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Research Article

A Tauberian Theorem for Double Cesàro Summability Method

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We have generalized Littlewood Tauberian theorems for (C,k,r) summability of double sequences by using oscillating behavior and de la Vallée-Poussin mean. Further, the generalization of (C,r) summability from (C,k,r) summability is given as corollaries which were earlier established by the authors.

1. Introduction

Let $u=(u_{mn})$ be a double real sequence. We define $\Delta_n u_{mn} = u_{mn} - u_{m,n-1}$, $\Delta_m u_{mn} = u_{mn} - u_{m-1,n}$, and $\Delta_{m,n} u_{mn} = u_{mn} - u_{m-1,n} - u_{m,n-1} + u_{m-1,n-1}$.

A double sequence (u_{mn}) is said to be bounded if there exists a finite real number $C>0$ such that $|u_{mn}| \le C$, $\forall m,n \in Z^+$. Let us write

$$\begin{aligned}\sigma_{mn}^{(11)}(u) &= \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} u_{pq} \\ &= \sum_{p=1}^{m}\sum_{q=1}^{n}\frac{V_{pq}^{(11)}}{pq} + u_{00}\end{aligned} \tag{1}$$

(see [1]),

$$\begin{aligned}\sigma_{mn}^{(10)}(u) &= \frac{1}{m+1}\sum_{p=0}^{m} u_{pn}, \\ \sigma_{mn}^{(01)}(u) &= \frac{1}{n+1}\sum_{q=0}^{n} u_{mq}.\end{aligned} \tag{2}$$

Then, we say that a double sequence $u=(u_{mn})$ is $(C,1,1)$ summable to s if $\sigma_{mn}^{(11)}(u)$ converges to s, as $m,n \to \infty$. Similarly, we say that it is $(C,1,0)$ summable to s if $\sigma_{mn}^{(10)}(u)$ converges to s as $m,n \to \infty$ and $(C,0,1)$ summable to s if $\sigma_{mn}^{(10)}(u)$ converges to s as $m,n \to \infty$. For all nonnegative integers k and r, we define $\sigma_{mn}^{(kr)}(u)$ as follows:

$$\sigma_{mn}^{(kr)}(u) = \begin{cases} \dfrac{1}{(m+1)(n+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n}\sigma_{pq}^{(k-1,r-1)}(u) & \text{for } k,r \ge 1, \\ u_{mn} & \text{for } k,r=0. \end{cases} \tag{3}$$

A double sequence $u=(u_{mn})$ is said to be (C,k,r) summable to s if $\sigma_{mn}^{(kr)}(u)$ converges to s. If $k=1$ and $r=1$, then (C,k,r) summability reduces to $(C,1,1)$ summability. Again, if $k \ne 0$ and $r=0$, then (C,k,r) summability reduces to $(C,k,0)$ summability. Further, if $k=0$ and $r \ne 0$, then (C,k,r) summability reduces to $(C,0,r)$ summability. Here, Cesàro summability of $u=(u_{mn})$ refers to (C,k,r) summability of $u=(u_{mn})$. It may also be noted that the convergence of a double sequence $u=(u_{mn})$ implies the Cesàro summability of $u=(u_{mn})$, but the converse is not generally true.

For example, consider a function $f(x,y)=e^{2x}\sin(3y)$: the sequence (u_{mn}), which is the sequence of coefficients in the Taylor series expansion of the function $f(x,y)$ about origin, is Cesàro summable but not convergent.

For the proof of the converse part, certain conditions are presented in terms of oscillatory behavior of double sequence $u=(u_{mn})$.

Let us define (u_{mn}) as

$$u_{mn} = V_{mn}^{(11)} + \sum_{p=1}^{m}\sum_{q=1}^{n}\frac{V_{pq}^{(11)}}{pq} + u_{00}, \quad (m,n = 1,2,\ldots), \tag{4}$$

where

$$\begin{aligned} u_{mn} - \sigma_{mn}^{11}(u) &= V_{mn}^{(11)}(\Delta u) \\ &= \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} pq\left(\Delta_{p,q}u_{pq}\right) \end{aligned} \tag{5}$$

(see [2]).

