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Research Article

Homotopic Chain Maps Have Equal s-Homology and d-Homology

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The homotopy of chain maps on preabelian categories is investigated and the equality of standard homologies and d-homologies of homotopic chain maps is established. As a special case, if X and Y are the same homotopy type, then their nth d-homology R-modules are isomorphic, and if X is a contractible space, then its nth d-homology R-modules for $n \neq 0$ are trivial.

1. Introduction and Preliminaries

It is known that the homotopic chain maps of abelian groups or more generally of R-modules have the same homologies; see [1, 2]. In this paper, the homotopy of chain maps on preabelian categories is investigated and it is proved that homotopic chain maps have the same s-homologies and the same d-homologies.

To this end, for a pointed category $\mathscr{C}$, following the notation of [3], we recall the following.

(i) For $f : A \to B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$, and $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$ are, respectively, the kernel, the cokernel, and the kernel pair of f; see [4, 5].

(ii) The image I_f of f is the coequalizer of the kernel pair of f. In a homological category, $I_f \cong \mathrm{Cker}(k_f)$; see [4].

(iii) For a pair of maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$, the maps $\mathrm{Equ}(f, g) \xrightarrow{\mathrm{equ}(f,g)} A$ and $B \xrightarrow{\mathrm{coe}(f,g)} \mathrm{Coe}(f, g)$ are, respectively, the equalizer and the coequalizer of (f, g).

(iv) Given the diagram below in which the squares are commutative and the rows are coequalizers, i is the unique map making the right square commute. Furthermore, i is a regular epi.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "r"'] & B \arrow[r, "q"] \arrow[d, "1_B"] & C \arrow[d, dotted, "i"] \\ A' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & B \arrow[r, "q'"'] & C' \end{tikzcd} \tag{1}$$

(v) For a category $\mathscr{C}$ with a zero object, kernels, kernel pairs, and coequalizers of kernel pairs, the arrow category $\overline{\mathscr{C}}$ of $\mathscr{C}$ has as objects the morphisms of $\mathscr{C}$ and as morphisms from $f : A \to B$ to $f' : A' \to B'$ the pairs (α, β) of morphisms of $\mathscr{C}$, making the following square commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \tag{2}$$

And the pair-chain category $\widehat{\mathscr{C}}$ of $\mathscr{C}$ has as objects the pair-chains, that is, the composable pairs, (f, g), of morphisms of $\mathscr{C}$, such that $gf = 0$, and as morphisms from (f, g) to (f', g') the triple (α, β, γ) of morphisms of $\mathscr{C}$, making the following squares commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd} \tag{3}$$

Being functor of the following items are investigated and established in [3].

(vi) The kernel functor $K : \overline{\mathscr{C}} \to \mathscr{C}$ takes $(\alpha, \beta) : f \to f'$ to the left vertical arrow in the following commutative diagram:

$$\begin{tikzcd} K_f \arrow[r, "k_f"] \arrow[d, "K(\alpha,\beta)"'] & A \arrow[d, "\alpha"] \\ K_{f'} \arrow[r, "k_{f'}"'] & A' \end{tikzcd} \tag{4}$$

(vii) The image functor $I : \overline{\mathscr{C}} \to \mathscr{C}$ takes $(\alpha, \beta) : f \to f'$ to the left vertical arrow in the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "e_f"] \arrow[d, "\alpha"'] & I_f \arrow[d, "I(\alpha,\beta)"] \\ A' \arrow[r, "e_{f'}"'] & I_{f'} \end{tikzcd} \tag{5}$$

(viii) The functor $j : \widehat{\mathscr{C}} \to \overline{\mathscr{C}}$ takes the object $(f, g) \in \widehat{\mathscr{C}}$ to j_{fg} and the morphism (α, β, γ) to $(I(\alpha, \beta), K(\beta, \gamma))$, and we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "e_f"] \arrow[rr, bend left, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "///"] & I_f \arrow[r, "j_{fg}"] \arrow[d, "I(\alpha,\beta)"'] \arrow[dr, phantom, "///"] & K_g \arrow[r, tail, "k_g"] \arrow[d, "K(\beta,\gamma)"'] \arrow[dr, phantom, "///"] & B \arrow[d, "\beta"] \\ A' \arrow[r, shift left, "e_{f'}"'] \arrow[rr, bend right, "f'"'] & I_{f'} \arrow[r, "j_{f'g'}"'] & K_{g'} \arrow[r, tail, "k_{g'}"'] & B' \end{tikzcd} \tag{6}$$

In a pointed regular (homological, semiabelian, or abelian) category, j_{fg} is monic; see [3, 4].

