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Research Article

Quasi-Positive Delta Sequences and Their Applications in Wavelet Approximation

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A sufficient literature is available for the wavelet error of approximation of certain functions in the L^2-norm. There is no work in context of multiresolution approximation of a function in the sense of sup-error. In this paper, for the first time, wavelet estimator for the approximation of a function f belonging to $\mathrm{Lip}_\alpha[a,b]$ class under supremum norm has been obtained. Working in this direction, four new theorems on the wavelet approximation of a function f belonging to Lip_α, $0<\alpha\le 1$ class using the projection $P_m f$ of its wavelet expansions have been estimated. The calculated estimator is best possible in wavelet analysis.

1. Introduction

Orthogonal wavelet is a new development in analysis but very useful in engineering and technology, especially in high-resolution images and signal processing due to their localization properties in time and frequency. Wavelet expansions are superior to classical orthogonal series like Fourier series. Some properties of wavelet expansions have been studied by Chui [1], Daubechies and Lagarias [2], Meyer [3], Walter [4, 5], Islam et al. [6], and so forth. The idea of approximation of various functional spaces under different norms is obtained by Lal and Kumar [7, 8], Abu-Sirhan [9], Coskun [10], and Shiri and Azadi Kenary [11] which gives the inspiration for the present work. But till now no work seems to have been done to obtain the wavelet approximation of a function $f\in \mathrm{Lip}_\alpha[a,b]$, $0<\alpha\le 1$ using the projection $(P_m f)$ of its wavelet expansion and to discuss its convergence. In an attempt to make an advance study in this direction, in this paper, the best possible wavelet sup-error $E_m f$ of a function $f\in \mathrm{Lip}_\alpha[a,b]$ by $P_m f$ of its wavelet expansion has been determined. The convergence of wavelet expansions has been also discussed.

2. Definitions and Preliminaries

2.1. Multiresolution Analysis. Let $\mathbb{Z}$ be the set of all integers. A multiresolution analysis of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j\in\mathbb{Z}$, with the following properties:

(1) $V_j\subset V_{j+1}$.

(2) $f(x)\in V_j \Leftrightarrow f(2x)\in V_{j+1}$.

(3) $f(x)\in V_0 \Leftrightarrow f(x+1)\in V_0$.

(4) $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j=-\infty}^{\infty} V_j=\{0\}$.

(5) Suppose a function $\phi\in V_0$ exists such that the collection $\{\phi(x-k);\ k\in\mathbb{Z}\}$ is a Riesz basis of V_0.

Let $\psi\in L^2(\mathbb{R})$, and

$$\begin{aligned}\psi_{j,k} &:= 2^{j/2}\psi\left(2^j.-k\right),\\ W_j &:= \operatorname{clos}\left\langle \psi_{j,k} : k\in\mathbb{Z}\right\rangle.\end{aligned}\tag{1}$$

Then this family of subspaces of $L^2(\mathbb{R})$ gives direct sum decomposition of $L^2(\mathbb{R})$ in the sense that every $f\in L^2(\mathbb{R})$ has unique decomposition:

$$\begin{aligned}f(x) = \cdots + g_{-2}(x) + g_{-1}(x) + g_0(x) + g_1(x)\\ + g_2(x) + \cdots,\end{aligned}\tag{2}$$

where $g_j\in W_j$ for all $j\in\mathbb{Z}$ and we describe this by writing

$$L^2(\mathbb{R}) = V_j \oplus_{i=j}^{\infty} W_i,\tag{3}$$

where

$$V_j := \oplus_{k=-\infty}^{j-1} W_k. \tag{4}$$

$\{\psi_{j,k};\ k \in \mathbb{Z}\}$ is a Riesz basis of W_j.

