Research Article
New Results on the (Super) Edge-Magic Deficiency of Chain Graphs

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Let G be a graph of order v and size e. An *edge-magic labeling* of G is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, v + e\}$ such that $f(x) + f(xy) + f(y)$ is a constant for every edge $xy \in E(G)$. An edge-magic labeling f of G with $f(V(G)) = \{1, 2, 3, \ldots, v\}$ is called a *super edge-magic labeling*. Furthermore, the *edge-magic deficiency* of a graph G, $\mu(G)$, is defined as the smallest nonnegative integer n such that $G \cup nK_1$ has an edge-magic labeling. Similarly, the *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is either the smallest nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such integer n. In this paper, we investigate the (super) edge-magic deficiency of chain graphs. Referring to these, we propose some open problems.

1. Introduction

Let G be a finite and simple graph, where $V(G)$ and $E(G)$ are its vertex set and edge set, respectively. Let $v = |V(G)|$ and $e = |E(G)|$ be the number of the vertices and edges, respectively. In [1], Kotzig and Rosa introduced the concepts of edge-magic labeling and edge-magic graph as follows: an *edge-magic labeling* of a graph G is a bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, v + e\}$ such that $f(x) + f(xy) + f(y)$ is a constant, called the *magic constant* of f, for every edge xy of G. A graph that admits an edge-magic labeling is called an *edge-magic graph*. A *super edge-magic labeling* of a graph G is an edge-magic labeling f of G with the extra property that $f(V(G)) = \{1, 2, 3, \ldots, e\}$. A *super edge-magic graph* is a graph that admits a super edge-magic labeling. These concepts were introduced by Enomoto et al. [2] in 1998.

In [1], Kotzig and Rosa introduced the concept of edge-magic deficiency of a graph. They define the *edge-magic deficiency* of a graph G, $\mu(G)$, as the smallest nonnegative integer n such that $G \cup nK_1$ is an edge-magic graph. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno et al. [3] introduced the concept of super edge-magic deficiency of a graph. The *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is defined as the smallest

nonnegative integer n such that $G \cup nK_1$ is a super edge-magic graph or $+\infty$ if there exists no such n.

A *chain graph* is a graph with blocks $B_1, B_2, \ldots, B_k$ such that, for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. We will denote the chain graph with k blocks $B_1, B_2, \ldots, B_k$ by $C[B_1, B_2, \ldots, B_k]$. If $B_1 = \cdots = B_t = B$, we will write $C[B_1, B_2, \ldots, B_k]$ as $C[B^{(t)}, B_{t+1}, \ldots, B_k]$. If, for every i, $B_i = H$ for a given graph H, then $C[B_1, B_2, \ldots, B_k]$ is denoted by kH-path. Suppose that $c_1, c_2, \ldots, c_{k-1}$ are the consecutive cut vertices of $C[B_1, B_2, \ldots, B_k]$. The *string* of $C[B_1, B_2, \ldots, B_k]$ is $(k - 2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$, where d_i is the distance between c_i and c_{i+1}, $1 \leq i \leq k - 2$. We will write $(d_1, d_2, \ldots, d_{k-2})$ as $(d^{(t)}, d_{t+1}, \ldots, d_{k-2})$, if $d_1 = \cdots = d_t = d$.

For any integer $m \geq 2$, let $L_m = P_m \times P_2$. Let TL_m and DL_m be the graphs obtained from the ladder L_m by adding a single diagonal and two diagonals in each rectangle of L_m, respectively. Thus, $|V(\mathrm{TL}_m)| = |V(\mathrm{DL}_m)| = 2m$, $|E(\mathrm{TL}_m)| = 4m - 3$, and $|E(\mathrm{DL}_m)| = 5m - 4$. TL_m and DL_m are called triangle ladder and diagonal ladder, respectively.

Recently, the author studied the (super) edge-magic deficiency of $k\mathrm{DL}_m$-path, $C[K_4^{(k)}, \mathrm{DL}_m, K_4^{(n)}]$, and kC_4-path with some strings. Other results on the (super) edge-magic

deficiency of chain graphs can be seen in [4]. The latest developments in this area can be found in the survey of graph labelings by Gallian [5]. In this paper, we further investigate the (super) edge-magic deficiency of chain graphs whose blocks are combination of TL_m and DL_m and K_4 and TL_m, as well as the combination of C_4 and L_m. Additionally, we propose some open problems related to the (super) edge-magic deficiency of these graphs. To present our results, we use the following lemmas.

