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Research Article

On Shift-Dependent Cumulative Entropy Measures

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Measures of cumulative residual entropy (CRE) and cumulative entropy (CE) about predictability of failure time of a system have been introduced in the studies of reliability and life testing. In this paper, cumulative distribution and survival function are used to develop weighted forms of CRE and CE. These new measures are denominated as weighted cumulative residual entropy (WCRE) and weighted cumulative entropy (WCE) and the connections of these new measures with hazard and reversed hazard rates are assessed. These information-theoretic uncertainty measures are shift-dependent and various properties of these measures are studied, including their connections with CRE, CE, mean residual lifetime, and mean inactivity time. The notions of weighted mean residual lifetime (WMRL) and weighted mean inactivity time (WMIT) are defined. The connections of weighted cumulative uncertainties with WMRL and WMIT are used to calculate the cumulative entropies of some well-known distributions. The joint versions of WCE and WCRE are defined which have the additive properties similar to those of Shannon entropy for two independent random lifetimes. The upper boundaries of newly introduced measures and the effect of linear transformations on them are considered. Finally, empirical WCRE and WCE are proposed by virtue of sample mean, sample variance, and order statistics to estimate the new measures of uncertainty. The consistency of these estimators is studied under specific choices of distributions.

1. Introduction

The concept of entropy was originally introduced in Shannon [1] in the context of communication theory. Since then, it has been of great theoretical and applied interest. Shannon characterized the properties of information sources and of communication channels to analyze the outputs of these sources. Statisticians have played a crucial role in the development of information theory and have shown that it provides a framework for dealing with a wide variety of problems in reliability.

Let X be a nonnegative absolutely continuous random variable describing a component failure time. The probability density function of X is denoted as $f(x)$, the failure distribution is denoted as $F(x) = P(X \leq x)$, and the survival function is denoted as $\overline{F}(x) = 1 - F(x)$. The Shannon entropy of X, which has been shown by $H(X)$ in the literature of communication, is defined as

$$H(X) = -\int_0^\infty f(x) \log f(x)\, dx, \tag{1}$$

where log denotes the natural logarithm. Entropy (1) is not scale invariant because $H(cX) = \log|c| + H(X)$, but it is translation invariant, so that $H(c+X) = H(X)$ for some constant c. The latter property can be interpreted as the shift independence of Shannon information.

Let X be random lifetime of a system with support set $(0, \infty)$; the Shannon entropy can be rewritten as

$$H(X) = 1 - E\left[\log(r(X))\right] = 1 - E\left[\log(\tau(X))\right]. \tag{2}$$

Recall that hazard rate (HR) and reversed hazard rate (RHR) of random lifetime X are defined as $r(t) = f(t)/(\overline{F}(t))$ and $\tau(t) = f(t)/F(t)$, respectively. The HR and RHR have been used in the literature of reliability in both theory and applications of them.

The notion of cumulative residual entropy (CRE) as an alternative measure of uncertainty was introduced in Wang et al. [2]. This measure is based on survival function and is defined as follows:

$$\mathscr{E}(X) = -\int_0^\infty \overline{F}(x) \log \overline{F}(x)\, dx = E\left(m_F(X)\right), \tag{3}$$

where

$$m_F(t) = E(X - t \mid X \geq t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x)\, dx \tag{4}$$

is the mean residual life (MRL) of X for $t \geq 0$. CRE has been applied to reliability engineering and computer vision in Rao et al. [3] and Wang et al. [2].

The role of CRE in residual lifetimes was considered in Asadi and Zohrevand [4]. The dynamic cumulative residual entropy (DCRE) of lifetime X at time $t \geq 0$ is defined by

$$\begin{aligned} \mathscr{E}(X;t) &= -\int_t^{\infty} \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{F}(x)}{\overline{F}(t)} dx \\ &= -\frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x) \log \overline{F}(x)\, dx \\ &\quad + m_F(t) \log \overline{F}(t). \end{aligned} \tag{5}$$

Entropy (5) is, in fact, the CRE for residual lifetime distribution of X at time $t > 0$.

Recently, the cumulative entropy (CE) has been proposed in Di Crescenzo and Longobardi [5] with properties similar to those of CRE. Formally, the cumulative entropy of a nonnegative random lifetime X is defined as

$$\mathscr{CE}(X) = -\int_0^{\infty} F(x) \log F(x)\, dx = E(\mu_F(X)), \tag{6}$$

where

$$\mu_F(t) = E(t - X \mid X \leq t) = \frac{1}{F(t)} \int_0^t F(x)\, dx \tag{7}$$

is the mean inactivity time (MIT) of X for $t \geq 0$. Entropy (6) measures the uncertainty about the inactivity time of X, which is the time elapsing between the failure time of a system and the time when it is found to be down. In other words, $\mathscr{CE}(X)$ is a suitable measure of information when the uncertainty is related to the past.

Furthermore, Di Crescenzo and Longobardi [5] introduced the dynamic cumulative past entropy (DCPE) in past lifetimes. DCPE of lifetime X at time $t \geq 0$ is defined by

$$\begin{aligned} \mathscr{CE}(X;t) &= -\int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx \\ &= -\frac{1}{F(t)} \int_0^t F(x) \log F(x)\, dx \\ &\quad + \mu_F(t) \log F(t). \end{aligned} \tag{8}$$

Entropy (8) measures the uncertainty about a system which is observed only at deterministic inspection times and is found to be down at time t; then the uncertainty relies on which instant in $(0, t)$ it has failed.

