 + a^k (b/a (a-1))}{1 + a^k} \tag{118}$$

in I_2 (that converges to $b/a(a-1)$ as k tends to infinity). Then

$$x_0 = \frac{b\left(a^k \left(\left(a^k - 1\right)/(a-1)\right) + \left(a^{2k} - 1\right)\right)}{a^2 \left(a^{2k} - 1\right)} \tag{119}$$

so $x_2 = x_{2+2k}$ by Theorem 17(2).

Example 19. For integer $k \geq 1$, redefine x_0 in Example 18 to be the weighted average

$$x_0 \equiv \frac{\left(b/(a-1) - b/a^2\right) + a^k (b/a)}{1 + a^k} \tag{120}$$

in I_4 (that converges to b/a as $k \to \infty$). Then $x_2 = x_{2+2k}$ as above since

$$x_0 = \frac{b\left(\left(a^k - 1\right)/(a-1) + \left(a^{2k} - 1\right)a\right)}{a^2 \left(a^{2k} - 1\right)} \tag{121}$$

in form (74). Note that, in this case, $b/(a-1) - x_0$ is the initial value in Example 18.

Theorem 17 may be applied to cryptography.

Example 20. Arbitrary vectors $\mathbf{v} = \langle v_0, \ldots, v_{p-1} \rangle$ of zeros and ones may be encoded by selecting any real numbers $a \geq 2$, $b > 0$, and λ in I_3 and defining

$$x_0 \equiv \frac{b \sum v_i a^i}{a^p - 1} \tag{122}$$

in form (74). By Theorem 17(2), the "cipher-set" $\{a, b, \lambda, x_0 \text{ (computed value)}\}$ may then be decoded for $\mathbf{v}$ as follows: if $x_0 = 0$, then $\mathbf{v}$ is a zero vector of indeterminate length. If $x_0 = b/(a-1)$, then $\mathbf{v}$ is a vector of all ones of indeterminate length.

Assume $x_0 \neq 0, b/(a-1)$. Compute enough terms of the periodic sequence x_n to determine its period p. Then p is the length of $\mathbf{v}$ and the proof of Theorem 17(2) shows that

$$v_i = 1 \text{ iff } x_{p-1-i} > \lambda \quad (i = 0, \ldots, p-1). \tag{123}$$

The following eventually periodic solution x_n with x_0 of form (74) is of type A.

Example 21. Let $a \geq 2$ and let $b > 0$ and, for integers $m \geq 1$ and $p > 2$, let x_0 be the weighted average

$$x_0 \equiv \frac{b a^{p-2}/\left(a^p - 1\right) + \left(a^{m-1} - 1\right)(b/a(a-1))}{a^{m-1}} \tag{124}$$

in I_2. Then x_0 is of form (74) with $\mathbf{p}_{p-1}(a) = a^{p-1}$ and $\mathbf{q}_{m-1}(a) = (a^{m-1} - 1)/(a-1)$. Choose λ in I_2 such that $x_0 \leq \lambda$. Then $x_m = ba^{p-1}/(a^p - 1)$:

$$x_1 = a x_0 = \frac{b a^{p-1}}{a^{m-1}\left(a^p - 1\right)} + \frac{b}{a^{m-1}} \left(\frac{a^{m-1} - 1}{a - 1}\right), \tag{125}$$

so the claim is true for $m = 1$.

Assume $m > 1$. We prove by induction that

$$x_i = \frac{b a^{p-1}}{a^{m-i}\left(a^p - 1\right)} + \frac{b}{a^{m-i}} \left(\frac{a^{m-i} - 1}{a - 1}\right) > \frac{b}{a} > \lambda \tag{126}$$

for $i = 1, \ldots, m-1$. By the form of x_1 above,

$$x_1 > \frac{b}{a^{m-1}} \left(\frac{a^{m-1} - 1}{a - 1}\right) \geq \frac{b}{a} > \lambda \tag{127}$$

since $a^{m-1} \geq a$; so (126) holds for $i = 1$.

Suppose that (126) is true for some $i < m - 1$. Then

$$\begin{aligned} x_{i+1} &= a x_i - b \\ &= \frac{b a^{p-1}}{a^{m-(i+1)}\left(a^p - 1\right)} + \frac{b}{a^{m-(i+1)}} \left(\frac{a^{m-(i+1)} - 1}{a - 1}\right) \\ &\geq \frac{b}{a^{m-(i+1)}} \left(\frac{a^{m-(i+1)} - 1}{a - 1}\right) \geq \frac{b}{a} > \lambda \end{aligned} \tag{128}$$

as in the case $i = 1$ since $m > i + 1$. Thus (126) holds for $i = 1, \ldots, m-1$.