Lemma 1 (see [6]). *A graph G is a super edge-magic graph if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, v\}$ such that the set $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of e consecutive integers.*

Lemma 2 (see [2]). *If G is a super edge-magic graph, then $e \leq 2v - 3$.*

2. Main Results

For $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = TL_m$ when j is odd and $B_j = DL_m$ when j is even. Thus G is a chain graph with $|V(G)| = (2m-1)k+1$ and $|E(G)| = (1/2)(k+1)(4m-3) + (1/2)(k-1)(5m-4)$ when k is odd, or $|E(G)| = (k/2)(4m-3) + (k/2)(5m-4)$ when k is even. By Lemma 2, it can be checked that G is not super edge-magic when $m \geq 3$ and k is even and when $m \geq 4$ and k is odd. As we can see later, when $m = 3$ and k is odd, G is super edge-magic. Next, we investigate the super edge-magic deficiency of G. Our first result gives its lower bound. This result is a direct consequence of Lemma 2, so we state the result without proof.

Lemma 3. *Let $k \geq 3$ be an integer. For any integer $m \geq 3$,*

$$\mu_s(G)$$
$$\geq \begin{cases} \left\lfloor \dfrac{1}{4}k(m-3) \right\rfloor + 1, & \text{if } k \text{ is even,} \\[2mm] \left\lfloor \dfrac{1}{4}(k(m-3)-(m-1)) \right\rfloor + 1, & \text{if } k \text{ is odd.} \end{cases} \quad (1)$$

Notice that the lower bound presented in Lemma 3 is sharp. We found that when m is odd, the chain graph G with particular string has the super edge-magic deficiency equal to its lower bound as we state in Theorem 4. First, we define vertex and edge sets of B_j as follows.

$V(B_j) = \{u_j^i, v_j^i : 1 \leq i \leq m\}$, for $1 \leq j \leq k$. $E(B_j) = \{u_j^i u_j^{i+1}, v_j^i v_j^{i+1} : 1 \leq i \leq m-1\} \cup \{e_j^i : \text{where } e_j^i \text{ is either } u_j^i v_j^{i+1} \text{ or } v_j^i u_j^{i+1}, 1 \leq i \leq m-1\} \cup \{u_j^i v_j^i : 1 \leq i \leq m\}$, for $1 \leq j \leq k$, when j is odd, and $E(B_j) = \{u_j^i u_j^{i+1}, v_j^i v_j^{i+1}, u_j^i v_j^{i+1}, v_j^i u_j^{i+1} : 1 \leq i \leq m-1\} \cup \{u_j^i v_j^i : 1 \leq i \leq m\}$, for $1 \leq j \leq k$, when j is even.

Theorem 4. *Let $k \geq 3$ be an integer and $G = C[B_1, B_2, \ldots, B_k]$ with string $(m-1, d_1, m-1, d_2, m-1, \ldots, d_{(1/2)(k-3)}, m-1)$ when k is odd or $(m-1, d_1, m-1, d_2, \ldots, m-1, d_{(1/2)(k-2)})$*

when k is even, where $d_1, d_2, \ldots, d_{\lfloor(1/2)(k-2)\rfloor} \in \{m-1, m\}$. For any odd integer $m \geq 3$,

$$\mu_s(G) = \begin{cases} \dfrac{1}{4}k(m-3) + 1, & \text{if } k \text{ is even,} \\[2mm] \dfrac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd.} \end{cases} \quad (2)$$

Proof. First, we define G as a graph with vertex set $V(G) = \bigcup_{j=1}^k V(B_j)$, where $u_j^m = v_{j+1}^1$, $1 \leq j \leq k-1$, and edge set $E(G) = \bigcup_{j=1}^k E(B_j)$. Under this definition, $u_j^m = v_{j+1}^1$, $1 \leq j \leq k-1$, are the cut vertices of G.

Next, for $1 \leq i \leq m$ and $1 \leq j \leq k$, define the labeling $f : V(G) \cup \alpha K_1 \rightarrow \{1, 2, 3, \ldots, (2m-1)k + 1 + \alpha\}$, where $\alpha = (1/4)k(m-3)+1$ when k is even or $\alpha = (1/4)(k-1)(m-3)$ when k is odd, as follows:

$f(x)$
$$= \begin{cases} \dfrac{1}{4}(j-1)(9m-7) + 2i - 1, & \text{if } x = u_j^i, \ j \text{ is odd,} \\[1.5mm] \dfrac{1}{4}(j-1)(9m-7) + 2i, & \text{if } x = v_j^i, \ j \text{ is odd,} \\[1.5mm] \beta + \dfrac{1}{2}(5i-3), & \text{if } x = u_j^i, \ i \text{ is odd, } j \text{ is even,} \\[1.5mm] \beta + \dfrac{1}{2}(5i-4), & \text{if } x = u_j^i, \ i \text{ is even, } j \text{ is even,} \\[1.5mm] \beta + \dfrac{1}{2}(5i-7), & \text{if } x = v_j^i, \ i \text{ is odd, } j \text{ is even,} \\[1.5mm] \beta + \dfrac{1}{2}(5i-6), & \text{if } x = v_j^i, \ i \text{ is even, } j \text{ is even,} \end{cases} \quad (3)$$

where $\beta = (1/4)(j-2)(9m-7) + 2m$.