A function $\phi \in L^2(\mathbb{R})$ is called a scaling function, if the subspaces V_j of $L^2\mathbb{R}$ defined by

$$V_j := \operatorname{clos}_{L^2\mathbb{R}} \left\{\phi_{j,k} : k \in \mathbb{Z}\right\}, \quad j \in \mathbb{Z} \tag{5}$$

satisfy properties (1) to (5) stated above in this section. It is important to note that the scaling function ϕ generates a multiresolution analysis $\{V_j\}$ of $L^2(\mathbb{R})$ (Debnath [12]).

Following Walter [4], each $f \in L^2(\mathbb{R})$ has two representations

$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle f, \psi_{m,n} \rangle \psi_{m,n}(t), \tag{6}$$

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}(t) \\ &\quad + \sum_{k=m}^{\infty} \sum_{n=-\infty}^{\infty} \langle f, \psi_{k,n} \rangle \psi_{k,n}(t) \\ &= (P_m f)(t) + (r_m f)(t), \end{aligned} \tag{7}$$

where convergence is in sense of $L^2(\mathbb{R})$. The function $(P_m f) \in V_m$ and in fact is the projection of f onto V_m. It is also given in terms of the kernel $K_m(x, y)$ of V_m as

$$(P_m f)(x) = \int_{-\infty}^{\infty} K_m(x, t) f(t)\, dt. \tag{8}$$

The function K_m is given by

$$K_m(x, y) = 2^m K(2^m x, 2^n y), \tag{9}$$

where

$$\begin{aligned} K(x, y) &= \sum_{n=-\infty}^{\infty} \phi(x - n) \overline{\phi}(y - n), \\ 1 &= \int_{-\infty}^{\infty} \phi(x)\, dx = \sum_{n=-\infty}^{\infty} \phi(x - n). \end{aligned} \tag{10}$$

It is remarkable to note that

$$K(x, y) = \sum_{n=-\infty}^{\infty} \phi(x - n) \overline{\phi}(y - n) \tag{11}$$

$$\begin{aligned} K_m(x, y) &= 2^m K(2^m x, 2^m y) \\ &= 2^m \sum_{n=-\infty}^{\infty} \phi(2^m x - n) \overline{\phi}(2^m y - n) \\ &= \sum_{n=-\infty}^{\infty} 2^{m/2} \phi(2^m x - n)\, 2^{m/2} \overline{\phi}(2^m y - n). \end{aligned} \tag{12}$$

Then

$$\begin{aligned} &\int_{-\infty}^{\infty} K_m(x, y) \\ &= \sum_{n=-\infty}^{\infty} 2^m \left(\int_{-\infty}^{\infty} \phi(2^m x - n)\, dx \right) \overline{\phi}(2^m y - n) \\ &= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \phi(t)\, dt \right) \overline{\phi}(2^m y - n) \\ &= \sum_{n=-\infty}^{\infty} \overline{\phi}(y - n) = 1. \end{aligned} \tag{13}$$

2.2. Quasi-Positive Delta Sequences. If

(1) there is $C > 0$ such that $\int_{-\infty}^{\infty} |\delta_m(x, y)| dx \le C$, $y \in \mathbb{R}$, $m \in \mathbb{N}$,

(2) $\int_{y-C}^{y+C} \delta_m(x, y) dx \to 1$ uniformly on compact subset of $\mathbb{R}$, as $m \to \infty$,

(3) for each $\gamma > 0$,

$$\sup_{|x-y| \ge \gamma} |\delta_m(x, y)| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty \tag{14}$$

then $\{\delta_m(\cdot, y)\}$ is known as quasi-positive delta sequence of function.

An example of a quasi-positive delta sequence is the Fejer kernel,

$$\begin{aligned} &F_m(x, y) \\ &= \frac{\sin^2(((m+1)/2)(x - y))}{2(m+1)\pi \sin^2((x - y)/2)} \chi_{[-\pi,\pi]}(x - y), \end{aligned} \tag{15}$$

while a delta sequence that is not quasi-positive is the Dirichlet kernel of Fourier series,

$$D_m(x, y) = \frac{\sin(m + 1/2)(x - y)}{2\pi \sin((x - y)/2)} \chi_{[-\pi,\pi]}(x - y) \tag{16}$$

because $\int_{-\infty}^{\infty} D_m(x, y) dx$ is not absolutely convergent.