(ix) With $S : \mathscr{C} \to \mathscr{C}$, the squaring functor, taking $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$, a kernel transformation is a natural transformation $d : S \circ K \to K : \overline{\mathscr{C}} \to \mathscr{C}$ such that for all (f, g) in $\widehat{\mathscr{C}}$, the pullback, $j^*_{fg} : R_{fg} \to K^2_g$, of j_{fg} along d_g and the coequalizer of the pair $j_1 = \mathrm{pr}_1 j^*_{fg}$ and $j_2 = \mathrm{pr}_2 j^*_{fg}$ exist, where pr_1 and pr_2 are the projection maps.

(x) The d-homology functor $H^d : \widehat{\mathscr{C}} \to \mathscr{C}$ takes $(f, g) \in \widehat{\mathscr{C}}$ to $H^d_{fg} = \mathrm{Coe}(j_1, j_2)$ and the morphism (α, β, γ) to $H^d(\alpha, \beta, \gamma)$. We have the following commutative diagram:

$$\begin{tikzcd} K_g \arrow[r, "q_1"] \arrow[d, "K(\beta,\gamma)"'] & H^d_{fg} \arrow[d, "H^d(\alpha,\beta,\gamma)"] \\ K_{g'} \arrow[r, "q'_1"'] & H^d_{f'g'} \end{tikzcd} \tag{7}$$

(xi) Let R be a commutative ring with unity. Any kernel transformation in R mod is of the form $d = +(r \times s)$, for some $r, s \in R$. In particular, any kernel transformation d in *Abgrp* is of the form $d = +(r \times s)$, for some $r, s \in \mathbb{Z}$.

(xii) Let $\mathscr{C} = R\mathrm{mod}$ and $d = r\mathrm{pr}_1 + s\mathrm{pr}_2 = +(r \times s)$ with $r, s \in R$. Let (f, g) be a pair-chain. Then, $R_{fg} = \{(a, b) \in K^2_g \mid ra + sb \in I_f\}$, j^* is the inclusion, and $H_{fg} = K_g/(j_1 - j_2)(R_{fg}) = \{[a] : a \in K_g\}$, where $[a] = \{b \mid r(a-b) \in (r+s)K_g + I_f\} = \{b \mid s(a-b) \in (r+s)K_g + I_f\}$ is the equivalence class under the equivalence relation $a \sim b$ if and only if $\exists m, n \in K_g$ such that $a - b = m - n$ and $rm + sn \in I_f$.

(xiii) As a special case of above example, for $d = +(r \times 1)$ or $d = +(1 \times r)$ with $r \in R$, we have $H^d_{fg} = K_g/((1+r)K_g + I_f)$.

We call the homology which is defined in [2, 4] the standard homology.

(xiv) The standard homology or s-homology functor H^s takes $(f, g) \in \widehat{\mathscr{C}}$ to $\mathrm{Coker}(j_{fg})$, and for a pair-chain map $(\alpha, \beta, \gamma) : (f, g) \to (f', g')$, we have the following commutative diagram:

$$\begin{tikzcd} K_g \arrow[r, "q"] \arrow[d, "k(\beta,\gamma)"'] & H^s_{fg} \arrow[d, "H^s(\alpha,\beta,\gamma)"] \\ K_{g'} \arrow[r, "q'"'] & H^s_{f'g'} \end{tikzcd} \tag{8}$$

where $q = \mathrm{coker}(j_{fg})$ and $q' = \mathrm{coker}(j_{f'g'})$.

2. Homotopy

Definition 1. Let $\mathscr{C}$ be an additive category. Two morphisms

$$(f, g) \underset{(\alpha', \beta', \gamma')}{\overset{(\alpha, \beta, \gamma)}{\rightrightarrows}} (f', g') \tag{9}$$

in $\widehat{\mathscr{C}}$ are said to be homotopic whenever there is a pair of morphisms (h, h') in $\mathscr{C}$, as in the diagram below, such that $f'h + h'g = \beta - \beta'$; see [2],