This paper is aimed at defining and assessing the weighted forms of CRE and CE. In Section 2, some properties of newly introduced measures are discussed, including the connections with reliability notions and determined various bounds. Section 3 is devoted to estimation of proposed measures by means of empirical distribution function and order statistics. Some conclusions are given in Section 4.

Throughout the remaining of this paper, all random variables are assumed as absolutely continuous.

2. Weighted Cumulative Measures of Information

In this section, two new measures of uncertainty are presented in nonnegative random variables and then some properties are discussed about these new measures.

In some practical situations of reliability and neurobiology, a shift-dependent measure of uncertainty is desirable. The notion of weighted entropy addresses this requirement. An important feature of the human visual system is that it can recognize objects in a scale- and transformation-invariant manner. To intercept or avoid moving objects successfully, a visual system must compensate for the sensorimotor delays associated with visual processing and motor movement. In spite of straightforwardness in the case of constant velocity motion, it is unclear how humans compensate for accelerations, as our visual system is relatively poor at detecting changes in velocity (see Wallis [6] and de Rugy et al. [7]). Neurophysiological evidence shows that some neurons in the macaque temporal cortical visual areas have responses which are invariant with respect to the position, size, and view of faces and objects and that these neurons show rapid processing and rapid learning. Wallis and Rolls [8] propose that neurons in these visual areas use a modified rule with a short-term memory trace to capture whatever can be captured at each stage which is invariant about objects as the object changes in retinal position, size, rotation, and view. Transformation-invariant measures have been attracted by researchers from finance and industry. In robotics and machinery analysis, line and screw systems are singular at particular geometric configurations. Otherwise, measures that describe how far they are from being so are required. Hartley and Kerr [9] proposed a new measure whose outcome is strictly invariant with respect to coordinate frame, origin, and unit of length. Kerr and Hartley [10] describe a general analytical method for determining the proximity to linear dependence of any system of lines and screws. Their method gives invariant scalars for n-system of screws. The robustness of optimal portfolio with respect to the choice of risk measure has been investigated in Adam et al. [11]. Argenti et al. [12] studied filtering of generalized signal-dependent noise which is performed and estimated in shift-invariant wavelet domains. They address the scheme which filtered pixel values obtaining as adaptive combinations of raw and local average values, driven by locally computed statistics. Ghosh et al. [13] analyze experimental data in order to characterize strange attractors in terms of invariant measures such as correlation, embedding, Lyapunov dimensions, and entropy. Misagh and Yari [14] studied some theoretic uncertainty measures which are shift-dependent. They introduced the weighted differential information measure for two-sided truncated random variables which is generalization of dynamic entropy measures.

In analogy with (3) and (6), Misagh et al. [15] defined the notions of weighted cumulative residual entropy (WCRE) and weighted cumulative entropy (WCE). The measures WCRE and WCE are defined for nonnegative random lifetime X as

$$\mathcal{E}^{\omega}(X) = -\int_0^{\infty} x\overline{F}(x)\log\overline{F}(x)\,dx, \tag{9}$$

$$\mathcal{CE}^{\omega}(X) = -\int_0^{\infty} xF(x)\log F(x)\,dx, \tag{10}$$

respectively. The designation of (9) and (10) as weighted entropies arises from coefficient x which emphasizes the importance of the occurrence of events $\{X > x\}$ and $\{X \le x\}$, respectively.

The definitions given in (9) and (10) are suitable modifications of the notions of weighted entropy functions introduced in Di Crescenzo and Longobardi [16]. Misagh et al. [15] studied various properties of these measures, including their connections with CRE and CE. They showed that, in some cases, there is a direct relation between variance and WCE. In such cases WCE may be used instead of variance. In addition, some extensions of weighted cumulative entropy are presented in Suhov and Yasaei Sekeh [17]. Furthermore, they defined the notion of weighted Kullback-Leibler divergence between two random lifetimes.

Remark 1. Due to (9) and (10), it can be shown that $0 \le \mathcal{E}^{\omega}(X)$ and $\mathcal{CE}^{\omega}(X) \le \infty$, with $\mathcal{E}^{\omega}(X) = 0$ or $\mathcal{CE}^{\omega}(X) = 0$, if and only if X follows a degenerate distribution.

Example 2. Suppose X and Y be random lifetimes of two systems with common support $(0,\infty)$ and density functions $f(x) = (1/3)\exp(-x/3)$ and $f(y) = (4\times 3^4)/(3+4x)^4$, respectively. From (3), $\mathcal{E}(X) = \mathcal{E}(Y) = 3$. Therefore, the expected cumulative residual uncertainties in the predictability of the residual lifetimes of X and Y are identical. By simple calculations, $\mathcal{E}^{\omega}(X) = 0.097$ and $\mathcal{E}^{\omega}(Y) = 0.73$. Hence, even though $\mathcal{E}(X) = \mathcal{E}(Y)$, the expected weighted cumulative residual uncertainty of the predictability of the failure time of component Y is larger than that of X.

