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Research Article

Greedy Expansions with Prescribed Coefficients in Hilbert Spaces

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Greedy expansions with prescribed coefficients, which have been studied by V. N. Temlyakov in Banach spaces, are considered here in a narrower case of Hilbert spaces. We show that in this case the positive result on the convergence does not require monotonicity of coefficient sequence $\mathscr{C}$. Furthermore, we show that the condition sufficient for the convergence, namely, the inclusion $\mathscr{C} \in \ell^2 \setminus \ell^1$, can not be relaxed at least in the power scale. At the same time, in finite-dimensional spaces, the condition $\mathscr{C} \in \ell^2$ can be replaced by convergence of $\mathscr{C}$ to zero.

1. Introduction

Expansion in Fourier series [1] is a classical and comprehensively studied tool of theoretical and applied mathematics which takes an expanded function as an input and constructs a sequence of its expansion coefficients. Greedy expansions [2, 3], which are equivalent in the simplest case to Fourier series reordered by decreasing norms of terms and known in statistics and signal processing as Projection Pursuit Regression [4, 5] and Matching Pursuit [6], respectively, perform parallel computation of expansion coefficients and selection of expansion elements from a predefined dictionary. V. N. Temlyakov [3, 7] (see also [8]) proposed a type of a greedy expansion that performs only selection of expansion elements, while coefficients are prescribed in advance. The definition proposed by V. N. Temlyakov for the case of Banach spaces, in the case of Hilbert spaces, takes the following form.

Definition 1. Let H be a Hilbert space over $\mathbb{R}$ with a scalar product $(\cdot,\cdot)$, D be a symmetric unit-normed dictionary in H (i.e., $\overline{\operatorname{span} D} = H$, all elements in D have a unit norm, and if $g \in D$, then $-g$ also belongs to D). In addition, let $f \in H$, $t \in (0;1]$, and $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. We define inductively a sequence of remainders $\{r_n\}_{n=0}^{\infty} \subset H$ and a sequence of expanding elements $\{e_n\}_{n=1}^{\infty} \subset D$. First, we set $r_0 = f$. Then, if $r_{n-1} \in H$ $(n \in 1, 2, 3, \ldots)$ has already been defined, we select e_n as an (arbitrary) element which satisfies the condition

$$\left(r_{n-1}, e_n\right) \geq t \sup_{e \in D} \left(r_{n-1}, e\right), \qquad (1)$$

and set $r_n = r_{n-1} - c_n e_n$.

The series $\sum_{n=1}^{\infty} c_n e_n(f)$ is called a greedy expansion of f in the dictionary D with the prescribed coefficients $\mathscr{C}$ and the weakness parameter t.

It immediately follows from the definition of a greedy expansion that $r_N = f - \sum_{n=1}^{N} c_n e_n(f)$ $(N \in \mathbb{N})$, and hence the convergence of the expansion to an expanded element is equivalent to the convergence of remainders r_n to zero as $n \longrightarrow \infty$.

As a selection of an expanding element e_n is potentially not unique, there may exist different realizations of a greedy expansion for a given dictionary D, weakness parameter t and sequence of coefficients $\mathscr{C}$. Furthermore, for $t = 1$ greedy expansion may turn out to be nonrealizable due to the absence of an element $e \in D$ which provides $\sup_{e \in D}(r_{n-1}, e)$.

V. N. Temlyakov showed [3, Theorem 2.1] that if a number of conditions hold which are equivalent in the case of Hilbert spaces to the divergence of the series $\sum_{n=1}^{\infty} c_n$ and the convergence of the series $\sum_{n=1}^{\infty} c_n^2$, a greedy expansion with prescribed coefficients $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ converges to an expanded element at least for a subsequence of indexes, i.e., $\liminf_{n\to\infty}\|r_n\| = 0$. Later V. N. Temlyakov proved the standard convergence (i.e., $\lim_{n\to\infty} r_n = 0$) under the additional condition of monotonicity of $\mathscr{C}$ [7, Theorem 4]. Yet, it remained unknown whether the condition $\mathscr{C} \in \ell^2$ and the monotonicity condition could be essentially relaxed without violating the guaranteed convergence to an expanded element.

2. Main Results

We start with a positive result which states that in Hilbert spaces the monotonicity is not required for the standard convergence. Namely, the following theorem holds.