Moreover, in analogy to Kronecker identity for a single sequence, we can write

$$\begin{aligned} V_{mn}^{(1,0)}(\Delta u) &= \frac{1}{(m+1)}\sum_{p=0}^{m} p\left(\Delta_p, u_{p,n}\right), \\ V_{mn}^{(0,1)}(\Delta u) &= \frac{1}{(n+1)}\sum_{q=0}^{n} q\left(\Delta_q, u_{m,q}\right) \end{aligned} \tag{6}$$

as the $(C,1,0)$ mean of the sequence $m\Delta_m u_{mn}$ and the $(C,0,1)$ mean of the sequence "$n\Delta_n u_{mn}$," respectively.

Further, as the sequence $V_{mn}^{(11)}(\Delta_{mn}u_{mn})$ is the $(C,1,1)$ mean of the sequence $mn(\Delta_{mn}u_{mn})$, the sequence $mn(\Delta_{mn}u_{mn})$ is $(C,1,1)$ summable to s whenever $V_{mn}^{(11)}(\Delta_{mn}u_{mn})$ converges to s as $m,n \to \infty$. For all nonnegative integers k and r, let us define $V_{mn}^{(kr)}(\Delta u)$ as follows:

$$V_{mn}^{(kr)}(\Delta u) = \begin{cases} \dfrac{1}{(m+1)(n+1)}\displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n} V_{pq}^{(k-1,r-1)}(\Delta u) & \text{for } k,r \geq 1, \\ mn\left(\Delta_{m,n}u_{mn}\right) & \text{for } k,r = 0. \end{cases} \tag{7}$$

The sequence $mn(\Delta_{mn}u_{mn})$ is said to be (C,k,r) summable to s if $V_{mn}^{(kr)}(\Delta_{mn}u_{mn})$ converges to s as $m,n \to \infty$. In particular, if $k = 1$ and $r = 1$, then (C,k,r) summability reduces to $(C,1,1)$ summability. Again, if $k \neq 0$ and $r = 0$, then (C,k,r) summability reduces to $(C,k,0)$ summability. Further, if $k = 0$ and $r \neq 0$, then (C,k,r) summability reduces to $(C,0,r)$ summability.

Then, the de la Vallée-Poussin mean of double real sequence (u_{mn}) is defined by

$$\tau_{mn}(u) = \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]} u_{ij}, \tag{8}$$

for sufficiently large nonnegative integers m,n and $\lambda > 1$, and

$$\tau_{mn}(u) = \frac{1}{(m-[\lambda m])(n-[\lambda n])}\sum_{i=\lambda m+1}^{m}\sum_{j=\lambda n+1}^{n} u_{ij}, \tag{9}$$

for sufficiently large nonnegative integers m,n and $0 < \lambda < 1$.

A single sequence $u = (u_n)$ is slowly oscillating [3] if

$$\lim_{\lambda\to 1^+}\limsup_{n}\max_{n\leq k\leq[\lambda n]}\left|u_k - u_n\right| = 0. \tag{10}$$

A double sequence $u = (u_{mn})$ is slowly oscillating [4] if

$$\lim_{\lambda\to 1^+}\limsup_{m,n}\max_{m+1,n+1\leq i,j\leq[\lambda m,\lambda n]}\left|\sum_{x=m+1}^{i}\sum_{y=n+1}^{j}\Delta_{x,y}u_{xy}\right| = 0. \tag{11}$$

In an earlier paper by Jena et. al. [5], a proof of the generalized Littlewood Tauberian theorem by Cesàro summability method has been established. For a proof of Littlewood Tauberian theorem differently, the paper of Çanak and Totur [6] and Çanak [7–9] can be referred to. Also, a similar result was introduced earlier by Çanak [10] under the consideration of improper integral. Recently, Totur [4] has introduced Littlewood Tauberian theorem by $(C,1,1)$ mean for double real sequence.

In the proposed paper, with certain novelty, we have generalized it for (C,k,r) summability of a double real sequence defined in (4).