Under the vertex labeling f, it can be checked that no labels are repeated, $f(u_j^m) = f(v_{j+1}^1)$, $1 \leq j \leq k-1$, $\{f(x) + f(y) : xy \in E(G)\}$ is a set of $|E(G)|$ consecutive integers, and the largest vertex label used is $(1/4)(k-2)(9m-7) + (1/2)(9m-3)$ when k is even or $(1/4)(k-1)(9m-7) + 2m$ when k is odd. Also, it can be checked that $f(u_j^i) + f(v_j^{i+1}) = f(v_j^i) + f(u_j^{i+1})$ when j is odd.

Next, label the isolated vertices in the following way.

Case k Is Odd. In this case, we denote the isolated vertices with $\{z_{2j-1}^l \mid 1 \leq l \leq (1/2)(m-3), 1 \leq j \leq (1/2)(k-1)\}$ and set $f(z_{2j-1}^l) = f(v_{2j-1}^m) + 5l$.

Case k Is Even. In this case, we denote the isolated vertices with $\{z_{2j-1}^l \mid 1 \leq l \leq (1/2)(m-3), 1 \leq j \leq k/2\} \cup \{z_0\}$ and set $f(z_{2j-1}^l) = f(v_{2j-1}^m) + 5l$ and $f(z_0) = f(v_k^m) + 1$.

By Lemma 1, f can be extended to a super edge-magic labeling of $G \cup \alpha K_1$ with the magic constant $(k/4)(27m-21)+5$ when k is even or $(1/4)(k-1)(27m-21) + 6m$ when k is odd. Based on these facts and Lemma 3, we have the desired result. $\square$

An example of the labeling defined in the proof of Theorem 4 is shown in Figure 1(a).

Notice that when $m = 3$ and k is odd, $\mu_s(G) = 0$. In other words, the chain graph G with string $(2, d_1, 2, d_2, 2, \ldots, d_{(1/2)(k-3)}, 2)$, where $d_i \in \{2, 3\}$, is super edge-magic
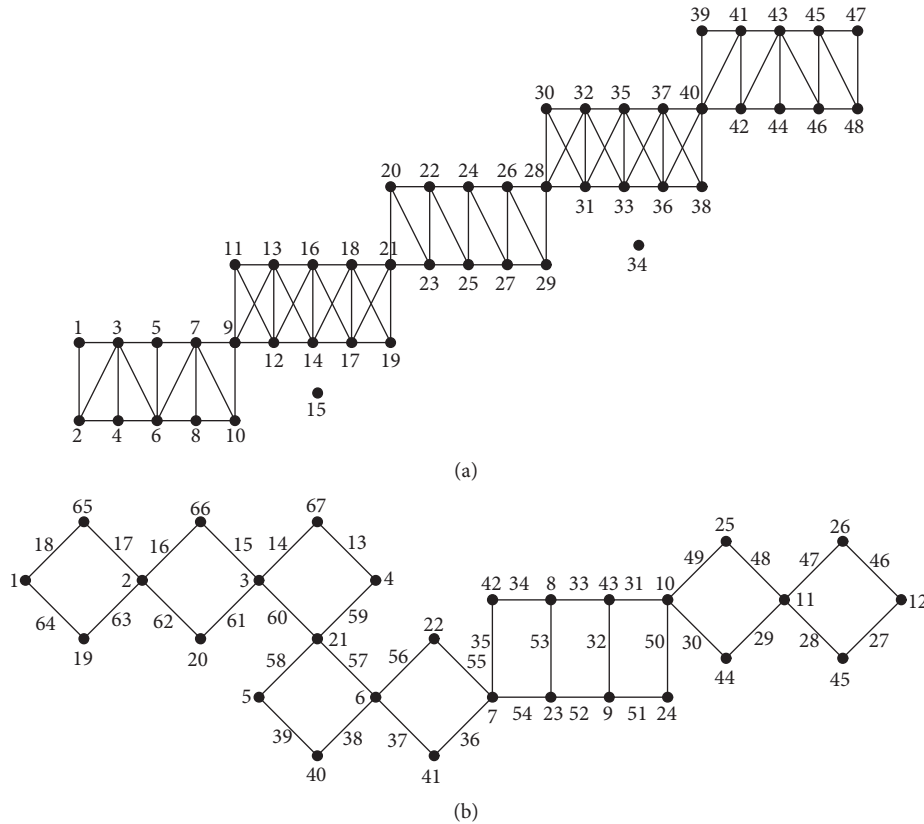
FIGURE 1: (a) Vertex labeling of $C[\mathrm{TL}_5, \mathrm{DL}_5, \mathrm{TL}_5, \mathrm{DL}_5, \mathrm{TL}_5] \cup 2K_1$ with string $(4, 5, 4)$. (b) Vertex and edge labelings of $c[C_4^{(3+2)}, L_4, C_4^{(2)}]$ with string $(2, 1^{(2)}, 2, 4, 2)$.