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\alpha'"] & B \arrow[r, "g"] \arrow[d, shift right, "\beta"'] \arrow[d, shift left, "\beta'"] & C \arrow[d, shift right, "\gamma"'] \arrow[d, shift left, "\gamma'"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] \arrow[l, "h"'] & C' \arrow[l, "h'"']
\end{tikzcd} \tag{10}$$

Theorem 2. *Let $\mathscr{C}$ be a preabelian category. If the maps (α, β, γ) and $(\alpha', \beta', \gamma')$ in $\widehat{\mathscr{C}}$ are homotopic, then $H^s(\alpha, \beta, \gamma) = H^s(\alpha', \beta', \gamma')$.*

Proof. Consider

$$\begin{aligned}
\left(k_{g'} j_{f'g'} e_{f'} h + h' g\right) k_g &= \left(\beta - \beta'\right) k_g, \\
k_{g'} j_{f'g'} e_{f'} h k_g &= k_{g'} \left(k(\beta, \gamma) - k\left(\beta', \gamma'\right)\right) \\
j_{f'g'} e_{f'} h k_g &= k(\beta, \gamma) - k\left(\beta', \gamma'\right)
\end{aligned} \tag{11}$$

since $q' j_{f'g'} = 0$ we have $q'(k(\beta, \gamma) - k(\beta', \gamma')) = 0$.

$(H^s(\alpha, \beta, \gamma) - H^s(\alpha', \beta', \gamma'))q = 0$, $H^s(\alpha, \beta, \gamma) - H^s(\alpha', \beta', \gamma') = 0$ since q is a regular epi. Therefore, $H^s(\alpha, \beta, \gamma) = H^s(\alpha', \beta', \gamma')$. □

Lemma 3. *Let $\mathscr{C}$ be a pointed category with pullbacks and pushouts and let d be a kernel transformation in $\mathscr{C}$. There is a natural transformation $p : H^s \to H^d : \widehat{\mathscr{C}} \to \mathscr{C}$. Furthermore, p is pointwise regular epic.*

Proof. Let $d : S \circ K \to K$ be a natural transformation. For $(f, g) \in \widehat{\mathscr{C}}$ and $h : I_f \to 0$, $I_f = K_h$. Factoring $f = m_f e_f$ and using the morphism $(m_f, 0) : h \to g$, naturality of d yields, $jd_h = d_g j^2$, and so $d_g \langle j, 0\rangle = d_g j^2 \langle 1, 0\rangle = jd_h \langle 1, 0\rangle$. Setting $r = d_h \langle 1, 0\rangle$, we get $d_g \langle j, 0\rangle = jr$. So, there exists a unique map ψ, such that the triangles in the following diagram commute:

$$\begin{tikzcd}
I_f \arrow[dr, "\psi"] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "<j,0>"'] & & \\
& R_{fg} \arrow[r, "d_g^*"] \arrow[d, "j^*"'] \arrow[dr, phantom, "\text{pb}"] & I_f \arrow[d, "j"] \\
& K_g^2 \arrow[r, "d_g"'] & K_g
\end{tikzcd} \tag{12}$$

Therefore, $j = \mathrm{pr}_1 \langle j, 0\rangle = \mathrm{pr}_1 j^* \psi = j_1 \psi$ and $0 = \mathrm{pr}_2 \langle j, 0\rangle = \mathrm{pr}_2 j^* \psi = j_2 \psi$ so that $qj = q\mathrm{pr}_1 j^* \psi = q\mathrm{pr}_2 j^* \psi = q0 = 0$. Then, there is a unique morphism $p_{fg} : H^s_{fg} \to H^d_{fg}$ such that $p_{fg} c_j = q$; that is, the following diagram commutes:

$$\begin{tikzcd}
I_f \arrow[r, shift left, "j"] \arrow[r, shift right, "0"'] \arrow[d, "\psi"'] & K_g \arrow[r, "c_j"] \arrow[d, "1_{K_g}"] & H^s_{fg} \arrow[d, dotted, "p_{fg}"] \\
R_{fg} \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & K_g \arrow[r, "q"'] & H^d_{fg}
\end{tikzcd} \tag{13}$$

Furthermore, by a mentioned point in preliminaries, p_{fg} is regular epic.