2. Known Results

Theorem 1 (see [4]). *If the sequence (u_{mn}) is $(C,1,1)$ summable to s and (u_{mn}) is slowly oscillating (in the sense of $(1,1)$), then $\lim_{m,n\to\infty}u_{mn} = s$.*

Corollary 2 (see [5]). *If the sequence (u_n) is $(C,1)$ summable to s and (u_n) is slowly oscillating, then $\lim_{n\to\infty}u_n = s$.*

Theorem 3 (see [4]). *If the sequence $u = (u_{mn})$ is $(C,1,1)$ summable to s and $V_{mn}^{(11)}(\Delta_{mn}u_{mn})$ is slowly oscillating, then $\lim_{m,n\to\infty}u_{mn} = s$.*

Corollary 4 (see [5]). *If the sequence (u_n) is $(C,1)$ summable to s and $V_{mn}^{(1,0)}(\Delta u)$ is slowly oscillating, then $\lim_{n\to\infty}u_n = s$.*

3. Main Result

Theorem 5. *If (u_{mn}) is (C,k,r) summable to s and (u_{mn}) is slowly oscillating, then $\lim_{m,n\to\infty}u_{mn} = s$.*

To prove the above theorem, we need the help of the following lemmas.

Lemma 6. *A double sequence $u = (u_{mn})$ is slowly oscillating if and only if $V_{mn}^{(11)}$ is slowly oscillating and bounded.*

Proof. Let $u = (u_{mn})$ be slowly oscillating. Initially, let us show that $V_{mn}^{(11)} = O(1)$.

We have by definition of slow oscillation, for $\lambda > 1$,

$$\lim_{\lambda\to 1^+}\limsup_{m,n}\max_{m+1,n+1\leq i,j\leq[\lambda m,\lambda n]}\left|\sum_{x=m+1}^{i}\sum_{y=n+1}^{j}\Delta_{x,y}u_{xy}\right| = 0, \tag{12}$$

and let us rewrite the finite sum $\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta u_{ij}$ as the series $\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{m/2^{x+1},n/2^{y+1}\le i,j<m/2^{x},n/2^{y}} ij\Delta u_{ij}$.

Clearly,

$$\begin{aligned}\left|\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta u_{ij}\right| &\le \left|\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\sum_{m/2^{x+1},n/2^{y+1}\le i,j<m/2^{x},n/2^{y}} ij\Delta u_{ij}\right| \\ &\le \left(\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{mn}{2^{x+y}}\right) \\ &\quad\cdot \max_{m/2^{x+1}+1,n/2^{y+1}+1\le i,j\le[\lambda m/2^{i+1},\lambda n/2^{j+1}]}\left|\sum_{x=m/2^{x+1}+1}^{i}\sum_{y=n/2^{y+1}+1}^{j}\Delta_{x,y}u_{xy}\right| \\ &\le mnC\left(\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}\frac{1}{2^{x+y}}\right) = mnC^{*},\end{aligned} \tag{13}$$

where $C^{*} > 0$.

Consequently, we have

$$\begin{aligned}V_{mn}^{(11)}(\Delta u) &= \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n} pq\left(\Delta_{p,q}u_{pq}\right) \\ &= O(1), \quad \text{as } m,n\longrightarrow\infty.\end{aligned} \tag{14}$$

Since $\{\sigma_{mn}^{(11)}(u) = \sum_{p=1}^{m}\sum_{q=1}^{n}(V_{pq}^{(11)}/pq) + u_{00}\}$ is slowly oscillating, $V_{mn}^{(11)}$ is slowly oscillating.

To prove the converse part, consider $V_{mn}^{(11)}$ to be bounded and slowly oscillating. Now, the boundedness of $V_{mn}^{(11)}$ implies that $\sigma_{mn}^{(11)}(u)$ is slowly oscillating. Further, $V_{mn}^{(11)}$ is slowly oscillating, so, by Kronecker identity (5), (u_{mn}) is slowly oscillating.

This completes the proof of Lemma 6. □

Next, we represent the difference $(u_{mn} - \sigma_{mn}^{(11)}(u))$ under two different cases in the following lemma.