Theorem 2. *Let H be a Hilbert space, D be a symmetric unit-normed dictionary in H, $t \in (0,1]$, $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ be a sequence of positive numbers which satisfies the conditions*

$$\sum_{n=1}^{\infty} c_n = \infty; \qquad \sum_{n=1}^{\infty} c_n^2 < \infty \tag{2}$$

(i.e., $\mathscr{C} \in \ell^2 \setminus \ell^1$). Then for every element $f \in H$ all realizations of its greedy expansion in the dictionary D with the prescribed coefficients $\mathscr{C}$ and the weakness parameter t converge to f.

It is clear that if the first condition on $\mathscr{C}$ is violated, then there is no convergence to an expanded element for all f with the norm exceeding the sum $\sum_{n=1}^{\infty} c_n$. The significance of the second condition on $\mathscr{C}$ follows from the following theorem.

Theorem 3. *There exist a Hilbert space H, a symmetric unit-normed dictionary $D \subset H$, an element $f \in H$ and a sequence of positive numbers $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ such that*

$$\sum_{n=1}^{\infty} c_n = \infty, \qquad c_n \leqslant \frac{1}{\sqrt{n}} \quad \forall n \in \{1, 2, 3 \ldots\}, \tag{3}$$

but a greedy expansion of f in the dictionary D with the prescribed coefficients $\mathscr{C}$ and the weakness parameter $t = 1$ does not converge to f.

As for the second condition of Theorem 2 a boundary in the power scale is $1/\sqrt{n}$, Theorem 3 in fact shows that this condition in Theorem 2 can not be relaxed at least in the power scale.

However, the question about a possibility of a more delicate relaxation of the condition $\mathscr{C} \in \ell^2$ remains open. This question can be stated as follows: is it true that for every sequence $\mathscr{C}$ of positive numbers that converges to zero but does not belong to ℓ^2 there exist a Hilbert space H, a symmetric unit-normed dictionary $D \subset H$ and an expanded element $f \in H$ such that at least one realization of greedy expansion of f in D with the prescribed coefficients $\mathscr{C}$ (and, e.g., the weakness parameter $t = 1$) does not converge to f?

We note that assertions similar to Theorems 2 and 3 have been announced by O. Rassudova in her conference talk [9], but the proofs have not been published. To the best of our knowledge, in her proof of an analogue of Theorem 3 O. Rassudova used a modification of the construction [10, Theorem 3] which is based on analytical estimates and does not have a clear geometric interpretation. The construction presented in our work is geometrically demonstrative.

We also note that at least for $t = 1$ for the natural class of monotonic coefficients in case of finite-dimensional Hilbert spaces the condition $\sum_{n=1}^{\infty} c_n^2 < \infty$ in Theorem 2 can be replaced by an essentially weaker condition $c_n \to 0$ $(n \to \infty)$. The proof of this fact is presented in section *The case of finite-dimensional spaces*.

3. Proof of Theorem 2

The theorem can be easily derived from the equality $\liminf_{n\to\infty}\|r_n\| = 0$, which holds due to the aforementioned result by V. N. Temlyakov [3, Theorem 2.1]. From the definition of the greedy expansion it immediately follows that

$$\begin{aligned} \|r_m\|^2 &= (r_{m-1} - c_m e_m, r_{m-1} - c_m e_m) \\ &= \|r_{m-1}\|^2 - 2c_m (r_{m-1}, e_m) + c_m^2. \end{aligned} \tag{4}$$

As coefficients c_n are positive and the dictionary is symmetric, $2c_m(r_{m-1}, e_m) \geqslant 0$. Hence

$$\|r_m\|^2 \leqslant \|r_{m-1}\|^2 + c_m^2, \tag{5}$$

and thus

$$\|r_{m+k}\|^2 \leqslant \|r_m\|^2 + \sum_{j=m+1}^{m+k} c_j^2. \tag{6}$$

The condition $\mathscr{C} = \{c_n\}_{n=1}^{\infty} \in \ell^2$ implies that

$$\forall \varepsilon > 0 \ \ \exists N_1 > 0 : \ \ \forall m > N_1 \quad \sum_{j=m+1}^{\infty} c_j^2 < \frac{\varepsilon}{2}. \tag{7}$$