2.3. Function of $\operatorname{Lip}_\alpha[a, b]$ *Class.* A function $f \in \operatorname{Lip}_\alpha[a, b]$ if

$$|f(x) - f(y)| = O\left(|x - y|^\alpha\right), \quad \text{for } 0 < \alpha \le 1, \tag{17}$$

(Titchmarsh [13], p. 406).

Examples

(1) If $f(x) = |x|\ \forall x \in [0, 1]$, $f \in \operatorname{Lip}_\alpha$.

(2) If $f(x) = 1/x\ \forall x \in (0, 1]$, $f \notin \operatorname{Lip}_1(0, 1]$.

Let N be a positive integer. Let $x = 1/N$, $y = 1/(N + 1)$ $x, y \in (0, 1]$.

Then $x - y = 1/N - 1/(1+N) < 1/N$ and $1/(x - y) > N$.

Now, $|f(x) - f(y)|/|x - y| = |N - (N + 1)|/|x - y| = 1/|x - y| \ge N$

$|f(x) - f(y)| > N|x - y|$

$\therefore f \notin \operatorname{Lip}_1(0, 1]$.

Remarks

(1) If $f \in \mathrm{Lip}_\alpha[a,b]$ $\alpha > 0$, then f is continuous, indeed, uniformly continuous on $[a,b]$.

(2) Lip_α class is a linear space over $\mathbb{R}$ or $\mathbb{C}$.

(3) If $f \in \mathrm{Lip}_\alpha$ $\alpha > 1$, then f is constant function.

2.4. Projection $P_m f$. $P_m f$, the orthogonal projection of $L^2(\mathbb{R})$ onto V_m, is defined by

$$P_m f = \sum_{k=-\infty}^{\infty} \langle f, \phi_{m,k} \rangle \phi_{m,k}, \quad m \in \mathbb{Z}, \tag{18}$$

(Sweldens and Piessens [14]).

2.5. Wavelet Approximation. The wavelet approximation $E_m(f)$ of a function f under supremum norm is defined by

$$E_m(f) = \|f - P_m f\|_\infty = \sup_x |(f(x) - P_m f(x))| \tag{19}$$

(Zygmund [15], p. 114).

If $E_m(f) \to 0$ as $m \to \infty$ then $E_m(f)$ is called the best approximation of f of order m (Zygmund [15], p. 115).

3. Theorems

In this paper, we prove the following theorems.

Theorem 11. *Let $\phi \in L^2(\mathbb{R})$ be a scaling function and $\psi \in L^2(\mathbb{R})$ be a basic wavelet satisfying the admissibility condition*

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty. \tag{20}$$

If a function $f \in \mathrm{Lip}_\alpha[a,b]$ is represented by its wavelet expansion as

$$\begin{aligned} f(\cdot) &= \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}(\cdot) \\ &\quad + \sum_{k=m}^{\infty} \sum_{n=-\infty}^{\infty} \langle f, \psi_{m,n} \rangle \psi_{m,n}(\cdot) \\ &= (P_m f)(\cdot) + (r_m f)(\cdot), \end{aligned} \tag{21}$$

then the wavelet approximation $E_m(f)$ of f by $(P_m f)$ satisfies

$$\begin{aligned} E_m(f) &= \|f - P_m f\|_\infty = \sup_{y \in [a,b]} |(f(y)) - P_m f(y)| \\ &= O\left(\frac{1}{(|m|+1)^\alpha}\right). \end{aligned} \tag{22}$$

Theorem 12. *If a function $f \in \mathrm{Lip}_\alpha[a,b]$ and $P_m f$ is the projection of f onto V_m then $P_m f \to f$ as $m \to \infty$ uniformly on $[a,b]$.*