when $m = 3$ and k is odd. Based on this fact and previous results, we propose the following open problems.

Open Problem 1. Let $k \geq 3$ be an integer. For $m = 2$, decide if there exists a super edge-magic labeling of G. Further, for any even integer $m \geq 2$, find the super edge-magic deficiency of G.

Next, we investigate the super edge-magic deficiency of the chain graph $H = C[K_4^{(p)}, \mathrm{TL}_m, K_4^{(q)}]$ with string $(1^{(p-1)}, d, 1^{(q-1)})$, where $d \in \{m - 1, m\}$. H is a graph of order $3(p + q) + 2m$ and size $6(p + q) + 4m - 3$. We define the vertex and edge sets of H as follows: $V(H) = \{a_i, b_i : 1 \leq i \leq p\} \cup \{c_i : 1 \leq i \leq p + 1\} \cup \{u_j, v_j : 1 \leq j \leq m\} \cup \{x_t, y_t : 1 \leq t \leq q\} \cup \{z_t : 1 \leq t \leq q + 1\}$, where $c_{p+1} = u_1$ and $v_m = z_1$, and $E(H) = \{a_i b_i, a_i c_i, a_i c_{i+1}, b_i c_i, b_i c_{i+1}, c_i c_{i+1} : 1 \leq i \leq p\} \cup \{u_j v_j \mid 1 \leq j \leq m\} \cup \{u_j u_{j+1}, v_j v_{j+1} : 1 \leq j \leq m - 1\} \cup \{e_j : e_j \text{ is either } u_j v_{j+1} \text{ or } v_j u_{j+1}, 1 \leq j \leq m - 1\} \cup \{x_t y_t, x_t z_t, x_t z_{t+1}, y_t z_t, y_t z_{t+1}, z_t z_{t+1} : 1 \leq t \leq q\}$. Hence, the cut vertices of H are c_i, $2 \leq i \leq p + 1$, and z_t, $1 \leq t \leq q$. Notice that H has string $(1^{(p-1)}, m - 1, 1^{(q-1)})$, if at least one of e_j is $u_j v_{j+1}$, and its string is $(1^{(p-1)}, m, 1^{(q-1)})$, if $e_j = v_j u_{j+i}$ for every $1 \leq j \leq m - 1$.

Theorem 5. For any integers $p, q \geq 1$ and $m \geq 2$, $\mu_s(H) = 0$.

Proof. Define a bijective function $g : V(H) \rightarrow \{1, 2, 3, \ldots, 3(p + q) + 2m\}$ as follows:

$$g(x)$$
$$= \begin{cases} 3i - 2, & \text{if } x = a_i, \ 1 \leq i \leq p, \\ 3i, & \text{if } x = b_i, \ 1 \leq i \leq p, \\ 3i - 1, & \text{if } x = c_i, \ 1 \leq i \leq p + 1, \\ 3p + 2j, & \text{if } x = u_j, \ 1 \leq j \leq m, \\ 3p + 2j - 1, & \text{if } x = v_j, \ 1 \leq j \leq m, \\ 3p + 2m + 3t - 2, & \text{if } x = x_t, \ 1 \leq t \leq q, \\ 3p + 2m + 3t, & \text{if } x = y_t, \ 1 \leq t \leq q, \\ 3p + 2m + 3t - 4, & \text{if } x = z_t, \ 1 \leq t \leq q + 1. \end{cases} \quad (4)$$

Under the labeling g, it can be checked that $g(c_{p+1}) = g(u_1)$ and $g(v_m) = g(z_1)$. Also, it can be checked that $g(u_j) + g(v_{j+1}) = g(v_j) + g(u_{j+1})$, $1 \leq j \leq m - 1$, and $\{g(x) + g(y) \mid xy \in E(H)\} = \{3, 4, 5, \ldots, 6(p + q) + 4m - 1\}$. By Lemma 1, g can be extended to a super edge-magic labeling of H with the magic constant $9(p + q) + 6m$. Hence, $\mu_s(H) = 0$. $\square$

Open Problem 2. For any integers $p, q \geq 1$ and $m \geq 2$, find the super edge-magic deficiency of $C[K_4^{(p)}, \mathrm{TL}_m, K_4^{(q)}]$ with string $(1^{(p-1)}, d, 1^{(q-1)})$, where $d \in \{1, 2, 3, \ldots, m - 2\}$.