To show naturality, given $(\sigma, \delta, \zeta) : (f, g) \to (f', g')$ in $\widehat{\mathscr{C}}$, consider the following diagram that, by the above diagram the triangles commute, with the functors H^s and H^d implying the square and the left parallelogram are commutative:

$$\begin{tikzcd}
K_g \arrow[rr, "q_{fg}"] \arrow[dr, "c_j"] \arrow[dd, "K(\delta,\zeta)"'] & & H^d \arrow[dd, "H^d(\sigma,\delta,\zeta)"] \\
& H^s_{fg} \arrow[ur, "p_{fg}"] \arrow[dd] & \\
K_{g'} \arrow[rr, "q_{f'g'}", "H^s(\sigma,\delta,\zeta)"'] \arrow[dr, "c_{j'}"] & & H^d_{f'g'} \\
& H^s_{f'g'} \arrow[ur, "p_{f'g'}"'] &
\end{tikzcd} \tag{14}$$

We have $p_{f'g'} H^s(\sigma, \delta, \zeta) c_j = p_{f'g'} c_{j'} K(\delta, \zeta) = q_{f'g'} K(\delta, \zeta) = H^d(\sigma, \delta, \zeta) q_{fg} = H^d(\sigma, \delta, \zeta) p_{fg} c_j$. Since c_j is epic, $p_{f'g'} H^s(\sigma, \delta, \zeta) = H^d(\sigma, \delta, \zeta) p_{fg}$ and the right parallelogram commutes. Then, the following diagram commutes:

$$\begin{tikzcd}
H^s_{fg} \arrow[r, "p_{fg}"] \arrow[d, "H^s(\sigma,\delta,\zeta)"'] & H^d_{fg} \arrow[d, "H^d(\sigma,\delta,\zeta)"] \\
H^s_{f'g'} \arrow[r, "p_{f'g'}"'] & H^d_{f'g'}
\end{tikzcd} \tag{15}$$

and so $p : H^s \to H^d : \widehat{\mathscr{C}} \to \mathscr{C}$ is a natural transformation. □

Theorem 4. *Let $\mathscr{C}$ be a preabelian category. If (α, β, γ) and $(\alpha', \beta', \gamma')$ are homotopic, then $H^d(\alpha, \beta, \gamma) = H^d(\alpha', \beta', \gamma')$.*

Proof. The equalities $p_{fg}c_j = q$, $p_{f'g'}c_{j'} = q'$, $H^d(\alpha,\beta,\gamma)q = q'K(\alpha,\beta)$, and $H^d(\alpha',\beta',\gamma')q = q'K(\alpha',\beta')$ and the facts that p is regular epic and standard homologies are equal imply $H^d(\alpha,\beta,\gamma) = H^d(\alpha',\beta',\gamma')$ as desired. □

Definition 5. Let $\mathscr{C}$ be a pointed category. A chain complex in $\mathscr{C}$ is a differential graded object of $\mathscr{C}$ of degree −1 as

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\nu_{n+1}} C_n \xrightarrow{\nu_n} C_{n-1} \longrightarrow \cdots \tag{16}$$

in which $\nu_n\nu_{n+1} = 0$ for all $n \in Z$ and a chain map $f_* : C_* \to D_*$ is a graded map $\{f_n : C_n \to D_n : n \in Z\}$ as in the following diagram, in which all the squares commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\nu_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\nu_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\nu_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\mu_{n+1}"'] & D_n \arrow[r, "\mu_n"'] & D_{n-1} \arrow[r, "\mu_{n-1}"'] & \cdots
\end{tikzcd} \tag{17}$$

These chain complexes and chain maps form a category that is denoted by $C_*(\mathscr{C})$.

Definition 6. Let $\mathscr{C}$ be an additive category and $C_* \underset{g_*}{\overset{f_*}{\rightrightarrows}} D_*$ two maps in $C_*(\mathscr{C})$; one says f_* is chain homotopic to g_* if there is a morphism $h_* : C_* \to D_*$ of degree 1 as

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\nu_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\nu_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\nu_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\mu_{n+1}"'] & D_n \arrow[r, "\mu_n"'] & D_{n-1} \arrow[r, "\mu_{n-1}"'] & \cdots
\end{tikzcd} \tag{18}$$

such that $\mu_*h_* + h_*\nu_* = f_* - g_*$.