Lemma 7 (see [1]). *Let $u = (u_{mn})$ be a sequence of real numbers with m, n sufficiently large; then one has the following:*

(i) *For $\lambda > 1$,*

$$\begin{aligned}u_{mn} - \sigma_{mn}^{(11)}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left(\sigma_{[\lambda m],[\lambda n]}^{(11)}(u)\right. \\ &\quad \left.-\sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{mn}^{(11)}(u)\right) \\ &\quad + \frac{[\lambda m]+1}{[\lambda n]-m}\left(\sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,n}^{(11)}(u)\right) \\ &\quad + \frac{[\lambda n]+1}{[\lambda m]-m}\left(\sigma_{m,[\lambda n]}^{(11)}(u) - \sigma_{mn}^{(11)}(u)\right) \\ &\quad - \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}\left(u_{ij} - u_{mn}\right).\end{aligned} \tag{15}$$

(ii) *For $0 < \lambda < 1$,*

$$\begin{aligned}u_{mn} - \sigma_{mn}^{(1)}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])}\left(\sigma_{mn}^{(11)}(u)\right. \\ &\quad \left.-\sigma_{[\lambda m],n}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u) + \sigma_{[\lambda m],[\lambda n]}^{(11)}(u)\right) \\ &\quad + \frac{[\lambda m]+1}{m-[\lambda m]}\left(\sigma_{m,n}^{(11)}(u) - \sigma_{[\lambda m],n}^{(11)}(u)\right) \\ &\quad + \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_{mn}^{(11)}(u) - \sigma_{m,[\lambda n]}^{(11)}(u)\right) \\ &\quad - \frac{1}{(m-[\lambda m])(n-[\lambda n])} \\ &\quad \cdot \sum_{i=[\lambda m]+1}^{m}\sum_{j=[\lambda n]+1}^{n}\left(u_{mn} - u_{ij}\right).\end{aligned} \tag{16}$$

Proof of Theorem 5. Let $u = (u_{mn})$ be slowly oscillating; then, $\sigma_{mn}^{(k,r)}(u)$ is slowly oscillating (by Lemma 6). Further, since $u = (u_{mn})$ is (C,k,r) summable to s, by Theorem 1,

$$\lim_{m,n\to\infty}\sigma_{mn}^{(k,r)}(u) = s. \tag{17}$$

Next, from the definition,

$$\sigma_{mn}^{(k,r)}(u) = \sigma_{mn}^{(1,1)}(u)\left(\sigma_{mn}^{(k-1,r-1)}(u)\right). \tag{18}$$

Clearly, (17) and (18) imply that $u = (u_{mn})$ is $(C,k-1,r-1)$ summable to s. Again, $(\sigma_{mn}^{(k-1,r-1)}(u))$ is also slowly oscillating (by Lemma 6).

Thus, by Theorem 1, we have

$$\lim_{m,n\to\infty}\sigma_{mn}^{(k-1,r-1)}(u) = s. \tag{19}$$

Continuing in this way, we get $\lim_{m,n\to\infty}(u_{mn}) = s$. □

Remark 8. If $k = 0$ and $r \neq 0$, then (C,k,r) summability reduces to $(C,0,r)$ summability. Again, for $k \neq 0$ and $r = 0$, (C,k,r) summability reduces to $(C,k,0)$ summability and, consequently, the following corollary is generated from the main result.

Corollary 9 (see [5]). *If $u = (u_n)$ is $(C,0,r)$ or $(C,k,0)$ summable to s and $u = (u_n)$ is slowly oscillating, then $\lim_{n\to\infty}u_n = s$.*

Theorem 10. *If (u_{mn}) is (C,k,r) summable to s and $V_{mn}^{(11)}(\Delta u)$ is slowly oscillating, then $\lim_{m,n\to\infty}u_{mn} = s$.*