Theorem 13. *If $f \in \mathrm{Lip}_\alpha[a,b]$ and*

$$f(\cdot) = (P_m f)(\cdot) + (r_m f)(\cdot) \tag{23}$$

then

$$\|r_m f\|_\infty = O\left(\frac{1}{(|m|+1)^\alpha}\right), \quad 0 < \alpha \le 1. \tag{24}$$

Theorem 14. *If $f \in \mathrm{Lip}_\alpha[a,b]$,*

$$f(\cdot) = (P_m f)(\cdot) + (r_m f)(\cdot), \tag{25}$$

then $r_m f \to 0$ as $m \to \infty$ uniformly on $[a,b]$.

4. Lemma

For the proof of our theorems, the following lemma is required.

Lemma 15. *If $K_m(x,y) = 2^m K(2^m x, 2^m y)$, $x, y \in [a,b]$, then $\{K_m(\cdot, y)\}_{-\infty}^{\infty}$ is a quasi-positive delta sequence of function in $L^1(\mathbb{R})$.*

4.1. Proof of Lemma 15. Following the proof of Walter ([4] p. 113),

$$\int_{-\infty}^{\infty} |K_m(x,y)| \, dx = C. \tag{26}$$

Let $c > 0$; then

$$\begin{aligned} \int_{y-c}^{y+c} |K_m(x,y)| \, dx &= 1 - \int_{-\infty}^{y-2^m c} |K(t,y)| \, dt \\ &\quad - \int_{y+2^m c}^{\infty} |K(t,y)| \, dt \end{aligned} \tag{27}$$

$\int_{-\infty}^{y-2^m c} |K(t,y)| dt \to 0$ as $m \to \infty$ and $\int_{y+2^m c}^{\infty} |K(t,y)| dt \to 0$ as $m \to \infty$.

Thus, $\int_{y-c}^{y+c} |K_m(x,y)| dx \to 1$ as $m \to \infty$ uniformly on $[a,b] \subset \mathbb{R}$ and $\sup_{|x-y| \ge 1/(|m|+1)} |K_m(x,y)| \to 0$ as $m \to \infty$, [4].

Therefore $\{K_m(x,y)\}_{-\infty}^{\infty}$ is a quasi-positive delta sequence of functions.

5. Proofs

5.1. Proof of Theorem 11. $(P_m f)$ is given by

$$\begin{aligned} (P_m f)(y) &= \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}(y) \\ &= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) \overline{\phi}_{m,n}(x) \, dx \right) \phi_{m,n}(y) \\ &= \int_{-\infty}^{\infty} f(x) \left(\sum_{n=-\infty}^{\infty} \phi_{m,n}(y) \overline{\phi_{m,n}(x)} \right) dx \\ &= \int_{-\infty}^{\infty} f(x) \\ &\quad \cdot \left(\sum_{n=-\infty}^{\infty} 2^{m/2} \phi(2^m y - n) \, 2^{m/2} \overline{\phi(2^m x - n)} \right) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f(x) \\
&\cdot \left(\sum_{n=-\infty}^{\infty} 2^m \phi(2^m y - n) \overline{\phi(2^m x - n)} \right) dx \\
&= \int_{-\infty}^{\infty} f(x) 2^m K(2^m x, 2^m y)\, dx \\
&= \int_{-\infty}^{\infty} K_m(x, y) f(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
f(y) - (P_m f)(y) &= f(y) - \int_{-\infty}^{\infty} K_m(x, y) f(x)\, dx \\
&= f(y) \int_{-\infty}^{\infty} K_m(x, y)\, dx - \int_{-\infty}^{\infty} K_m(x, y) \\
&\cdot f(x)\, dx, \quad \int_{-\infty}^{\infty} K_m(x, y)\, dx = 1 \\
&= \int_{-\infty}^{\infty} K_m(x, y)(f(y) - f(x))\, dx \\
&= \int_{-\infty}^{y-1/(|m|+1)} K_m(x, y)(f(y) - f(x)) \\
&+ \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} K_m(x, y)(f(y) - f(x))\, dx \\
&+ \int_{y+1/(|m|+1)}^{\infty} K_m(x, y)(f(y) - f(x)) \\
|f(y) - (P_m f)(y)| &\le \int_{-\infty}^{y-1/(|m|+1)} |K_m(x, y)| \\
&\cdot |f(y) - f(x)| + \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x, y)| \\
&\cdot |(f(y) - f(x))|\, dx + \int_{y+1/(|m|+1)}^{\infty} |K_m(x, y)| \\
&\cdot |(f(y) - f(x))|\, dx = I_1 + I_2 + I_3 \text{ say.}
\end{aligned} \tag{28}
$$