Due to the equality $\liminf_{n\to\infty}\|r_n\| = 0$ we have that

$$\forall \varepsilon > 0 \ \ \forall N > 0 \ \ \exists m > N : \ \ \|r_m\|^2 < \frac{\varepsilon}{2}. \tag{8}$$

From two last assertions we obtain that for every $\varepsilon > 0$ there exists $m > 0$ such that the following two conditions simultaneously hold:

$$\sum_{j=m+1}^{\infty} c_j^2 < \frac{\varepsilon}{2}, \qquad \|r_m\|^2 < \frac{\varepsilon}{2}. \tag{9}$$

Thus using estimate (6) we get that $\|r_{m+k}\|^2 < \varepsilon$ for all $k \in \mathbb{N}$. But according to the definition of the limit it directly means that $\lim_{n\to\infty}\|r_n\| = 0$. The proof of Theorem 2 is complete.

We note that for monotonic coefficients and the weakness parameter $t = 1$ the statement of Theorem 2, which is covered in this case by [7, Theorem 4], can be also derived as a corollary of the same result by V. N. Temlyakov about the convergence for a subsequence of indexes [3, Theorem 2.1] and the following lemma. We find this lemma to be interesting on its own.

Lemma 4. *Let $t = 1$ and c_n monotonically converge to zero as $n \longrightarrow \infty$. Then for every realization of greedy expansion with the prescribed coefficients $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ the sequence of norms of its remainders $\{\|r_n\|\}_{n=0}^{\infty}$ converges.*

We begin our proof of this lemma with an estimate of a possible increase of the remainder norms. Let M denote the set of all indexes n for which $\|r_{n+1}\| > \|r_n\|$. If the set M is finite, then starting from a certain index the sequence $\{\|r_n\|\}_{n=0}^{\infty}$ is monotonic and thus convergent. Hence it remains to consider the case of countably infinite M.

For the sake of brevity we denote the scalar products (r_n, e_{n+1}) by x_{n+1} $(n = 0, 1, 2, \ldots)$. It follows from the definition of the greedy expansion and the symmetric property of the dictionary that all x_n are nonnegative.

The Pythagorean theorem implies that for all indexes n the equality

$$\|r_{n+1}\|^2 = \|r_n\|^2 - x_{n+1}^2 + (c_{n+1} - x_{n+1})^2 \tag{10}$$

holds. If $n \in M$, then $c_{n+1} > 2x_{n+1}$, $\|r_{n+1}\|^2 \leqslant \|r_n\|^2 + c_{n+1}^2$ and thus

$$x_{n+2} = \sup_{e \in D} (r_{n+1}, e) \geqslant -(r_{n+1}, e_{n+1}) = c_{n+1} - x_{n+1}. \tag{11}$$

Consequently

$$\begin{aligned}
\|r_{n+2}\|^2 &= \|r_{n+1}\|^2 - x_{n+2}^2 + (c_{n+2} - x_{n+2})^2 \\
&= \|r_n\|^2 - x_{n+1}^2 - x_{n+2}^2 + (c_{n+1} - x_{n+1})^2 \\
&\quad + (c_{n+2} - x_{n+2})^2 \\
&= \|r_n\|^2 + c_{n+1}^2 + c_{n+2}^2 - 2c_{n+1}x_{n+1} - 2c_{n+2}x_{n+2} \\
&\leqslant \|r_n\|^2 + c_{n+1}^2 + c_{n+2}^2 - 2c_{n+1}x_{n+1} \\
&\quad - 2c_{n+2}(c_{n+1} - x_{n+1}) \\
&= \|r_n\|^2 + (c_{n+1} - c_{n+2})^2 - 2x_{n+1}(c_{n+1} - c_{n+2}) \\
&\leqslant \|r_n\|^2 + (c_{n+1} - c_{n+2})^2.
\end{aligned} \tag{12}$$

At the same time

$$\begin{aligned}
\|r_{n+2}\|^2 &= \|r_{n+1}\|^2 + c_{n+2}^2 - 2c_{n+2}x_{n+2} \\
&= \|r_{n+1}\|^2 + c_{n+2}(c_{n+2} - 2x_{n+2}) \\
&\leqslant \|r_{n+1}\|^2 + c_{n+2}(c_{n+2} - 2(c_{n+1} - x_{n+1})) \\
&= \|r_{n+1}\|^2 \\
&\quad + c_{n+2}((c_{n+2} - c_{n+1}) + (2x_{n+1} - c_{n+1})) \\
&\leqslant \|r_{n+1}\|^2.
\end{aligned} \tag{13}$$

It means that if the remainder norm increased at the transition from r_n to r_{n+1}, then the increase of the norm square does not exceed c_{n+1}^2, for the next expansion step the increase is impossible, and the joint increase of the square of remainder norm for two steps of the expansion does not exceed $(c_{n+1} - c_{n+2})^2$ and hence does not exceed $C(c_{n+1} - c_{n+2})$, where C can be set to $2c_1$.