Next, we study the edge-magic deficiency of ladder L_m and chain graphs whose blocks are combination of C_4 and L_m with some strings. In [6], Figueroa-Centeno et al. proved that the ladder L_m is super edge-magic for any odd m and suspected that L_m is super edge-magic for any even $m > 2$. Here, we can prove that L_m is edge-magic for any $m \geq 2$ by showing its edge-magic deficiency is zero. The result is presented in Theorem 6.

Theorem 6. *For any integer $m \geq 2$, $\mu(L_m) = 0$.*

Proof. Let $V(L_m) = \{u_i, v_i : 1 \leq i \leq m\}$ and $E(G) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq m-1\} \cup \{u_i v_i : 1 \leq i \leq m\}$ be the vertex set and edge set, respectively, of L_m. It is easy to verify that the labeling $h : V(L_m) \cup E(L_m) \rightarrow \{1, 2, 3, \ldots, 5m-2\}$ is a bijection and, for every $xy \in E(L_m)$, $h(x) + h(xy) + h(y) = 6m$.

$$
h(x) = \begin{cases}
i, & \text{if } x = u_i, \ i \text{ is odd}, \\
3m + \dfrac{1}{2}(i-2), & \text{if } x = u_i, \ i \text{ is even}, \\
m + \dfrac{1}{2}(i+1), & \text{if } x = v_i, \ i \text{ is odd}, \\
i, & \text{if } x = v_i, \ i \text{ is even}, \\
3m - \dfrac{1}{2}(3i-1), & \text{if } x = u_i u_{i+1}, \ i \text{ is odd}, \\
3m - \dfrac{3}{2}i, & \text{if } x = u_i u_{i+1}, \ i \text{ is even}, \\
5m - \dfrac{3}{2}(i+1), & \text{if } x = v_i v_{i+1}, \ i \text{ is odd}, \\
5m - \dfrac{1}{2}(3i+2), & \text{if } x = v_i v_{i+1}, \ i \text{ is even}, \\
5m - \dfrac{1}{2}(3i+1), & \text{if } x = u_i v_i, \ i \text{ is odd}, \\
3m - \dfrac{1}{2}(3i-2), & \text{if } x = u_i v_i, \ i \text{ is even}.
\end{cases}
\tag{5}
$$

Thus, $\mu(L_m) = 0$ for every $m \geq 2$. ☐

Theorem 7. *Let p and $q \geq 1$ be integers.*

(a) *If $m \geq 2$ is an even integer and $F_1 = C[C_4^{(p)}, L_m, c_4^{(q)}]$ with string $(2^{(p-1)}, m, 2^{(q-1)})$, then $\mu(F_1) = 0$.*

(b) *If $m \geq 3$ is an odd integer and $F_2 = C[C_4^{(p)}, L_m, c_4^{(q)}]$ with string $(2^{(p-1)}, m-1, 2^{(q-1)})$, then $\mu(F_2) = 0$.*

Proof. (a) First, we introduce a constant λ as follows: $\lambda = 1$, if m is odd and $\lambda = 2$, if m is even. Next, we define F_1 as a graph with $V(F_1) = \{a_i, b_i : 1 \leq i \leq p\} \cup \{c_i : 1 \leq i \leq p+1\} \cup \{u_j, v_j : 1 \leq j \leq m\} \cup \{x_t, y_t : 1 \leq t \leq q\} \cup \{z_t : 1 \leq t \leq q+1\}$, where $c_{p+1} = v_1$ and $u_m = z_1$, and $E(H) = \{c_i a_i, c_i b_i, a_i c_{i+1}, b_i c_{i+1} : 1 \leq i \leq p\} \cup \{u_j v_j \mid 1 \leq j \leq m\} \cup \{u_j u_{j+1}, v_j v_{j+1} : 1 \leq j \leq m-1\} \cup \{z_t x_t, z_t y_t, x_t z_{t+1}, y_t z_{t+1} : 1 \leq t \leq q\}$. The cut vertices of F_1 are c_i, $2 \leq i \leq p+1$, and z_t, $1 \leq t \leq q$.