Let $x, y \in [a, b]$. Select m such that

$$
\begin{aligned}
y + \frac{1}{|m| + 1} &< b, \\
x - \frac{1}{|m| + 1} &> a.
\end{aligned} \tag{29}
$$

Since it is known that

$$
\begin{aligned}
&\sup_{x \in [y-1/(|m|+1), y+1/(|m|+1)]} |f(x) - f(y)| \\
&\le \left| f\left(y + \frac{1}{|m| + 1}\right) - f(y) \right| \\
&+ \left| f\left(y - \frac{1}{|m| + 1}\right) - f(y) \right| \\
&= M_1 \left(\frac{1}{|m| + 1} \right)^{\alpha} + M_2 \left(\frac{1}{|m| + 1} \right)^{\alpha}, \quad f \in \mathrm{Lip}_{\alpha} \\
&= (M_1 + M_2) \frac{1}{(|m| + 1)^{\alpha}} \quad \text{where } M_1 > 0,\ M_2 > 0
\end{aligned} \tag{30}
$$

therefore

$$
\begin{aligned}
I_2 &= \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x, y)|\, |(f(x) - f(y))|\, dx \\
&\le \frac{M_1 + M_2}{(|m| + 1)^{\alpha}} \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x, y)|\, dx \\
&\le \frac{M_1 + M_2}{(|m| + 1)^{\alpha}} \int_{-\infty}^{\infty} |K_m(x, y)|\, dx = \frac{(M_1 + M_2) C}{(|m| + 1)^{\alpha}}, \\
&\qquad C > 0 \text{ by Lemma 15.}
\end{aligned} \tag{31}
$$

Next,

$$
\begin{aligned}
I_1 + I_3 &= \int_{-\infty}^{y-1/(|m|+1)} |K_m(x, y)|\, |f(y) - f(x)|\, dx \\
&+ \int_{y+1/(|m|+1)}^{\infty} |K_m(x, y)|\, |f(y) - f(x)|\, dx \\
&= \int_{a}^{y-1/(|m|+1)} |K_m(x, y)|\, |f(y) - f(x)|\, dx \\
&+ \int_{y+1/(|m|+1)}^{b} |K_m(x, y)|\, |f(y) - f(x)|\, dx \\
&\le \sup_{|x-y| \ge 1/(|m|+1)} |K_m(x, y)| \\
&\cdot \frac{M_3}{(|m| + 1)^{\alpha}} \left(\int_{a}^{y-1/(|m|+1)} dx + \int_{y+1/(|m|+1)}^{b} dx \right), \\
&\qquad M_3 > 0 \\
&\le \sup_{|x-y| \ge 1/(|m|+1)} |K_m(x, y)| \\
&\cdot \frac{M_3}{(|m| + 1)^{\alpha}} \left\{ (b - a) - \frac{2}{|m| + 1} \right\}, \\
&\le \frac{M_3 (b - a) C_1}{(|m| + 1)^{\alpha}}, \quad C_1 > 0.
\end{aligned} \tag{32}
$$

$\{K_m(x, y)\}$ is a quasi-positive delta sequence of function.