The forthcoming proposition is analogous to (3) and (6); the proof is given in Misagh et al. [15] and here it is omitted. First definitions of weighted mean residual lifetime (WMRL) and weighted mean inactivity time (WMIT) are given.

Definition 3. The WMRL and WMIT of a nonnegative random variable X are given by

$$m_F^*(t) = \frac{1}{\overline{F}(t)}\int_t^{\infty} x\overline{F}(x)\,dx, \tag{11}$$

$$\mu_F^*(t) = \frac{1}{F(t)}\int_0^{t} xF(x)\,dx, \tag{12}$$

respectively.

Proposition 4. *Let X be a nonnegative random variable with WMRL $m_F^*(t)$ and WMIT $\mu_F^*(t)$. Then*

(a) $\mathcal{E}^{\omega}(X) = E(m_F^*(X))$,

(b) $\mathcal{CE}^{\omega}(X) = E(\mu_F^*(X))$.

Example 5. (i) If X is distributed as exponential with mean λ, then $m_F^*(t) = \lambda t + \lambda^2$. Hence, $\mathcal{E}^{\omega}(X) = 2\lambda^2$.

(ii) If X has power distribution with density function,

$$f(x) = \begin{cases} \left(\dfrac{\beta}{\alpha}\right)\cdot\left(\dfrac{x}{\alpha}\right)^{\beta-1}, & 0 \le x \le \alpha,\ \alpha > 0,\ \beta > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

then $\mu_F^*(t) = t^2/(\beta+2)$ and $\mathcal{CE}^{\omega}(X) = (1/(\beta+2))E(X^2) = \beta(\alpha/(\beta+2))^2$.

(iii) If X is distributed uniformly on $(0,a)$, $a > 0$, then $m_F^*(t) = (1/2)a(a+t) - (1/3)(a^2 + at + t^2)$ and $\mu_F^*(t) = (1/3)t^3$. From Proposition 4, $\mathcal{E}^{\omega}(X) = (1/4)a^2 - (1/9)a^3$ and $\mathcal{CE}^{\omega}(X) = (1/9)a^2$.

WCRE is based on survival function and then a close relationship between it and mean residual life is expected. The same can be argued about WCE and MIT.

Proposition 6. *For nonnegative random variable X, there holds*

(a) $\mathcal{E}^{\omega}(X) = \int_0^{\infty}\overline{F}(t)(\mathcal{E}(X;t) - m_F(t)\log\overline{F}(t))dt$,

(b) $\mathcal{CE}^{\omega}(X) = \int_0^{\infty} F(t)(\mathcal{CE}(X;t) - \mu_F(t)\log F(t))dt$.

Proof. Part (a) is proven in Misagh et al. [15]. The second part is proven in a similar way. □

For independent random variables X and Y, $H(X,Y) = H(X) + H(Y)$, where $H(X,Y) = -E(\log f(X,Y))$ is the two-dimensional Shannon entropy. In the following proposition, similar properties are presented for weighted cumulative entropies.

Proposition 7. *Let X and Y be two nonnegative independent random variables with finite WMRL and WMIT. Then*

(a) $\mathcal{E}^{\omega}(X,Y) = (\int_0^{\infty} x\overline{F}_X(x)dx)\mathcal{E}^{\omega}(Y) + (\int_0^{\infty} y\overline{F}_Y(y)dy)\mathcal{E}^{\omega}(X)$,

(b) $\mathcal{CE}^{\omega}(X,Y) = (\int_0^{\infty} xF_X(x)dx)\mathcal{CE}^{\omega}(Y) + (\int_0^{\infty} yF_Y(y)dy)\mathcal{CE}^{\omega}(X)$.

Proof. The proof is straightforward. For part (a),

$$\begin{aligned} \mathcal{E}^{\omega}(X,Y) &= -\iint_0^{\infty} xy\overline{F}(x,y)\log\overline{F}(x,y)\,dx\,dy \\ &= -\iint_0^{\infty} xy\overline{F}_X(x)\overline{F}_Y(y)\log\overline{F}_X(x)\,dx\,dy \\ &\quad -\iint_0^{\infty} xy\overline{F}_X(x)\overline{F}_Y(y)\log\overline{F}_Y(y)\,dx\,dy, \end{aligned} \tag{14}$$

where the second equality comes from the independence of random variables. □

Remark 8. If the support sets of X and Y are limited to finite sets (a_X, b_X) and (a_Y, b_Y), respectively, from Proposition 7,

$$\begin{aligned} \mathscr{E}^{\omega}(X,Y) &= m_F^*(a_X)\,\mathscr{E}^{\omega}(Y) + m_G^*(a_Y)\,\mathscr{E}^{\omega}(X), \\ \mathscr{C}\mathscr{E}^{\omega}(X,Y) &= \mu_F^*(b_X)\,\mathscr{C}\mathscr{E}^{\omega}(Y) + \mu_G^*(b_Y)\,\mathscr{C}\mathscr{E}^{\omega}(X). \end{aligned} \tag{15}$$