In particular, $x_m = a x_{m-1} - b = b a^{p-1}/(a^p - 1) > b/a > \lambda$.

Finally $x_{m+p} = x_m$: $x_{m+1} = a x_m - b = b/(a^p - 1) \leq b/a^2 < \lambda$ since $p > 2$, and, by inducton, $x_{m+i} = b a^{i-1}/(a^p - 1) \leq b/a^2 < \lambda$ for $i = 1, \ldots, p-2$. Hence, $x_{m+(p-1)} = a x_{m+(p-2)} = b a^{p-2}/(a^p - 1) < x_0 \leq \lambda$ by the form of x_0 and the choice of λ. Therefore, $x_{m+p} = b a^{p-1}/(a^p - 1) = x_m$.

Unfortunately, not all type A solutions x_n with x_0 of form (74) are bounded.

Example 22. Let $a \geq 2$ and let $b > 0$ and

$$\begin{aligned} \frac{b}{a^2} < x_0 \equiv \frac{b\left(a^2 + 1\right)}{a\left(a^3 - 1\right)} < \lambda \equiv \frac{ba}{(a-1)\left(a^2 + 1\right)} \\ < \frac{b}{a(a-1)}. \end{aligned} \tag{129}$$

Then x_0 is of form (74) with $m = 1$, $p = 3$, $\mathbf{p}_{p-1}(a) = a^2 + 1$, and $\mathbf{q}_{m-1}(a) = 0$. Moreover,

$$\begin{aligned} x_1 &= a x_0 = \frac{b\left(a^2 + 1\right)}{a^3 - 1} > \frac{b}{a} > \lambda, \\ x_2 &= a x_1 - b = \frac{b(a+1)}{a^3 - 1} > \lambda, \\ x_3 &= a x_2 - b = \frac{b\left(\left(a^3 - 1\right)/(a-1) - a^3\right)}{a^3 - 1} < 0 \end{aligned} \tag{130}$$

since $a \geq 2$. Therefore, $x_{3+j} = f_-(3, j)$ for all $j \geq 0$.

Suppose that x_0 is defined by (74). By Theorems 13 and 17, if $a \geq 2$ and x_n is bounded, then x_n is eventually periodic. In fact, if $a = 2$ and λ is in I_3, then $x_m = x_{m+p}$ by treating the cases $\mathbf{p}_{p-1}(a) \neq 0$ and $\mathbf{p}_{p-1}(a) = 0$ separately. However, if $a < 2$ and λ is in I_3, then x_n is bounded (Lemma 6) but may not be eventually periodic.

Example 23. Assume that $1 < a = r/s < \phi$, where r and s are odd and even integers, respectively; $b > 0$ and $x_0 \equiv ba/(a^2 - 1)$. Then x_0 is of form (74) with $m = 0$, $p = 2$, and $\mathbf{p}_{p-1}(a) = a$; and x_0 is in I_3 since $a(a-1) < 1$. Choose λ in I_3 such that $x_0 \leq \lambda$. Then $\lambda \neq b/a$ and $\mathbf{p}_{p-1}(a) \neq 0$, but x_n is not eventually periodic: with $r_0 \equiv r$ and $r_1 \equiv r^2$, we have as in Example 9, for $n \geq 0$,

$$x_n = \frac{b}{a-1} \frac{r_n}{s^n (r+s)}, \tag{131}$$

where r_n is odd. If x_n is eventually periodic, then $x_u = x_{u+v}$ for some $u \geq 0$ and $v \geq 1$; and therefore $r_{u+v} = s^v r_u$ is both even and odd.

5. Symmetric Solutions

Theorems 13 and 17 indicate symmetry about the midpoint of $(0, b/(a-1))$ between pairs of solutions of (18). Moreover, if x_0 satisfies (74), then so does $b/(a-1) - x_0$: if

$$x_0 = \frac{b\left(\mathbf{p}_{p-1}(a) + (a^p - 1)\,\mathbf{q}_{m-1}(a)\right)}{a^m (a^p - 1)}, \tag{132}$$

then

$$\frac{b}{a-1} - x_0 = \frac{b\left(\left((a^p - 1)/(a-1) - \mathbf{p}_{p-1}(a)\right) + (a^p - 1)\left((a^m - 1)/(a-1) - \mathbf{q}_{m-1}(a)\right)\right)}{a^m (a^p - 1)}. \tag{133}$$

Therefore, in view of Theorem 17, a natural question is, if x_n and y_n are given by (18), where $y_0 \equiv b/(a-1) - x_0$, does it follow that $y_n = b/(a-1) - x_n$ for all n? We show affirmative answers in general for at least two choices of λ, the first of which requires that λ is not a term of x_n.