Corollary 7. *Let $\mathscr{C}$ be a preabelian category and let f_* and g_* be homotopic in $C_*(\mathscr{C})$. Then, $H^s(f_*) = H^s(g_*)$ and $H^d(f_*) = H^d(g_*)$.*

Proof. Suppose $h_* : C_* \to D_*$ is a homotopy of the pair $C_* \underset{g_*}{\overset{f_*}{\rightrightarrows}} D_*$; that is, there is the following diagram in which for all n we have $\mu_{n+1}h_n + h_{n-1}\nu_n = f_n - g_n$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\nu_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\nu_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\nu_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\mu_{n+1}"'] & D_n \arrow[r, "\mu_n"'] & D_{n-1} \arrow[r, "\mu_{n-1}"'] & \cdots
\end{tikzcd} \tag{19}$$

By the above theorems, $H_n^s(f_*) = H_n^s(g_*)$ and $H_n^d(f_*) = H_n^d(g_*)$, where $H_n(f_*) = H(f_{n+1}, f_n, f_{n-1})$.

Let Top, $\mathrm{Set}^{\Delta^{op}}$, $R\mathrm{mod}^{\Delta^{op}}$, $C_*R\mathrm{mod}$, and $gR\mathrm{mod}$ be, respectively, categories of topological spaces, simplicial sets, simplicial R-modules, chain complexes of R-modules, and graded R-modules, and let $S_{\Delta_*} : \mathrm{Top} \to \mathrm{Set}^{\Delta^{op}}$, $F^{\Delta^{op}} : \mathrm{Set}^{\Delta^{op}} \to R\mathrm{mod}^{\Delta^{op}}$, $C_* : R\mathrm{mod}^{\Delta^{op}} \to C_*R\mathrm{mod}$, and $H_*^d : C_*R\mathrm{mod} \to gR\mathrm{mod}$ be, respectively, singular functor, induced functor by free generator functor $F : \mathrm{Set} \to R\mathrm{mod}$, chain complexes generator functor, and d-homology functor. If $X \in \mathrm{Top}$, then the singular d-homology of X is $H_*^d(X) = H_*^d o C_* o F^{\Delta^{op}} o S_{\Delta_*}(X)$. R is PID. □

Example 8. If $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is homotopic continuous maps, then $H_*^d(f) = H_*^d(g) : H_*^d(X) \to H_*^d(Y)$.

Example 9. Let 1 be a terminal object in Top. Since $H_n^s(1) = 0$ for $n \neq 0$ and the natural transformation $p : H^s \to H^d : \widehat{\mathscr{C}} \to \mathscr{C}$ is pointwise regular epic, $H_n^d(1) = 0$ for $n \neq 0$ and $H_0^d(1) = R/(j_1 - j_2)(R_{00})$ in which $R_{00} = \mathrm{Ker}(d_0)$.

Let $\mathrm{Sim}Cx$ be simplicial complexes and let $S_* : \mathrm{Sim}Cx \to \mathrm{Set}^{\Delta^{op}}$ be the simplicial functor. If $(K,S) \in \mathrm{Sim}Cx$, then the simplicial d-homology of (K,S) is $H_*^d(X) = H_*^d o C_* o F^{\Delta^{op}} o S_*(K,S)$.

Example 10. Let $(1, 1)$ be a terminal object in $\mathrm{Sim}Cx$. Then, $H_n^d(1, 1) = 0$ for $n \neq 0$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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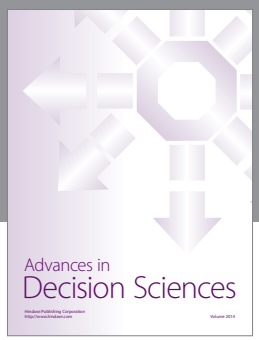

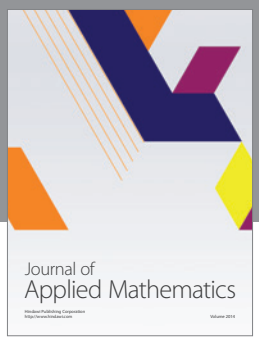

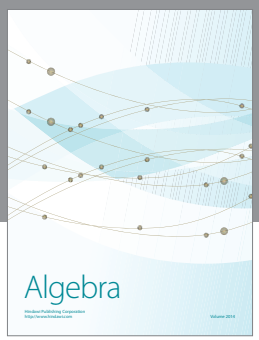

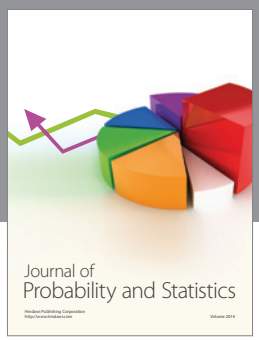

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