Furthermore, for

$$\begin{aligned} m_G^*(a_Y) &= m_F^*(a_X) = m^*, \\ \mu_G^*(b_Y) &= \mu_F^*(b_X) = \mu^*, \end{aligned} \tag{16}$$

it is obtained that

$$\begin{aligned} \mathscr{E}^{\omega}(X,Y) &= m^*\left(\mathscr{E}^{\omega}(Y) + \mathscr{E}^{\omega}(X)\right), \\ \mathscr{C}\mathscr{E}^{\omega}(X,Y) &= \mu^*\left(\mathscr{C}\mathscr{E}^{\omega}(Y) + \mathscr{C}\mathscr{E}^{\omega}(X)\right), \end{aligned} \tag{17}$$

which is similar to the property of Shannon entropy for two independent random variables. For instance, if X and Y have same uniform distribution in the interval $(0, \sqrt{3})$, then $\mu^* = 1$ and $\mathscr{C}\mathscr{E}^{\omega}(X,Y) = \mathscr{C}\mathscr{E}^{\omega}(X) + \mathscr{C}\mathscr{E}^{\omega}(Y)$.

In the following proposition, alternative expressions to (9) and (10) are provided in terms of double integrals of hazard and reversed hazard rates. A similar result for CE has been considered in Di Crescenzo and Longobardi [5].

Proposition 9. *Let X be a nonnegative random variable with finite WCE and WCRE; then*

(a) $\mathscr{E}^{\omega}(X) = E((X - 1/r(X))\widetilde{T}^2(X))$,

(b) $\mathscr{C}\mathscr{E}^{\omega}(X) = E((X + 1/\tau(X))T^2(X))$,

where

$$\begin{aligned} \widetilde{T}^2(x) &= -\int_0^x \log \overline{F}(t)\,dt = \int_0^x \int_0^t r(u)\,du\,dt, \\ T^2(x) &= -\int_x^{\infty} \log F(t)\,dt = \int_x^{\infty} \int_t^{\infty} \tau(u)\,du\,dt. \end{aligned} \tag{18}$$

Proof. By recalling (9),

$$\begin{aligned} \mathscr{E}^{\omega}(X) &= -\int_0^{\infty} x\overline{F}(x)\log\overline{F}(x)\,dx \\ &= \int_0^{\infty}\left(\int_x^{\infty}\left(\overline{F}(t) - tf(t)\right)dt\right)\log\overline{F}(x)\,dx \\ &= \int_0^{\infty}\int_0^t \overline{F}(t)\log\overline{F}(x)\,dx\,dt \\ &\quad - \int_0^{\infty}\int_0^t tf(t)\log\overline{F}(x)\,dx\,dt \\ &= \int_0^{\infty}\overline{F}(t)\left(\int_0^t \log\overline{F}(x)\,dx\right)dt \\ &\quad - \int_0^{\infty} tf(t)\left(\int_0^t \log\overline{F}(x)\,dx\right)dt \\ &= -\int_0^{\infty} f(t)\frac{\overline{F}(t)}{f(t)}\widetilde{T}^2(t)\,dt \\ &\quad + \int_0^{\infty} tf(t)\widetilde{T}^2(t)\,dt \\ &= E\left(\left(X - \frac{\overline{F}(X)}{f(X)}\right)\widetilde{T}^2(X)\right). \end{aligned} \tag{19}$$

Part (b) is proven in a similar way. Note that

$$\begin{aligned} \mathscr{C}\mathscr{E}^{\omega}(X) &= -\int_0^{\infty} xF(x)\log F(x)\,dx \\ &= -\int_0^{\infty}\left(\int_0^x \left(F(t) + tf(t)\right)dt\right)\log F(x)\,dx; \end{aligned} \tag{20}$$

this completes the proof. □

Example 10. (i) Let X be exponentially distributed with mean λ; then $\widetilde{T}^2(x) = x^2/2\lambda$ and $r(x) = 1/\lambda$. From Proposition 9, $\mathscr{E}^{\omega}(X) = E[(X-\lambda)(X^2/2\lambda)] = (1/2\lambda)E(X^3) - (1/2)E(X^2) = 2\lambda^2$.

(ii) Consider the random variable X with the following density function:

$$f(x) = \frac{2\alpha}{x^3}\exp\left(-\frac{\alpha}{x^2}\right), \quad x > 0,\ \alpha > 0. \tag{21}$$

Then $T^2(x) = \alpha/x$ and $\tau(x) = 2\alpha/x^3$ and, from Proposition 9, it is obtained that

$$\mathscr{C}\mathscr{E}^{\omega}(X) = \alpha + \frac{1}{2}E\left(X^2\right) = \alpha + \frac{1}{2}\alpha Ei\left(1, \frac{\alpha}{x^2}\right), \tag{22}$$

where

$$Ei(a,z) = \int_1^{\infty} x^a e^{-zx}dx \tag{23}$$

is the exponential integral function (see Abramowitz and Stegun [18]).

Proposition 11. *Let X be a random variable with finite support set (α, β) with $\beta > \alpha > 0$. Then, for θ in $(0, 1]$,*

(a) $\mathscr{E}^{\omega}(X) \le \theta((\beta^2 - \alpha^2)/2) - m_F^*(\alpha)\log(\theta e)$,

(b) $\mathscr{C}\mathscr{E}^{\omega}(X) \le \theta((\beta^2 - \alpha^2)/2) - \mu_F^*(\beta)\log(\theta e)$.