Having this estimate, let us complete the proof. We note that the series $\sum_{n \in M}(c_{n+1} - c_{n+2})$ converges: it can be easily derived either from the Leibniz's alternating series test or from the inequality

$$\begin{aligned}
\sum_{n \in M, n \leqslant K} (c_{n+1} - c_{n+2}) &\leqslant \sum_{n=0}^{K} (c_{n+1} - c_{n+2}) = c_1 - c_{K+2} \\
&\leqslant c_1.
\end{aligned} \tag{14}$$

Let us fix an arbitrary positive ε and find an index N_0 such that $\sum_{n \in M, n > N_0}(c_{n+1} - c_{n+2}) < \varepsilon/(4C)$ and simultaneously $\sup_{n > N_0} c_n^2 < \varepsilon/4$. Next we find an index $N_1 > N_0$ such that $\|r_{N_1}\|^2 < r^2 + \varepsilon/4$, where r denotes the infimum of the remainder norms $\{\|r_n\|\}_{n > N_0}$. Then for every $n > N_1$ we have that

$$\begin{aligned}
r^2 \leqslant \|r_n\|^2 &\leqslant \left\|r_{N_1}\right\|^2 + \sum_{n \in M, n \geqslant N_1} C(c_{n+1} - c_{n+2}) + c_n^2 \\
&< r^2 + \varepsilon.
\end{aligned} \tag{15}$$

Hence $\|r_n\|^2 \longrightarrow r^2$ $(n \longrightarrow \infty)$ and consequently $\|r_n\| \longrightarrow r$. The proof of Lemma 4 is complete.

4. Proof of Theorem 3

Our proof of Theorem 3 includes the following blocks: construction of a dictionary with simultaneous construction of coefficients $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$; description of realization of greedy expansion; proof of the absence of convergence to the expanded element; obtaining the required estimate of c_n. As a Hilbert space H we take an arbitrary infinite-dimensional separable space, e.g., ℓ^2.

Figure 1 illustrates certain steps of the proof.

4.1. Description of the Construction. We first present the structure of the example; i.e., we describe the construction of dictionary elements $\{e_n\}$ and coefficients $\{c_n\}$. As a part of this construction we also define the sequence of vectors (remainders) $\{r_n\}$, including the expanded element $f = r_0$.

Let f be an arbitrary non-zero element of H with $\|f\| \leqslant 1/2$, $r_0 = f$. We define dictionary elements e_{-1} and e_0 as arbitrary (unequal) unit vectors such that e_{-1}, e_0 and r_0 lie in one plane and the angle α_0 between e_0 and r_0 equals

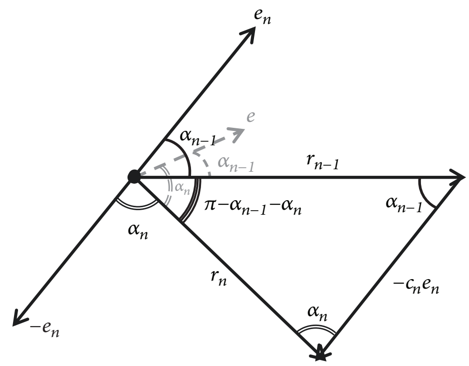

Figure 1: An illustration for certain steps of the proof of Theorem 3. All vectors except e lie in the vertical plane; e (as well as r_{n-1}) lies in the horizontal plane; e represents any vector from the set $\{e_{-1}, e_0, -e_1, -e_2, \ldots, -e_{n-1}\}$. The spherical law of cosines is applied to angles formed by the vectors r_{n-1}, r_n, e. The law of sines is applied to the triangle formed by vectors $r_{n-1}, r_n, -c_n e_n$.

the angle between e_{-1} and r_0 and belongs to the interval $(\pi/4, \pi/2)$.