Theorem 24. *Let x_n satisfy difference equation (18) with respect to $a > 1$, $b > 0$, $0 < \lambda, x_0 < b/(a-1)$, and $F_\lambda \equiv F$; and let y_n be defined by (18) in terms of $a \equiv a$, $b \equiv b$, $\lambda_y \equiv b/(a-1) - \lambda$, $y_0 \equiv b/(a-1) - x_0$, and the corresponding F denoted by F_{λ_y}. Then*

$$y_1 = \begin{cases} \dfrac{b}{a-1} - x_1, & \text{if } x_0 \neq \lambda \\ a\left(\dfrac{b}{a-1}\right) - x_1, & \text{if } x_0 = \lambda. \end{cases} \tag{134}$$

Moreover, if $x_0 = \lambda$ and $\lambda \geq b/a(a-1)$, then $x_{1+j} = f_+(1, j)$ for all $j \geq 0$.

Proof. Suppose first that $x_0 > \lambda$. Then $x_1 = ax_0 - b = b/(a-1) - a(b/(a-1) - x_0)$ and $y_0 = b/(a-1) - x_0 < b/(a-1) - \lambda = \lambda_y$ so that $y_1 = ay_0 = b/(a-1) - x_1$.

Next, assume that $x_0 \leq \lambda$. Then $x_1 = ax_0$. If $x_0 < \lambda$, then $y_0 = b/(a-1) - x_0 > b/(a-1) - \lambda = \lambda_y$ and therefore $y_1 = ay_0 - b = b/(a-1) - ax_0 = b/(a-1) - x_1$. And if $x_0 = \lambda$, then $y_0 = b/(a-1) - \lambda = \lambda_y$ and $y_1 = ay_0 = a(b/(a-1)) - x_1$.

Finally, if $x_0 = \lambda \geq b/a(a-1)$, then $x_{1+j} = f_+(1, j)$ by Lemma 6(2). □

For the solutions x_n in the following examples (with the conditions imposed), $y_n = b/(a-1) - x_n$ for all n by Theorem 24 since $x_k \neq \lambda$ for all k: Examples 8 (if $\lambda \neq b/2(a-1)$), 12, 21 (if $x_0 < \lambda$), and 22.

Let x_n and y_n be given as in Theorem 24 and suppose further that $x_k \neq \lambda$ for all k. By Remark 2, $x_n = x(n)$, where $x = x(t)$ satisfies

$$x'(t) = Ax(t) + Bx([t]) + CF_\lambda(x([t])) \tag{135}$$

for constants A, B, and C such that $b/(a-1) = -C/(A+B)$. Therefore, $y_n = y(n)$, where $y(t) \equiv b/(a-1) - x(t)$ is the symmetric solution about the line $x = b/2(a-1)$ of the differential equation

$$y'(t) = Ay(t) + By([t]) + CF_{\lambda_y}\left(y([t])\right) \tag{136}$$

with $y(0) = b/(a-1) - x_0$; and $F_{\lambda_y}(y([t])) = 1 - F_\lambda(x([t]))$ since $x([t]) \neq \lambda$, and hence $x([t]) \leq \lambda$ if and only if $y([t]) > \lambda_y$.

For type C solutions, where $\lambda \neq b/a$, there may be another symmetric solution obtained by reflecting x_0 about $b/2(a-1)$.

Theorem 25. *Let x_n satisfy (18) with respect to $a > 2$, $b > 0$, and $0 < x_0 < b/(a-1)$ such that either*

(1) *$b/a(a-1) \leq \lambda < b/2(a-1)$ and x_n is bounded below*

or

(2) *$b/2(a-1) \leq \lambda < b/a$ and x_n is bounded above.*

If z_n satisfies (18) in terms of $a \equiv a$, $b \equiv b$, $\lambda \equiv \lambda$, and $z_0 \equiv b/(a-1) - x_0$, then $z_n = b/(a-1) - x_n$ for all n.

Proof. Suppose that (1) holds. If $x_0 \leq \lambda$, then $x_1 = ax_0$ and $z_0 = b/(a-1) - x_0 \geq b/(a-1) - \lambda > b/(a-1) - b/2(a-1) = b/2(a-1) > \lambda$. Thus, $z_1 = az_0 - b = b/(a-1) - ax_0 = b/(a-1) - x_1$ in this case.