Next, define a bijection $f_1 : V(F_1) \cup E(F_1) \rightarrow \{1, 2, 3, \ldots, 7(p+q) + 5m - 2\}$ as follows:

$f_1(x)$

$$
= \begin{cases}
4(p+q) + 3m + i - 1, & \text{if } x = a_i, \ 1 \leq i \leq p, \\
p + q + m + i, & \text{if } x = b_i, \ 1 \leq i \leq p, \\
i, & \text{if } x = c_i, \ 1 \leq i \leq p+1, \\
5p + 4q + 3m + \dfrac{1}{2}(j-1), & \text{if } x = u_j, \ j \text{ is odd}, \\
p + j, & \text{if } x = u_j, \ j \text{ is even}, \\
p + j, & \text{if } x = v_j, \ j \text{ is odd}, \\
2p + q + m + \dfrac{j}{2}, & \text{if } x = v_j, \ j \text{ is even}, \\
5p + 4q + \gamma_1 + t, & \text{if } x = x_t, \ 1 \leq t \leq q, \\
2p + q + \gamma_2 + t, & \text{if } x = y_t, \ 1 \leq t \leq q, \\
p + m + t - 1, & \text{if } x = z_t, \ 1 \leq t \leq q+1, \\
4(p+q) + 3m + 1 - 2i, & \text{if } x = c_i a_i, \ 1 \leq i \leq p, \\
7(p+q) + 5m - 2i, & \text{if } x = c_i b_i, \ 1 \leq i \leq p, \\
4(p+q) + 3m - 2i, & \text{if } x = a_i c_{i+1}, \ 1 \leq i \leq p, \\
7(p+q) + 5m - 1 - 2i, & \text{if } x = b_i c_{i+1}, \ 1 \leq i \leq p, \\
2p + 4q + 3m - \dfrac{1}{2}(3j+1), & \text{if } x = u_j u_{j+1}, \ j \text{ is odd}, \\
2p + 4q + 3m - \dfrac{1}{2}(3j), & \text{if } x = u_j u_{j+1}, \ j \text{ is even}, \\
5p + 7q + 5m - \dfrac{1}{2}(3j+1), & \text{if } x = v_j v_{j+1}, \ j \text{ is odd}, \\
5p + 7q + 5m - \dfrac{1}{2}(3j+2), & \text{if } x = v_j v_{j+1}, \ j \text{ is even}, \\
2p + 4q + 3m - \dfrac{1}{2}(3i-1), & \text{if } x = u_j v_j, \ j \text{ is odd}, \\
5p + 7q + 5m - \dfrac{3}{2}j, & \text{if } x = u_j v_j, \ j \text{ is even}, \\
2p + 4q + \gamma_3 - 2t & \text{if } x = z_t x_t, \ 1 \leq t \leq q, \\
5p + 7q + \gamma_4 - 2t, & \text{if } x = z_t y_t, \ 1 \leq t \leq q, \\
2p + 4q + \gamma_5 - 2t, & \text{if } x = x_t z_{t+1}, \ 1 \leq t \leq q, \\
5p + 7q + \gamma_6 - 2t, & \text{if } x = y_t z_{t+1}, \ 1 \leq t \leq q,
\end{cases}
\tag{6}
$$

where $\gamma_1 = (1/2)(\lambda-1)(7m-2) - (1/2)(\lambda-2)(7m-1)$, $\gamma_2 = (1/2)(\lambda-1)(3m) - (1/2)(\lambda-2)(3m-1)$, $\gamma_3 = (1/2)(\lambda-1)(3m+4) - (1/2)(\lambda-2)(3m+3)$, $\gamma_4 = (1/2)(\lambda-1)(7m+2) - (1/2)(\lambda-2)(7m+3)$, $\gamma_5 = (1/2)(\lambda-1)(3m+2) - (1/2)(\lambda-2)(3m+1)$, and $\gamma_6 = (1/2)(\lambda-1)(7m) - (1/2)(\lambda-2)(7m+1)$. It is easy to verify that, for every edge $xy \in E(F_1)$, $f(x) + f(xy) + f(y) = 8(p+q) + 6m$.

(b) We define F_2 as graph with $V(F_2) = V(F_1)$, where $c_{p+1} = v_1$ and $v_m = z_1$, and $E(F_2) = E(F_1)$. Under this definition, the cut vertices of F_2 are c_i, $2 \leq i \leq p+1$, and z_t, $1 \leq t \leq q$. Next, we define a bijection $f_2 : V(F_2) \cup E(F_2) \rightarrow \{1, 2, 3, \ldots, 7(p+q) + 5m - 2\}$, where $f_2(x) = f_1(x)$ for all $x \in V(F_2) \cup E(F_2)$. It can be checked that f_2 is an edge-magic labeling of F_2 with the magic constant $8(p+q) + 6m$. ☐

Open Problem 3. Let p and $q \geq 1$ be integers.