Proof. From Taylor expansion (see Walker [19]), it can be seen that, for all θ in $(0, 1]$,

$$-x\log x \le \theta - x\left(1 + \log\theta\right). \tag{24}$$

Now, from (9),

$$\begin{aligned}\mathscr{E}^{\omega}(X) &\le \int_{\alpha}^{\beta} x\left(\theta - \overline{F}(x)(1+\log\theta)\right)dx \\ &= \theta\int_{\alpha}^{\beta} x\,dx - (1+\log\theta)\int_{\alpha}^{\beta} x\overline{F}(x)\,dx \\ &= \theta\frac{\beta^2-\alpha^2}{2} - m_F^*(\alpha)\log(\theta e).\end{aligned} \tag{25}$$

Similarly, from (10),

$$\begin{aligned}\mathscr{C}\mathscr{E}^{\omega}(X) &\le \theta\int_{\alpha}^{\beta} x\,dx - (1+\log\theta)\int_{\alpha}^{\beta} xF(x)\,dx \\ &= \theta\frac{\beta^2-\alpha^2}{2} - \mu_F^*(\beta)\log(\theta e).\end{aligned} \tag{26}$$

This completes the proof. □

Remark 12. From Proposition 11, we get, for all θ in $(0,1]$,

$$\begin{aligned}\mathscr{E}^{\omega}(X) + \mathscr{C}\mathscr{E}^{\omega}(X) &\le \theta\left(\beta^2-\alpha^2\right) \\ &\quad - \left[m_F^*(\alpha) + \mu_F^*(\beta)\right]\log(\theta e).\end{aligned} \tag{27}$$

Remark 13. The right-hand sides of (a) and (b) in Proposition 11 are minimized at the points $\theta = 2m_F^*(\alpha)/(\beta^2-\alpha^2)$ and $\theta = 2\mu_F^*(\beta)/(\beta^2-\alpha^2)$, respectively.

The following proposition considers the effect of linear transformations on WCE and WCRE.

Proposition 14. *Let X be a nonnegative random variable. Then, for positive constants a and b,*

(a) $\mathscr{E}^{\omega}(aX+b) = a^2\mathscr{E}^{\omega}(X) + ab\mathscr{E}(X)$,

(b) $\mathscr{C}\mathscr{E}^{\omega}(aX+b) = a^2\mathscr{C}\mathscr{E}^{\omega}(X) + ab\mathscr{C}\mathscr{E}(X)$.

Proof. The proof is straightforward. Note that

$$\begin{aligned}&\mathscr{E}^{\omega}(aX+b) \\ &= -\int_b^{\infty} yP\left(X > \frac{y-b}{a}\right)\log P\left(X > \frac{y-b}{a}\right)dy \\ &= -\int_0^{\infty} a(ax+b)P(X>x)\log P(X>x)\,dx \\ &= -a^2\int_0^{\infty} x\overline{F}(x)\log\overline{F}(x)\,dx \\ &\quad - ab\int_0^{\infty}\overline{F}(x)\log\overline{F}(x)\,dx.\end{aligned} \tag{28}$$

Part (b) is proven in a similar way. □

According to Proposition 14, it is realized that there may exist a close relation between WCE, WCRE, and variance.

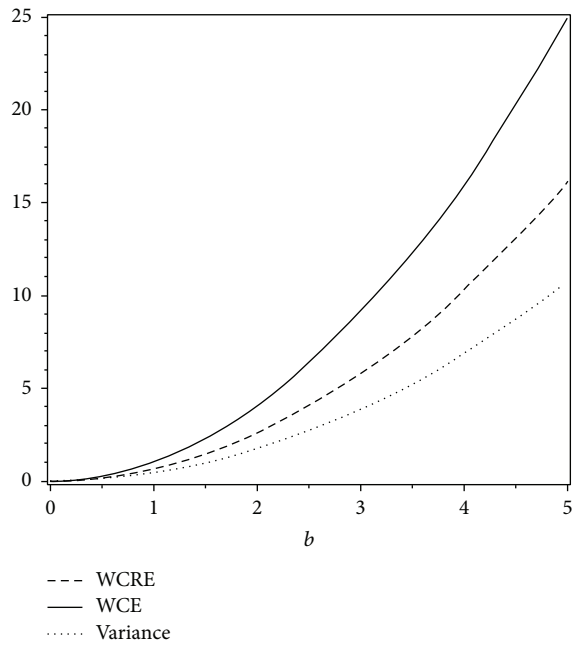

Figure 1: WCRE, WCE, and variance for Weibull distribution.

Example 15. Let X be a nonnegative random variable with Weibull distribution with scale parameter b and shape parameter 2. Its probability density function is given by

$$f(x) = \frac{x}{b^2}e^{-1/2}\exp\left(\frac{x^2}{b^2}\right), \quad b>0,\ x>0. \tag{29}$$

The plot of $\mathrm{Var}(X) = (2-(1/2)\pi)b^2$, $\mathscr{E}^{\omega}(X)$, and $\mathscr{C}\mathscr{E}^{\omega}(X)$ are given in Figure 1 which shows a direct relation between variance and cumulative entropies. Furthermore, $\mathrm{Var}(X) \le \mathscr{C}\mathscr{E}^{\omega}(X) \le \mathscr{E}^{\omega}(X)$. In such cases, WCRE and WCE may be used instead of variance as a discrepancy measure. It should be noticed that as b grows, the tendency of WCE and WCRE to overestimate the variance of X increases.