Next, assume that $x_0 > \lambda$. Then $x_1 = ax_0 - b = b/(a-1) - a(b/(a-1) - x_0) = b/(a-1) - az_0$. We claim that $z_0 \le \lambda$. By way of contradiction, assume that $z_0 > \lambda \ge b/a(a-1)$. Then $x_0 = b/(a-1) - z_0 < b/(a-1) - \lambda \le b/(a-1) - b/a(a-1) = b/a$. Hence, $x_1 = ax_0 - b < a(b/a) - b = 0$ and x_n is not bounded below by Lemma 5(1), contrary to (1). Therefore $z_0 \le \lambda$ and $z_1 = az_0 = b/(a-1) - x_1$.

It follows in either case that $z_1 = b/(a-1) - x_1$; and, by induction, $z_n = b/(a-1) - x_n$ for all n.

Suppose now that (2) holds. If $x_0 > \lambda$, then $x_1 = ax_0 - b$ and $z_0 = b/(a-1) - x_0 < b/(a-1) - \lambda \le b/(a-1) - b/2(a-1) = b/2(a-1) \le \lambda$. Thus $z_1 = az_0 = ab/(a-1) - ax_0 = ab/(a-1) - (x_1 + b) = b/(a-1) - x_1$ in this case.

Finally assume that $x_0 \le \lambda$. We claim that $z_0 > \lambda$. Suppose otherwise that $z_0 \le \lambda$. Then $x_0 = b/(a-1) - z_0 \ge b/(a-1) - \lambda > b/(a-1) - b/a = b/a(a-1)$ and $x_1 = ax_0 > a(b/a(a-1)) = b/(a-1)$; so x_n is not bounded above by Lemma 5(2), contradicting (2). Therefore $z_0 > \lambda$ and $z_1 = az_0 - b = b/(a-1) - ax_0 = b/(a-1) - x_1$.

Hence, $z_1 = b/(a-1) - x_1$ in both cases; and, by induction, $z_n = b/(a-1) - x_n$ for all n. □

Note that Theorem 25 may fail when $\lambda = b/a$: let $a > 2$ and let $b > 0$ and let $x_0 \equiv b/a = \lambda$. Then $x_1 = ax_0 = b$ and $z_0 = b/(a-1) - x_0 = b/a(a-1) < \lambda$. Thus $z_1 = az_0 = b/(a-1) \ne b/(a-1) - x_1 = b/(a-1) - b$.

The solution x_n in Example 11 satisfies $y_n = b/(a-1) - x_n$ for all n whenever $\lambda \ne ba^{p-2}/(a^p - 1)$ by Theorem 24 since λ is thus not a term of x_n. Furthermore, in this example, if $a > 2$ and $\lambda \ne b/a$ is in I_3, then $z_n = b/(a-1) - x_n$ for all n by Theorem 25 since x_n is eventually periodic and therefore bounded.

Let x_n and z_n be given as in Theorem 25. By Remark 2, $x_n = x(n)$, where $x = x(t)$ satisfies (2) for constants A, B, and C such that $b/(a-1) = -C/(A+B)$. Therefore, $z_n = z(n)$, where $z(t) \equiv b/(a-1) - x(t)$ is the symmetric solution about $x = b/2(a-1)$ of the same differential equation

$$z'(t) = Az(t) + Bz([t]) + CF(z([t])) \tag{137}$$

but with $z(0) = b/(a-1) - x_0$, since, by the proof of Theorem 25, $F(z([t])) = 1 - F(x([t]))$ (i.e., $x([t]) \le \lambda$ if and only if $z([t]) > \lambda$).

If x_0 is given by (74), then $b/(a-1) - x_0$ is given by (133) and Theorem 17 may be applied to Theorems 24 and 25.