Now, collecting (50), (31), and (32), we have

$$
\begin{aligned}
|f(y) - (P_m f)(y)| &\le \frac{M_3 (b - a) C_1}{(|m| + 1)^{\alpha}} \\
&+ \frac{(M_1 + M_2) C}{(|m| + 1)^{\alpha}}
\end{aligned}
$$

$$= \frac{M_3 (b-a) C_1 + (M_1 + M_2) C}{(|m|+1)^\alpha} = O\left(\frac{1}{(|m|+1)^\alpha}\right). \tag{33}$$

Thus

$$\|f - P_m f\|_\infty = \sup_{y\in[a,b]} |f(y) - P_m f(y)| = O\left(\frac{1}{(|m|+1)^\alpha}\right). \tag{34}$$

Remark 16 (the converse of Theorem 11 is also true). Choose $x, y \in [a,b]$ such that $x - 1/(|m|+1)$, $y + 1/(|m|+1) \in [a,b]$ and $|y - x| = d/(|m|+1)$, $d > 0$.

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - (P_m f)(x) + (P_m f)(x) \\
&\quad - (P_m f)(y) + (P_m f)(y) - f(y)| \le |f(x) \\
&\quad - (P_m f)(x)| + |(P_m f)(x) - (P_m f)(y)| \\
&\quad + |(P_m f)(y) - f(y)| \le \sup_{x\in[a,b]} |f(x) - (P_m f)(x)| \\
&\quad + \sup_{x,y\in[a,b]} |(P_m f)(x) - (P_m f)(y)| \\
&\quad + \sup_{y\in[a,b]} |(P_m f)(y) - f(y)| = \|f - P_m f\|_\infty \\
&\quad + \sup_{x,y\in[a,b]} |(P_m f)(x-y)| + \|P_m f - f\|_\infty \\
&= O\left(\frac{1}{(|m|+1)^\alpha}\right) + O\left(|x-y|^\alpha\right) \\
&\quad + O\left(\frac{1}{(|m|+1)^\alpha}\right) = O\left(\frac{1}{(|m|+1)^\alpha}\right) \\
&\quad + O\left(\frac{1}{(|m|+1)^\alpha}\right) + O\left(\frac{1}{(|m|+1)^\alpha}\right) \\
&= O\left(|x-y|^\alpha\right).
\end{aligned} \tag{35}$$

Since $|f(x) - f(y)| = O|x-y|^\alpha$, therefore $f \in \mathrm{Lip}_\alpha[a,b]$.

Hence, Theorem 11 is completely established.

5.2. Proof of Theorem 12. By the proof of Theorem 11

$$\|P_m f - f\|_\infty = O\left(\frac{1}{(|m|+1)^\alpha}\right). \tag{36}$$

Since,

$$\frac{1}{(|m|+1)^\alpha} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{37}$$

therefore,

$$\|P_m f - f\|_\infty \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{38}$$

Thus,

$$P_m f \longrightarrow f \quad \text{as } m \longrightarrow \infty \text{ uniformly on } [a,b]. \tag{39}$$

Hence, Theorem 12 is completely established.

5.3. Proof of Theorem 13. Since,

$$f(x) = P_m f(x) + r_m f(x) \tag{40}$$

therefore,

$$\begin{aligned}
r_m f(x) &= f(x) - P_m f(x), \\
\|r_m f\|_\infty &= \|f - P_m f\|_\infty \\
&= O\left(\frac{1}{(|m|+1)^\alpha}\right), \text{by Theorem 11.}
\end{aligned} \tag{41}$$

Hence, Theorem 13 is completely established.

5.4. Proof of Theorem 14. Following the proof of Theorem 13,

$$\|r_m f\|_\infty = O\left(\frac{1}{(|m|+1)^\alpha}\right). \tag{42}$$

Since,

$$\|r_m f\|_\infty \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{43}$$

therefore,

$$r_m f \longrightarrow 0 \quad \text{as } m \longrightarrow \infty \text{ uniformly on } [a,b]. \tag{44}$$

Hence, Theorem 14 is completely established.