Hereafter, the weighted cumulative measures of information are studied when the lifetimes have proportional hazard (PH) and reversed proportional hazard (RPH) rates.

Two random variables X and Y with survival functions $\overline{F}$ and $\overline{G}$ are said to have PH model if there exists $\theta > 0$ such that $\overline{G}(x) = [\overline{F}(x)]^{\theta}$. The PH model, introduced by Cox [20], plays an important role in reliability and survival analysis. The RPH model is based on the assumption that the cumulative distribution functions of X and Y are related with $G(x) = [F(x)]^{\theta}$ with $\theta > 0$. Some results on these models are presented in Ebrahimi and Kirmani [21], Di Crescenzo [22], R. C. Gupta and R. D. Gupta [23], and Gupta et al. [24].

Proposition 16. *Let X and Y be two nonnegative random variables; then, for $\theta \ge 1$ and for PH and RPH models, $\mathscr{E}^{\omega}(Y) \le \mathscr{E}^{\omega}(\sqrt{\theta}X)$ and $\mathscr{C}\mathscr{E}^{\omega}(Y) \le \mathscr{C}\mathscr{E}^{\omega}(\sqrt{\theta}X)$, respectively. For $0<\theta<1$, the inequalities are reversed.*

Proof. The proof is easy and is omitted. Note that $F(x) \ge [F(x)]^{\theta}$ and $\overline{F}(x) \ge [\overline{F}(x)]^{\theta}$, $x \ge 0$, when $\theta > 1$. □

Example 17. Let $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_n = \max\{X_1, X_2, \ldots, X_n\}$, where X_i, $i = 1, 2, \ldots, n$, are nonnegative independently and identically distributed with common distribution F. Suppose also that Y_j, $j = 1, 2$, has distribution function G_j, $j = 1, 2$, where $\overline{G}_1(x) = [\overline{F}(x)]^n$ and $G_n(x) = [F(x)]^n$. From Proposition 16, it is obtained that $\mathscr{E}^{\omega}(Y_1) \leq \mathscr{E}^{\omega}(\sqrt{n}X)$ and $\mathscr{CE}^{\omega}(Y_n) \leq \mathscr{CE}^{\omega}(\sqrt{n}X)$.

3. Empirical WCRE and WCE

Suppose $X_1, X_2, \ldots, X_n$ be a random sample with distribution F. The empirical distribution is a discrete probability distribution with probability function generated from the given sample. The empirical distribution and survival functions of random sample $X_1, X_2, \ldots, X_n$ at point x are given by $F_n(x) = (1/n)\sum_{i=1}^{n} I\{X_i \leq x\}$ and $\overline{F}_n = 1 - F_n$, respectively, where

$$I\{X \leq x\} = \begin{cases} 1, & X \leq x, \\ 0, & X > x \end{cases} \tag{30}$$

is the indicator function of event $\{X \leq x\}$.

Definition 18. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a population having distribution function $F(x)$. The empirical WCRE (EWCRE) and empirical WCE (EWCE) are defined as

$$\mathscr{E}^{\omega}(F_n) = -\int_0^{\infty} x\overline{F}_n(x)\log\overline{F}_n(x)\,dx, \tag{31}$$

$$\mathscr{CE}^{\omega}(F_n) = -\int_0^{\infty} xF_n(x)\log F_n(x)\,dx, \tag{32}$$

respectively.

The following proposition presents alternative expressions of (31) and (32) in terms of sample mean $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$, sample variance $S^2 = (1/(n-1))\sum_{i=1}^{n}(X_i - \overline{X})^2$, and order statistics.

Proposition 19. *Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a population having order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$; then*

(a)

$$\mathscr{E}^{\omega}(F_n) = \left[\overline{X}^2 - X_{(1)}^2 + \left(1 - \frac{1}{n}\right)S^2\right]\log\sqrt{n} - \sum_{j=1}^{n-1} C_j(n-j)\log(n-j), \tag{33}$$

(b)

$$\mathscr{CE}^{\omega}(F_n) = \left(X_{(n)}^2 - \overline{X}^2 - \left(1 - \frac{1}{n}\right)S^2\right)\log\sqrt{n} - \sum_{j=1}^{n-1} C_j j\log j, \tag{34}$$

where $C_j = (1/2)(X_{(j+1)}^2 - X_{(j)}^2)$.

Proof. From (31), we get

$$\mathscr{E}^{\omega}(F_n) = -\sum_{j=1}^{n-1}\int_{X_{(j)}}^{X_{(j+1)}} x\overline{F}_n(x)\log\overline{F}_n(x)\,dx. \tag{35}$$

Recalling that, for $X_{(j)} \leq x < X_{(j+1)}$,

$$\overline{F}_n(x) = 1 - \frac{j}{n}, \quad j = 1, 2, \ldots, n-1, \tag{36}$$

so

$$\begin{aligned}\mathscr{E}^{\omega}(F_n) &= -\sum_{j=1}^{n-1}\int_{X_{(j)}}^{X_{(j+1)}} x\left(\frac{n-j}{n}\right)\log\left(\frac{n-j}{n}\right)dx \\ &= -\sum_{j=1}^{n-1}\left(\frac{X_{(j+1)}^2 - X_{(j)}^2}{2}\right)\left(\frac{n-j}{n}\right)\log\left(\frac{n-j}{n}\right) \\ &= -\frac{1}{n}\sum_{j=1}^{n-1} C_j(n-j)\log\left(\frac{n-j}{n}\right),\end{aligned} \tag{37}$$

where $C_j = (1/2)(X_{(j+1)}^2 - X_{(j)}^2)$.