Corollary 26. *Let x_n satisfy (18), where $a \ge 2$, $b > 0$, λ is in I_3, and x_0 is given by (74) for some integers $m \ge 0$ and $p \ge 1$ and some polynomials $\mathbf{p}_{p-1}(a)$ in $\mathbb{P}_{p-1}(a)$ and $\mathbf{q}_{m-1}(a)$ in $\mathbb{P}_{m-1}(a)$.*

(1) *If $\lambda \ne b/a(a-1)$ or $\mathbf{p}_{p-1}(a) \ne (a^p - 1)/(a-1)$ or $a = 2$, then $y_m = y_{m+p}$. Moreover, if x_n is bounded with no stationary states, then $y_n = b/(a-1) - x_n$ for all n. In particular, if $\mathbf{p}_{p-1}(a) \ne 0, (a^p - 1)/(a-1)$, then $y_n = b/(a-1) - x_n$ for all n.*

(2) *If $\lambda \ne b/a$ or $\mathbf{p}_{p-1}(a) \ne (a^p - 1)/(a-1)$, then $z_m = z_{m+p}$. Moreover, if $a > 2$, $\lambda \ne b/a$, and x_n is bounded, then $z_n = b/(a-1) - x_n$ for all n.*

Proof. Suppose that $a \ge 2$, $b > 0$, λ is in I_3, and x_0 is given by (74). The first lines of (1) and (2) follow directly from (133) and Theorem 17(2).

(1) Assume x_n is bounded with no stationary states. Then $x_0 \ne 0, b/(a-1)$ and $0 < x_0 < b/(a-1)$ by (81). By Theorem 13, x_k is not in I_3 so $x_k \ne \lambda$ for all k. Thus $y_n = b/(a-1) - x_n$ by Theorem 24.

In particular, suppose that $\mathbf{p}_{p-1}(a) \ne 0, (a^p - 1)/(a-1)$. Then x_n is eventually periodic, and hence bounded, by Theorem 17(2); and, by the proof of this theorem, the only possible stationary states are when

(i) $\mathbf{q}_{m-1}(a) = 0$ and $x_0 = b\mathbf{p}_{p-1}(a)/a^m(a^p - 1) = 0$ ($\le b/a(a-1)$) or

(ii) $x_m = b\mathbf{p}_{p-1}(a)/(a^p - 1)$ is 0 or $b/(a-1)$,

that is, when $\mathbf{p}_{p-1}(a)$ is 0 or $(a^p - 1)/(a-1)$, which are ruled out. Thus, by the previous case, $y_n = b/(a-1) - x_n$ for all n.

(2) Next, assume that $a > 2$, $\lambda \ne b/a$, and x_n is bounded. By (81), $0 \le x_0 \le b/(a-1)$. If $x_0 = 0$, then $x_n = 0$ for all n; and hence $z_0 = b/(a-1)$ and thus $z_n = b/(a-1) = b/(a-1) - x_n$ for all n. Similarly, if $x_0 = b/(a-1)$, then $z_n = b/(a-1) - x_n$ for all n.

Therefore, we may assume $0 < x_0 < b/(a-1)$ and the desired result is now immediate from Theorem 25. □

Example 18 (revisited). In this example, $a \ge 2$, $b > 0$, λ is in I_3, and x_0 is of form (74) with $m = 2$, $p = 2k \ge 2$, $\mathbf{p}_{p-1}(a) = a^k((a^k - 1)/(a-1))$, and $\mathbf{q}_{m-1}(a) = 1$. Moreover, $x_2 = x_{2+2k}$. By Corollary 26, $y_n = b/(a-1) - x_n$ for all n and $z_2 = z_{2+2k}$. Furthermore, if $a > 2$ and $\lambda \ne b/a$, then $z_n = b/(a-1) - x_n$ for all n.

Example 27. Let $a \ge 2$ and let $b > 0$ and let λ be in I_3. For positive integer k, consider the weighted average

$$x_0 \equiv \frac{b/a + a^k\left(b/(a-1) - b/a^2\right)}{1 + a^k} \tag{138}$$

in I_4 (that converges to $b/(a-1) - b/a^2$ as k tends to infinity). Then x_0 is of form (74) with $m = 1$, $p = 2k \ge 2$, $\mathbf{p}_{p-1}(a) = a^{k-1}((a^k - 1)/(a-1))$, and $\mathbf{q}_{m-1}(a) = 1$ so $x_1 = x_{1+2k}$ by Theorem 17(2). Therefore, as above,

(i) $y_n = b/(a-1) - x_n$ for all n;

(ii) $z_1 = z_{1+2k}$;

(iii) if $a > 2$ and $\lambda \ne b/a$, then $z_n = b/(a-1) - x_n$ for all n.

Moreover,

$$\begin{aligned} \frac{b}{a-1} - x_0 &= \frac{b/a(a-1) + a^k\left(b/a^2\right)}{1 + a^k} \\ &= \frac{b\left(\left(\left(a^k - 1\right)/(a-1)\right)\left(a^k - a^{k-1} + 1\right) + \left(a^{2k} - 1\right)0\right)}{a\left(a^{2k} - 1\right)}. \end{aligned} \tag{139}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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