6. Notes

(1) $E_m(f) \to 0$ as $m \to \infty$ in Theorem 11; the wavelet approximations determined in this theorem are best possible in wavelet analysis (Zygmund, [15], p. 115).

(2) Define a function $\phi : [0,1] \to \mathbb{R}$ by

$$\begin{aligned}
\phi(t) &= |t-1|^\alpha - |t^\alpha - 1| \quad \forall t \in [0,1],\ 0 < \alpha < 1 \\
&= (1-t)^\alpha - (1 - t^\alpha) \\
&= (1-t)^\alpha + t^\alpha - 1 \quad \forall t \in [0,1].
\end{aligned}$$

$$\begin{aligned}
\therefore \phi'(t) &= -\alpha(1-t)^{\alpha-1} + \alpha t^{\alpha-1} \quad \forall t \in (0,1) \\
&= \alpha\left(\frac{1}{t^{1-\alpha}} - \frac{1}{(1-t)^{1-\alpha}}\right) \\
\therefore \phi'(t) &> 0, \quad \forall t \in \left(0, \frac{1}{2}\right) \\
&= 0, \quad t = \frac{1}{2} \\
&< 0, \quad \forall t \in \left(\frac{1}{2}, 1\right).
\end{aligned} \tag{45}$$

Therefore ϕ is monotonic increasing in $(0, 1/2)$ and monotonic decreasing in $(1/2, 1)$

$$\therefore \phi(t) \geq \phi(0) = \phi(1) = 0 \quad \forall t \in (0,1)$$

$$\therefore |t-1|^{\alpha} - |t^{\alpha} - 1| \geq 0 \tag{46}$$

$$\therefore |t-1|^{\alpha} \geq |t^{\alpha} - 1|.$$

Thus,

$$\left|\frac{x}{y} - 1\right|^{\alpha} \geq \left|\left(\frac{x}{y}\right)^{\alpha} - 1\right|, \quad t = \frac{x}{y}, \quad 0 < x, y < 1. \tag{47}$$

Hence $|x-y|^{\alpha} \geq |x^{\alpha} - y^{\alpha}|$

$$\text{that is } |x^{\alpha} - y^{\alpha}| \leq |x-y|^{\alpha} \quad 0 < x, y < 1. \tag{48}$$

A function $f : [a,b] \to \mathbb{R}$ is defined by $f(x) = (1-x^2)^{\alpha}$, $0 < \alpha < 1$, $\forall x \in [a,b] \subset [0,1]$.

$$\begin{aligned}
|f(x) - f(y)| &= \left|\left(1-x^2\right)^{\alpha} - \left(1-y^2\right)^{\alpha}\right| \\
&\leq \left|\left(1-x^2\right) - \left(1-y^2\right)\right|^{\alpha}, \quad |x^{\alpha} - y^{\alpha}| \leq |x-y|^{\alpha} \\
&= \left|y^2 - x^2\right|^{\alpha} = |x-y|^{\alpha} |x+y|^{\alpha} \\
&\leq |x-y|^{\alpha} \left\{|x|^{\alpha} + |y|^{\alpha}\right\} \\
&\leq \max\left\{|2a|^{\alpha}, |2b|^{\alpha}\right\} |x-y|^{\alpha} \\
&\leq 2^{\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right) |x-y|^{\alpha} \\
&= O\left(|x-y|^{\alpha}\right)
\end{aligned} \tag{49}$$

$\therefore f \in \mathrm{Lip}_{\alpha}[a,b]$, $0 < \alpha < 1$.