In addition,

$$\begin{aligned}\sum_{j=1}^{n-1} C_j(n-j) &= \frac{n}{2}\sum_{j=1}^{n-1}\left(X_{(j+1)}^2 - X_{(j)}^2\right) \\ &\quad - \frac{1}{2}\sum_{j=1}^{n-1} j\left(X_{(j+1)}^2 - X_{(j)}^2\right) \\ &= \frac{n}{2}\left(\overline{X^2} - X_{(1)}^2\right),\end{aligned} \tag{38}$$

where

$$\overline{X^2} = \frac{1}{n}\sum_{j=1}^{n} X_{(j)}^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2. \tag{39}$$

Now, by virtue of $(1 - 1/n)S^2 = \overline{X^2} - \overline{X}^2$, (33) is obtained. Part (b) is proven in a similar way. Note that $\sum_{j=1}^{n-1} C_j j = (n/2)(X_{(n)}^2 - \overline{X^2})$. □

Here, the consistency of EWCRE and EWCE is studied under specific choices of function F. According to (33) and (34), the estimators $\mathscr{E}^{\omega}(F_n)$ and $\mathscr{CE}^{\omega}(F_n)$ are calculated for $n = 100$ simulated observations of distribution F and the process is repeated for 1000 times. Table 1 shows mean and mean squares of errors (MSE) of EWCRE and EWCE. Relevant values of $E[\mathscr{E}^{\omega}(F_{100})]$ and $E[\mathscr{CE}^{\omega}(F_{100})]$ are the mean of $\mathscr{E}^{\omega}(F_{100})$ and $\mathscr{CE}^{\omega}(F_{100})$ values generated in every step of simulation process. The numbers in brackets indicate the corresponding real values of $\mathscr{E}^{\omega}(X)$ and $\mathscr{CE}^{\omega}(X)$ calculated from (9) and (10). The MSE values are the mean of squared errors between empirical and real entropies. According to Table 1, the estimates are almost equal to real values of weighted cumulative information measures and MSE values are nearly zero.

Table 1: Mean and MSE of the empirical cumulative entropy for random samples of some familiar distributions.

Distribution	Survival function	$E[\mathcal{CE}^{\omega}(F_{100})]$	$E[\mathcal{E}^{\omega}(F_{100})]$	$\mathrm{MSE}[\mathcal{CE}^{\omega}(F_{100})]$	$\mathrm{MSE}[\mathcal{E}^{\omega}(F_{100})]$
Finite range	$(1-(1/2)x)^3$, $0<x<2$	0.15377 (0.15477)	0.26319 (0.27)	0.00087	0.00127
Uniform	$1-x$, $0<x<1$	0.11018 (0.11111)	0.13698 (0.13889)	0.00005	0.00004
Power	$1-(x/5)^2$, $0<x<5$	3.10340 (3.125)	3.09780 (3.125)	0.01958	0.01778
Exponential	$\exp(-2x)$, $x>0$	0.21012 (0.21175)	0.46532 (0.5)	0.00271	0.02267

4. Conclusion

Reliability and survival analysis is a branch of statistics which deals with death in biological organisms and failure in mechanical systems. There are several uncertainty measures that play a central role in understanding and describing reliability. Most of these information measures do not take into account the values of a random variable. Two shift-dependent measures of uncertainty are considered related to cumulative distribution and survival functions so that higher weight is assigned to large values of observed random variables. These measures are called weighted cumulative residual entropy (WCRE) and weighted cumulative entropy (WCE) with properties similar to those of the legacy entropies. Several propositions and examples for WCRE and WCE have been presented, some of which parallel those for results presented in Asadi and Zohrevand [4] and Di Crescenzo and Longobardi [5, 16]. It must be remarked that the empirical WCRE and WCE and the relationship with well-known statistics have been assessed. Of course, many properties of these measures are still waiting to be discovered.

Competing Interests

The author declares that there are no competing interests.

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References

[1] C. E. Shannon, "A mathematical theory of communication," *The Bell System Technical Journal*, vol. 27, pp. 379–423, 1948.

[2] F. E. Wang, B. C. Vemuri, M. Rao, and Y. Chen, "A new and robust information theoretic measure and its application to image alignment," in *Proceedings of the International Conference on Information Processing in Medical Imaging*, pp. 3880–4000, Springer, Ambleside, UK, 2003.

[3] M. Rao, Y. Chen, B. C. Vemuri, and F. Wang, "Cumulative residual entropy: a new measure of information," *IEEE Transactions on Information Theory*, vol. 50, no. 6, pp. 1220–1228, 2004.

[4] M. Asadi and Y. Zohrevand, "On the dynamic cumulative residual entropy," *Journal of Statistical Planning and Inference*, vol. 137, no. 6, pp. 1931–1941, 2007.