Proof. As $V_{mn}^{(11)}(\Delta u)$ is slowly oscillating, setting $u = (u_{mn})$ in place of $V_{mn}^{(11)}(\Delta u)$, $\sigma_{mn}^{(k,r)}(V_{mn}^{(11)}(\Delta u))$ is slowly oscillating by Lemma 6. Again, as $V_{mn}^{(11)}(\Delta u)$ is (C,k,r) summable to s, by Theorem 3, we have

$$\lim_{m,n\to\infty}\sigma_{mn}^{(k,r)}\left(V_{mn}^{(11)}(\Delta u)\right) = s. \tag{20}$$

By definition,

$$\begin{aligned}&\sigma_{mn}^{(k,r)}\left(V_{mn}^{(11)}(\Delta u)\right)\\&\quad=\sigma_{mn}^{(1,1)}\left(V_{mn}^{(11)}(\Delta u)\right)\left(\sigma_{mn}^{(k-1,r-1)}\left(V_{mn}^{(11)}(\Delta u)\right)\right).\end{aligned}\tag{21}$$

From (20) and (21), we have that $V_{mn}^{(11)}(\Delta u)$ is $(C,k-1,r-1)$ summable to s. Again, by Lemma 6, since $(\sigma_{mn}^{(k-1,r-1)}(V_{mn}^{(11)}(\Delta u)))$ is slowly oscillating, we have $\lim_{m,n\to\infty}\sigma_{mn}^{(k-1,r-1)}(V_{mn}^{(11)}(\Delta u)) = s$ (by Theorem 3). Continuing in this way, we get $\lim_{m,n\to\infty}(V_{mn}^{(11)}(\Delta u)) = s$. □

Remark 11. If $k = 0$ and $r \neq 0$, then (C,k,r) summability reduces to $(C,0,r)$ summability. Again, for $k \neq 0$ and $r = 0$, (C,k,r) summability reduces to $(C,k,0)$ summability and consequently the following corollaries are generated from the main result.

Corollary 12 (see [5]). *If $u = (u_n)$ is $(C,0,r)$ summable to s and $V_{mn}^{(1,0)}(\Delta u)$ is slowly oscillating, then* $\lim_{n\to\infty}u_n = s$.

Corollary 13 (see [5]). *If $u = (u_n)$ is $(C,k,0)$ summable to s and $V_{mn}^{(0,1)}(\Delta u)$ is slowly oscillating, then* $\lim_{n\to\infty}u_n = s$.

Competing Interests

The authors declare that there are no competing interests regarding publication of this paper.

References

[1] F. Móricz, "Tauberian theorems for Cesàro summable double sequences," *Studia Mathematica*, vol. 110, no. 1, pp. 83–96, 1994.

[2] K. Knopp, "Limitierungs—Umkehrsätze für Doppelfolgen," *Mathematische Zeitschrift*, vol. 45, pp. 573–589, 1939.

[3] C. V. Stanojević, "Analysis of divergence control and management of divergent process," in *Graduate Research Seminar Lecture Notes*, İ. Çanak, Ed., University of Missouri-Rolla, Fall, 1998.

[4] Ü. Totur, "Classical tauberian theorems for the (C; 1; 1) summability method," *Annals of theAlexandru Ioan Cuza University—Mathematics*, 2014.

[5] B. B. Jena, S. K. Paikray, and U. K. Misra, "A proof of Tauberian theorem for Cesàro summability method," *Asian Journal of Mathematics and Computer Research*, vol. 8, pp. 272–276, 2016.

[6] I. Çanak and Ü. Totur, "Some tauberian conditions for cesaro summability method," *Mathematica Slovaca*, vol. 62, no. 2, pp. 271–280, 2012.

[7] İ. Çanak, "A theorem on the Cesàro summability method," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1162–1166, 2011.

[8] I. Çanak, "On means (C; 1) of sequences," *Computers & Mathematics with Applications*, vol. 62, no. 9, pp. 3446–3448, 2011.

[9] İ. Çanak, "An extended Tauberian theorem for the (C,1) summability method," *Applied Mathematics Letters*, vol. 21, no. 1, pp. 74–80, 2008.

[10] İ. Çanak and Ü. Totur, "A Tauberian theorem for Cesàro summability of integrals," *Applied Mathematics Letters*, vol. 24, no. 3, pp. 391–395, 2011.