Next, we set dictionary element e_1 to an arbitrary unit vector with the following two properties: its orthogonal projection on the plane $< e_{-1}, e_0 >$ lies on the line with directing vector r_0, and the angle between this vector and r_0 also equals α_0. We also find the coefficient c_1 such that for $r_1 = r_0 - c_1 e_1$ scalar products $(r_1, -e_1), (r_1, e_0), (r_1, e_{-1})$ are equal or, equivalently, angles between r_1 and the vectors e_{-1}, e_0, $-e_1$ are equal. We denote this angle by α_1. Note that $\alpha_1 \in (\alpha_0, \pi/2)$: the formal justification of this fact can be based, e.g., on the equality $\cos\alpha_1 = \cos\alpha_0 \cos(\pi - \alpha_0 - \alpha_1)$, which directly follows from the spherical law of cosines.

Similarly, we set e_2 to an arbitrary unit vector with an orthogonal projection on the subspace $< e_{-1}, e_0, e_1 >$ lying on the line with directing vector r_1 and the angle between this vector and r_1 equal α_1, and select the coefficient c_2 in such a way that for $r_2 = r_1 - c_2 e_2$ scalar products $(r_2, -e_2)$, $(r_2, -e_1)$, (r_2, e_0), (r_2, e_{-1}) are equal or, equivalently, angles between r_2 and the vectors e_{-1}, e_0, $-e_1$, $-e_2$ are equal. We denote this angle by α_2. Again it is easy to see that $\alpha_2 \in (\alpha_1, \pi/2)$, as due to the spherical law of cosines $\cos\alpha_2 = \cos\alpha_1 \cos(\pi - \alpha_1 - \alpha_2)$.

We continue the construction inductively. Namely, after constructing $\{r_j\}_{j=0}^{n-1}$, $\{e_j\}_{j=-1}^{n-1}$, $\{c_j\}_{j=1}^{n-1}$ and $\{\alpha_j\}_{j=0}^{n-1}$ we set e_n to an arbitrary unit vector with an orthogonal projection on the subspace $< e_{-1}, e_0, \ldots, e_{n-1} >$ lying on the line with directing vector r_{n-1} and angle between this vector and r_{n-1} equal α_{n-1}, and select the coefficient c_n in such a way that for $r_n = r_{n-1} - c_n e_n$ scalar products $(r_n, -e_n), (r_n, -e_{n-1}), \ldots, (r_n, -e_1), (r_n, e_0), (r_n, e_{-1})$ are equal or, equivalently, angles between r_n and the vectors e_{-1}, e_0, $-e_1$, $-e_2$, …, $-e_{n-1}$, $-e_n$ are equal. We denote this angle by α_n and note that as $\cos\alpha_n = \cos\alpha_{n-1}\cos(\pi - \alpha_{n-1} - \alpha_n)$ due to the spherical law of cosines, $\alpha_n \in (\alpha_{n-1}, \pi/2)$.

Let us justify formally that it is possible to find e_n and c_n with the required properties. Let θ_n denote an arbitrary unit vector orthogonal to the subspace $< e_{-1}, e_0, \ldots, e_{n-1} >$ (which also contains vector r_{n-1}), and E_n denote the set of vectors $\{e_{-1}, e_0, -e_1, -e_2, \ldots, -e_{n-1}\}$. As e_n we can take any of two unit vectors from the plane $< r_{n-1}, \theta_n >$ that have an angle with r_{n-1} equal to α_{n-1}. Let $e_n = a_n r_{n-1} + b_n \theta_n$; note that $a_n > 0$ as the angle α_{n-1} is acute. By construction for all $e \in E_n$ the scalar product (r_{n-1}, e) equals $\|r_{n-1}\| \cos\alpha_{n-1}$. For the sake of brevity we denote $\|r_{n-1}\| \cos\alpha_{n-1}$ by β_n. Hence for all $e \in E$, as $\theta_n \perp e$, scalar products of e_n and e are the same and equal $a_n\beta_n$. Consequently, for an arbitrary positive c and every $e \in E_n$ we have equalities $(r_{n-1} - c e_n, -e_n) = -\beta_n + c$, $(r_{n-1} - c e_n, e) = \beta_n - c a_n \beta_n$. Thus, it remains to set c_n to the solution of the linear equation $-\beta_n + c = \beta_n - c a_n \beta_n$, i.e., to $2\beta_n/(1 + a_n\beta_n)$.