For this function,

$$\begin{aligned}
&|f(y) - (P_m f)(y)| \\
&\leq \int_{-\infty}^{y-1/(|m|+1)} |K_m(x,y)| \, |f(y) - f(x)| \, dx \\
&\quad + \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x,y)| \, |(f(y) - f(x))| \, dx \\
&\quad + \int_{y+1/(|m|+1)}^{\infty} |K_m(x,y)| \, |(f(y) - f(x))| \, dx \\
&= I_1 + I_2 + I_3 \text{ say.}
\end{aligned} \tag{50}$$

Then

$$\begin{aligned}
I_2 &= \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x,y)| \, |x-y|^{\alpha} |x+y|^{\alpha} \, dx \\
&\leq 2^{\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right) \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x,y)| \, |x-y|^{\alpha} \, dx, \\
&\qquad |x+y|^{\alpha} \leq 2^{\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right) \\
&\leq \frac{2^{2\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right)}{(|m|+1)^{\alpha}} \int_{y-1/(|m|+1)}^{y+1/(|m|+1)} |K_m(x,y)| \, dx, \\
&\qquad |x-y| \leq \frac{2}{|m|+1} \\
&\leq \frac{2^{2\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right)}{(|m|+1)^{\alpha}} \int_{-\infty}^{\infty} |K_m(x,y)| \, dx \\
&= \frac{2^{2\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right)}{(|m|+1)^{\alpha}}, \quad \int_{-\infty}^{\infty} |K_m(x,y)| \, dx = 1 \\
&= O\left(\frac{1}{(|m|+1)^{\alpha}}\right)
\end{aligned} \tag{51}$$

$$\begin{aligned}
I_1 + I_3 &= \int_{a}^{y-1/(|m|+1)} |K_m(x,y)| \, |f(y) - f(x)| \, dx \\
&\quad + \int_{y+1/(|m|+1)}^{b} |K_m(x,y)| \, |f(y) - f(x)| \, dx \\
&\leq \sup_{|x-y| \geq 1/(|m|+1)} |K_m(x,y)| \frac{2^{\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right)}{(|m|+1)^{\alpha}} \\
&\quad \cdot \left(\int_{a}^{y-1/(|m|+1)} dx + \int_{y+1/(|m|+1)}^{b} dx\right) \\
&\leq \sup_{|x-y| \geq 1/(|m|+1)} |K_m(x,y)| \frac{2^{\alpha}\left(|a|^{\alpha} + |b|^{\alpha}\right)}{(|m|+1)^{\alpha}} (b-a) \\
&= O\left(\frac{1}{(|m|+1)^{\alpha}}\right).
\end{aligned} \tag{52}$$

Hence

$$\sup_{y \in [a,b]} |f(y) - P_m f(y)| = O\left(\frac{1}{(|m|+1)^{\alpha}}\right). \tag{53}$$

$\|f(y) - P_m f(y)\|_{\infty} = O(1/(|m|+1)^{\alpha})$.

Thus, the result of Theorem 11 is verified for a function f defined by $f(x) = (1-x^2)^{\alpha}$, $0 < \alpha < 1$.

(3) *Hölder class*: Let $C_{2\pi}$ denote the Banach spaces of all 2π periodic continuous functions under sup-norm. For each $0 < \alpha \leq 1$ and some positive constant k, the function space H_{α} is defined by

$$H_{\alpha} = \left\{f \in C_{2\pi} : |f(x) - f(y)| \leq k|x-y|^{\alpha}\right\} \tag{54}$$

(Das et al. [16] p. 83).

Let $f \in H_{\alpha}$. Then

$$|f(x) - f(y)| = k|x-y|^{\alpha} = O\left(|x-y|^{\alpha}\right) \tag{55}$$

$\therefore f \in \mathrm{Lip}_{\alpha}$.

The function $f(x) = (1 - x^3)^\alpha$, $0 < \alpha < 1$, $\forall x \in [0, 1]$. It is continuous and belonging to Lip_α but $f(x + 2\pi) \neq f(x)$. Then $f \notin H_\alpha$.

Thus $H_\alpha \subsetneqq \mathrm{Lip}_\alpha$.

Theorem 11 is also valid for a function $f \in H_\alpha$.

Competing Interests

The authors declare that they have no competing interests.

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