(a) If $m \geq 3$ is an odd integer, find the super edge-magic deficiency of $C[C_4^{(p)}, L_m, c_4^{(q)}]$ with string $(2^{(p-1)}, m, 2^{(q-1)})$.

(b) If $m \geq 2$ is an even integer, find the super edge-magic deficiency of $C[C_4^{(p)}, L_m, c_4^{(q)}]$ with string $(2^{(p-1)}, m-1, 2^{(q-1)})$.

Theorem 8. *Let $p, q \geq 2$ and $r \geq 1$ be integers.*

(a) *If $m \geq 2$ is an even integer and $H_1 = C[C_4^{(p+q)}, L_m, c_4^{(r)}]$ with string $(2^{(p-2)}, 1^{(2)}, 2^{(q-1)}, m, 2^{(r-1)})$, then $\mu(H_1) = 0$.*

(b) *If $m \geq 3$ is an odd integer and $H_2 = C[C_4^{(p+q)}, L_m, c_4^{(r)}]$ with string $(2^{(p-2)}, 1^{(2)}, 2^{(q-1)}, m-1, 2^{(r-1)})$, then $\mu(H_2) = 0$.*

Proof. (a) First, we define H_1 as a graph with $V(H_1) = \{a_i : 1 \leq i \leq 2p\} \cup \{b_i : 1 \leq i \leq p+1\} \cup \{u_j : 1 \leq j \leq 2q\} \cup$ $\{v_j : 1 \leq j \leq q+1\} \cup \{w_s : 1 \leq s \leq 2m\} \cup \{x_t : 1 \leq t \leq 2r\} \cup \{y_t : 1 \leq t \leq r+1\}$, where $a_{2p} = u_1$, $v_{q+1} = w_1$, and $w_{2m} = y_1$, and $E(H_1) = \{b_i a_i, b_i a_{p+i}, a_i b_{i+1}, a_{p+i} b_{i+1} : 1 \leq i \leq p\} \cup \{v_j u_j, v_j u_{q+j}, u_j v_{j+1}, u_{q+j} v_{j+1} \mid 1 \leq j \leq q\} \cup \{w_s w_{s+1}, w_{m+s} w_{m+s+1} : 1 \leq s \leq m-1\} \cup \{w_s w_{m+s} : 1 \leq s \leq m\} \cup \{y_t x_t, y_t x_{r+t}, x_t y_{t+1}, x_{r+t} y_{t+1} : 1 \leq t \leq r\}$.

Next, define a bijection $g_1 : V(H_1) \cup E(H_1) \rightarrow \{1, 2, 3, \ldots, 7(p + q + r) + 5m - 2\}$ as follows:

$$g_1(z) = \begin{cases} 6p + 7(q+r) + 5m + i - 2, & \text{if } z = a_i,\ 1 \leq i \leq p, \\ 3p + q + r + m + 1 + i, & \text{if } z = a_{p+i},\ 1 \leq i \leq p, \\ i, & \text{if } z = b_i,\ 1 \leq i \leq p+1, \\ 4p + q + r + m + j, & \text{if } z = u_j,\ 1 \leq j \leq q, \\ 4(p+q+r) + 3m + j - 1, & \text{if } z = u_{q+j},\ 1 \leq j \leq q, \\ p + 1 + j, & \text{if } z = v_j,\ 1 \leq j \leq q+1, \\ p + q + 1 + s, & \text{if } z = w_s,\ s \text{ is odd}, \\ 4p + 2q + r + m + \frac{1}{2}s, & \text{if } z = w_s,\ s \text{ is even}, \\ 4p + 5q + 4r + 3m + \frac{1}{2}(s-1), & \text{if } z = w_{m+s},\ s \text{ is odd}, \\ p + q + 1 + s, & \text{if } z = w_{m+s},\ s \text{ is even}, \end{cases}$$