4.2. Realization of Greedy Expansion. We note that a possible realization of greedy expansion in the dictionary

$$D = \{\pm e_{-1}, \pm e_0, \pm e_1, \pm e_2, \ldots\} \tag{16}$$

with the prescribed coefficients $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ (and the weakness parameter $t = 1$) is a realization in which e_n is selected as an expanding element at the n-th step, and hence n-th remainder coincides with r_n. Indeed, an angle between a vector and its orthogonal projection on a subspace does not exceed an angle between this vector and any non-zero vector from the subspace. Thus while for every $n \in \{1, 2, 3, \ldots\}$ and every $e \in \{\pm e_{-1}, \pm e_0, \pm e_1, \pm e_2, \ldots, \pm e_n\}$ we have an equality

$$\left|(r_{n-1}, e)\right| = \left\|r_{n-1}\right\| \cos\alpha_{n-1} \tag{17}$$

and for $e = e_n$ the scalar product (r_{n-1}, e) is positive, for $e = \pm e_{n+j}$ $(j \in \{1, 2, 3, \ldots\})$ we have an inequality

$$\left|(r_{n-1}, e)\right| \le \left\|r_{n-1}\right\| \cos\alpha_{n+j-1} \le \left\|r_{n-1}\right\| \cos\alpha_{n-1}. \tag{18}$$

4.3. Absence of Convergence. In this subsection we show that the greedy expansion $\sum_{n=1}^{\infty} c_n e_n$ does not converge to f or, equivalently, remainders r_n do not converge to zero.

First we find the limit of α_n. It exists as $\{\alpha_n\}_{n=1}^{\infty}$ is a nondecreasing sequence with all values belonging to the interval $(\pi/4, \pi/2)$. As noted above, for all $n \in \{0, 1, 2, \ldots\}$ the equality

$$\cos\alpha_{n+1} = \cos\alpha_n \cos\left(\pi - \alpha_n - \alpha_{n+1}\right) \tag{19}$$

holds. Consequently, if α denotes the limit $\lim_{n\longrightarrow\infty}\alpha_n$, then $\cos(\pi - 2\alpha)\cos\alpha = \cos\alpha$. It implies that $\alpha = \pi/2$.

Now we show that $\sum_{n=0}^{\infty}(1/\tan\alpha_n) = \infty$. Indeed,

$$\begin{aligned}
&\cos\left(\pi - \alpha_n - \alpha_{n+1}\right)\cos\alpha_n = \cos\alpha_{n+1} \\
&\quad\Longrightarrow \cos\left(\alpha_n + \alpha_{n+1}\right)\cos\alpha_n + \cos\alpha_{n+1} = 0 \\
&\quad\Longrightarrow \left(\cos\alpha_n\cos\alpha_{n+1} - \sin\alpha_n\sin\alpha_{n+1}\right)\cos\alpha_n \\
&\qquad + \cos\alpha_{n+1} = 0 \\
&\quad\Longrightarrow \left(\cos\alpha_n - \sin\alpha_n\tan\alpha_{n+1}\right)\cos\alpha_n + 1 = 0 \\
&\quad\Longrightarrow \cos^2\alpha_n - \sin\alpha_n\tan\alpha_{n+1}\cos\alpha_n + 1 = 0 \\
&\quad\Longrightarrow \tan\alpha_{n+1} = \frac{\cos^2\alpha_n + 1}{\sin\alpha_n\cos\alpha_n} = \frac{\left(\cos 2\alpha_n + 1\right)/2 + 1}{\left(\sin 2\alpha_n\right)/2} \\
&\quad= \frac{\cos 2\alpha_n + 3}{\sin 2\alpha_n}
\end{aligned}$$

$$\begin{aligned}&= \frac{\left(1-\tan^2\alpha_n\right)/\left(1+\tan^2\alpha_n\right)+3}{2\tan\alpha_n/\left(1+\tan^2\alpha_n\right)} = \frac{2\tan^2\alpha_n+4}{2\tan\alpha_n}\\ &= \tan\alpha_n + \frac{2}{\tan\alpha_n}.\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}\tan\alpha_{n+1} &= \tan\alpha_0\\ &\quad + 2\left(\frac{1}{\tan\alpha_0}+\frac{1}{\tan\alpha_1}+\dots+\frac{1}{\tan\alpha_n}\right).\end{aligned} \tag{21}$$

As $\alpha_n \longrightarrow \pi/2$, $\tan\alpha_n \longrightarrow \infty$ and hence $\sum_{n=0}^{\infty}(1/\tan\alpha_n) = \infty$.