[5] A. Di Crescenzo and M. Longobardi, "On cumulative entropies," *Journal of Statistical Planning and Inference*, vol. 139, no. 12, pp. 4072–4087, 2009.

[6] G. Wallis, "Using spatio-temporal correlations to learn invariant object recognition," *Neural Networks*, vol. 9, no. 9, pp. 1513–1519, 1996.

[7] A. de Rugy, W. Marinovic, and G. Wallis, "Neural prediction of complex accelerations for object interception," *Journal of Neurophysiology*, vol. 107, no. 3, pp. 766–771, 2012.

[8] G. Wallis and E. T. Rolls, "Invariant face and object recognition in the visual system," *Progress in Neurobiology*, vol. 51, no. 2, pp. 167–194, 1997.

[9] D. M. Hartley and D. R. Kerr, "Invariant measures of the closeness to linear dependence of six lines or screws," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, vol. 215, no. 10, pp. 1145–1151, 2001.

[10] D. R. Kerr and D. M. Hartley, "Invariant measures of closeness to linear dependency of screw systems," *Proceedings of the Institution of Mechanical Engineers, Part C*, vol. 220, no. 7, pp. 1033–1043, 2006.

[11] A. Adam, M. Houkari, and J.-P. Laurent, "Spectral risk measures and portfolio selection," *Journal of Banking and Finance*, vol. 32, no. 9, pp. 1870–1882, 2008.

[12] F. Argenti, G. Torricelli, and L. Alparone, "MMSE filtering of generalised signal-dependent noise in spatial and shift-invariant wavelet domains," *Signal Processing*, vol. 86, no. 8, pp. 2056–2066, 2006.

[13] A. Ghosh, M. V. Badiger, P. S. Tapadia, V. Ravi Kumar, and B. D. Kulkarni, "Characterization of chaotic dynamics-I: dynamical invariants of sheared polymer solutions," *Chemical Engineering Science*, vol. 56, no. 19, pp. 5635–5642, 2001.

[14] F. Misagh and G. H. Yari, "On weighted interval entropy," *Statistics and Probability Letters*, vol. 81, no. 2, pp. 188–194, 2011.

[15] F. Misagh, Y. Panahi, G. H. Yari, and R. Shahi, "Weighted cumulative entropy and its estimation," in *Proceedings of the IEEE International Conference on Quality and Reliability (ICQR '11)*, pp. 477–480, Bangkok, Thailand, September 2011.

[16] A. Di Crescenzo and M. Longobardi, "On cumulative entropies," *Journal of Statistical Planning and Inference*, vol. 139, no. 12, pp. 4072–4087, 2009.

[17] Y. Suhov and S. Yasaei Sekeh, "Weighted cumulative entropies: an extension of CRE and CE," http://arxiv.org/abs/1507.07051v1.

[18] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, USA, 1995.

[19] P. L. Walker, *Examples and Theorems in Analysis*, Springer, 2004.

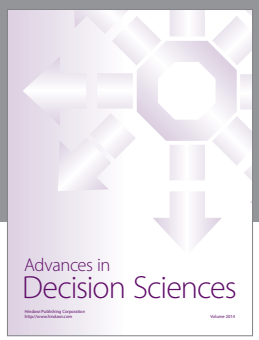
[20] D. R. Cox, "Regression models and life-tables," *Journal of the Royal Statistical Society. Series B. Methodological*, vol. 34, pp. 187–220, 1972.

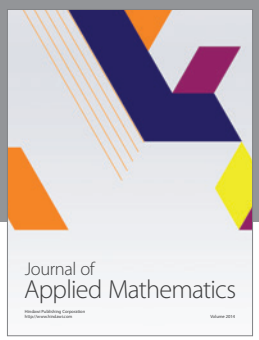
[21] N. Ebrahimi and S. N. U. A. Kirmani, "A measure of discrimination between two residual life-time distributions and its applications," *Annals of the Institute of Statistical Mathematics*, vol. 48, no. 2, pp. 257–265, 1996.

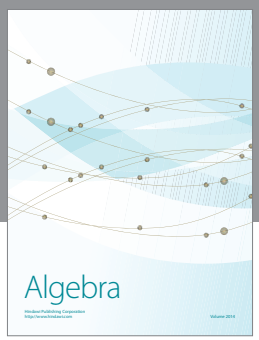
[22] A. Di Crescenzo, "Some results on the proportional reversed hazards model," *Statistics & Probability Letters*, vol. 50, no. 4, pp. 313–321, 2000.

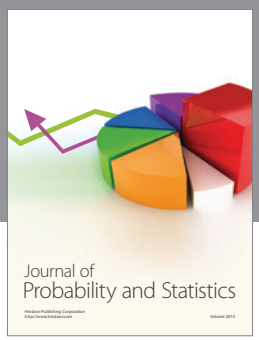
[23] R. C. Gupta and R. D. Gupta, "Proportional reversed hazard rate model and its applications," *Journal of Statistical Planning and Inference*, vol. 137, no. 11, pp. 3525–3536, 2007.

[24] R. C. Gupta, R. D. Gupta, and P. L. Gupta, "Modeling failure time data by Lehman alternatives," *Communications in Statistics-Theory and Methods*, vol. 27, no. 4, pp. 887–904, 1998.









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