$$g_1(z) = \begin{cases} 4p + 2q + r + \gamma_2 + t, & \text{if } z = x_t,\ 1 \leq t \leq r, \\ 4p + 5q + 4r + \gamma_1 + t, & \text{if } z = x_{r+t},\ 1 \leq t \leq r, \\ p + q + m + t, & \text{if } z = y_t,\ 1 \leq t \leq r+1, \\ 3p + q + r + m + 3 - 2i, & \text{if } z = b_i a_i,\ 1 \leq i \leq p, \\ 6p + 7(q+r) + 5m - 2i, & \text{if } z = b_i a_{p+i},\ 1 \leq i \leq p, \\ 3p + q + r + m + 2 - 2i, & \text{if } z = a_i b_{i+1},\ 1 \leq i \leq p, \\ 6p + 7(q+r) + 5m - 2i - 1, & \text{if } z = a_{p+i} b_{i+1},\ 1 \leq i \leq p, \\ 4p + 7(q+r) + 5m - 2j, & \text{if } z = v_j u_j,\ 1 \leq j \leq q, \\ 4(p+q+r) + 3m + 1 - 2j, & \text{if } z = v_j u_{q+j},\ 1 \leq j \leq q, \\ 4p + 7(q+r) + 5m - 2j - 1, & \text{if } z = u_j v_{j+1},\ 1 \leq j \leq q, \\ 4(p+q+r) + 3m - 2j, & \text{if } z = u_{q+j} v_{j+1},\ 1 \leq j \leq q, \\ 4p + 5q + 7r + 5m - \frac{1}{2}(3s+1), & \text{if } z = w_s w_{s+1},\ s \text{ is odd}, \\ 4p + 5q + 7r + 5m - \frac{1}{2}(3s+2), & \text{if } z = w_s w_{s+1},\ s \text{ is even}, \\ 4p + 2q + 4r + 3m - \frac{1}{2}(3s+1), & \text{if } z = w_{m+s} w_{m+s+1},\ s \text{ is odd}, \\ 4p + 2q + 4r + 3m - \frac{1}{2}(3s), & \text{if } z = w_{m+s} w_{m+s+1},\ s \text{ is even}, \\ 4p + 2q + 4r + 3m - \frac{1}{2}(3s-1), & \text{if } z = w_s w_{m+s},\ s \text{ is odd}, \\ 4p + 5q + 7r + 5m - \frac{3}{2}s, & \text{if } z = w_s w_{m+s},\ s \text{ is even}, \\ 4p + 5q + 7r + \gamma_4 - 2t, & \text{if } z = y_t x_t,\ 1 \leq t \leq r, \\ 4p + 2q + 4r + \gamma_3 - 2t, & \text{if } z = y_t x_{r+t},\ 1 \leq t \leq r, \\ 4p + 5q + 7r + \gamma_6 - 2t, & \text{if } z = x_t y_{t+1},\ 1 \leq t \leq r, \\ 4p + 2q + 4r + \gamma_5 - 2t, & \text{if } z = x_{r+t} y_{t+1},\ 1 \leq t \leq r, \end{cases}$$

(7)

where γ_1, γ_2, γ_3, γ_4, γ_5, γ_6, and λ are defined as in the proof of Theorem 7. It can be checked that, for every edge $xy \in E(H_1)$, $g_1(x)+g_1(xy)+g_1(y) = 9p+8(q+r)+6m+1$. Hence $\mu(H_1) = 0$.

An illustration of the labeling defined in the proof of Theorem 8 is given in Figure 1(b).

(b) We define H_2 as graph with $V(H_2) = V(H_1)$, where $a_{2p} = u_1$, $v_{q+1} = w_1$, and $w_m = y_1$, and $E(H_2) = E(H_1)$. It can be checked that $g_2 : V(H_2) \cup E(H_2) \rightarrow \{1, 2, 3, \ldots, 7(p+q+r)+ 5m - 2\}$ defined by $g_2(x) = g_1(x)$, for all $x \in V(H_2) \cup E(H_2)$, is an edge-magic labeling of H_2 with the magic constant $9p + 8(q + r) + 6m + 1$. □

Open Problem 4. Let $p, q \geq 2$ and $r \geq 1$ be integers.

(a) If $m \geq 3$ is an odd integer, find the edge-magic deficiency of $C[C_4^{(p)}, c_4^{(q)}, L_m, c_4^{(r)}]$ with string $(2^{(p-2)}, 1^{(2)}, 2^{(q-1)}, m, 2^{(r-1)})$.

(b) If $m \geq 2$ is an even integer, find the edge-magic deficiency of $C[C_4^{(p)}, c_4^{(q)}, L_m, c_4^{(r)}]$ with string $(2^{(p-2)}, 1^{(2)}, 2^{(q-1)}, m - 1, 2^{(r-1)})$.
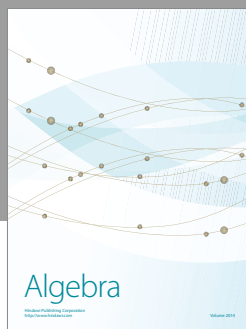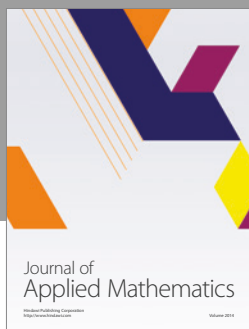
Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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