The law of sines gives the equalities

$$\begin{aligned}&\frac{\|r_0\|}{\sin\alpha_1} = \frac{\|r_1\|}{\sin\alpha_0} \Longrightarrow\\ &\|r_1\| = \frac{\sin\alpha_0}{\sin\alpha_1}\|r_0\|;\\ &\frac{\|r_1\|}{\sin\alpha_2} = \frac{\|r_2\|}{\sin\alpha_1} \Longrightarrow\\ &\|r_2\| = \frac{\sin\alpha_1}{\sin\alpha_2}\|r_1\|;\\ &\vdots\\ &\frac{\|r_{n-1}\|}{\sin\alpha_n} = \frac{\|r_n\|}{\sin\alpha_{n-1}} \Longrightarrow\\ &\|r_n\| = \frac{\sin\alpha_{n-1}}{\sin\alpha_n}\|r_{n-1}\|.\end{aligned} \tag{22}$$

Consequently

$$\|r_n\| = \frac{\sin\alpha_{n-1}}{\sin\alpha_n}\frac{\sin\alpha_{n-2}}{\sin\alpha_{n-1}}\cdots\frac{\sin\alpha_0}{\sin\alpha_1}\|r_0\| = \frac{\|r_0\|\sin\alpha_0}{\sin\alpha_n}. \tag{23}$$

Taking into consideration the convergence $\alpha_n \longrightarrow \pi/2$ ($n \longrightarrow \infty$), we derive from this equality that

$$\|r_n\| \underset{n\longrightarrow\infty}{\not\longrightarrow} 0, \tag{24}$$

so the absence of convergence to the expanded element is proved.

4.4. Estimate of c_n. Here we first show that $\sum_{n=1}^{\infty} c_n = \infty$. Applying again the law of sines, but this time for the other pairs of angles, we get equalities

$$\begin{aligned}&\frac{\|r_0\|}{\sin\alpha_1} = \frac{c_1}{\sin\left(\pi-\alpha_0-\alpha_1\right)} \Longrightarrow\\ &c_1 = \|r_0\|\frac{\sin\left(\alpha_0+\alpha_1\right)}{\sin\alpha_1};\\ &\vdots\\ &\frac{\|r_n\|}{\sin\alpha_{n+1}} = \frac{c_{n+1}}{\sin\left(\pi-\alpha_n-\alpha_{n+1}\right)} \Longrightarrow\\ &c_{n+1} = \|r_n\|\frac{\sin\left(\alpha_n+\alpha_{n+1}\right)}{\sin\alpha_{n+1}}.\end{aligned} \tag{25}$$

Let us use equality (23):

$$\begin{aligned}c_{n+1} &= \|r_0\|\sin\alpha_0\frac{\sin\left(\alpha_n+\alpha_{n+1}\right)}{\sin\alpha_{n+1}\sin\alpha_n}\\ &= \|r_0\|\sin\alpha_0\frac{\sin\alpha_n\cos\alpha_{n+1}+\cos\alpha_n\sin\alpha_{n+1}}{\sin\alpha_{n+1}\sin\alpha_n}\\ &= \|r_0\|\sin\alpha_0\left(\frac{1}{\tan\alpha_n}+\frac{1}{\tan\alpha_{n+1}}\right).\end{aligned} \tag{26}$$

Taking into account the monotonicity of $\{\alpha_n\}$, it implies the inequality

$$c_{n+1} > \frac{2\|r_0\|\sin\alpha_0}{\tan\alpha_{n+1}}. \tag{27}$$

Consequently, as the series $\sum_{n=0}^{\infty}(1/\tan\alpha_n)$ diverges, the series $\sum_{n=1}^{\infty} c_n$ also diverges.

Then we show that $c_n \leqslant 1/\sqrt{n}$. The equality (26) and monotonicity of $\{\alpha_n\}$ imply that

$$c_{n+1} < \frac{2\|r_0\|\sin\alpha_0}{\tan\alpha_n} \quad (n \in \{0,1,2,\dots\}). \tag{28}$$

It remains to establish the inequality $\tan\alpha_n > \sqrt{n+1}$: as $\|r_0\| \leqslant 1/2$, it directly gives the required upper estimate of c_n. For $n = 0$ this inequality holds because $\alpha_0 \in (\pi/4, \pi/2)$. Next, from equality (20), taking into account that the minimum value of the function $f(x) = x + 2/x$ on the positive semi-axis equals $2\sqrt{2}$, we obtain that $\tan\alpha_1 \geqslant 2\sqrt{2} > \sqrt{2}$. Besides, $f(x)$ is an increasing function on $[\sqrt{2}, +\infty)$, and the justification of the inequality can be completed by induction:

$$\begin{aligned}\tan\alpha_{n+1} &= f\left(\tan\alpha_n\right) \geqslant f\left(\sqrt{n+1}\right)\\ &= \sqrt{n+1} + \frac{2}{\sqrt{n+1}} > \sqrt{n+2}.\end{aligned} \tag{29}$$

Justification of this inequality completes the proof of the upper estimate of c_n and in total the proof Theorem 3.

We note that the construction described in the proof of Theorem 3 can be straightforwardly adapted from the case of greedy expansions with prescribed coefficients to the generalized Approximate Weak Greedy Algorithm (see [10, Theorem 3]).

5. The Case of Finite-Dimensional Spaces

In this section we prove the following theorem.

Theorem 5. *Let the space H be finite-dimensional, and let the coefficients $\mathscr{C} = \{c_n\}_{n=1}^{\infty}$ converge to zero monotonically and* *satisfy the condition* $\sum_{n=1}^{\infty} c_n = \infty$. *Then for every symmetric unit-normed dictionary* $D \subset H$ *and any* $f \in H$ *all realizations of greedy expansion of* f *in* D *with the prescribed coefficients* $\mathscr{C}$ *and the weakness parameter* $t = 1$ *converge to* f.

Thus in the finite-dimensional case at least for $t = 1$ in the natural class of monotonic coefficient sequences conditions sufficient for the convergence of a greedy expansion to an expanded element include only the convergence of coefficients to zero and infinity of their sum. Clearly, convergence of coefficients to zero is also a necessary condition for the convergence of greedy expansion.

As the first step of the proof we note that due to Lemma 4 there exists a limit of norms of remainders of a greedy expansion with prescribed coefficients $\lim_{n\longrightarrow\infty}\|r_n\|$. Let us suppose that this limit is non-zero. Then starting from a certain index remainder norms are separated from zero. Due to the compactness of a sphere in a finite-dimensional space and the completeness and the symmetric property of the dictionary it implies that scalar products

$$x_{n+1} = \left(r_n, e_{n+1}\right) = \sup_{e\in D}\left(r_n, e\right) \tag{30}$$

are also separated from zero. In other words, there exist a positive number γ and an index N_0 such that for all $n > N_0$ the inequality $x_{n+1} > \gamma$ holds. In addition there exists such an index $N_1 > N_0$ that for all $n > N_1$ coefficient c_n does not exceed γ. But consequently for $n \geqslant N_1$ it follows from equality (10) that

$$\begin{aligned}\left\|r_{n+1}\right\|^2 &= \left\|r_n\right\|^2 - x_{n+1}^2 + \left(c_{n+1} - x_{n+1}\right)^2 \\ &= \left\|r_n\right\|^2 - c_{n+1}\left(2x_{n+1} - c_{n+1}\right) \\ &\leqslant \left\|r_n\right\|^2 - \gamma c_{n+1}.\end{aligned} \tag{31}$$

Hence

$$\left\|r_{N_1+K}\right\|^2 \leqslant \left\|r_{N_1}\right\|^2 - \gamma \sum_{n=N_1+1}^{N_1+K} c_n \longrightarrow -\infty \qquad (K \longrightarrow \infty). \tag{32}$$

This contradiction completes the proof of Theorem 5.

6. Conclusion

The main results of the paper state that in Hilbert spaces a greedy expansion with prescribed coefficients converges to an expanded element if coefficients satisfy certain relatively weak conditions that do not include monotonicity, and these conditions can not be essentially relaxed. At the same time we showed that for the finite-dimensional case the relaxation is possible.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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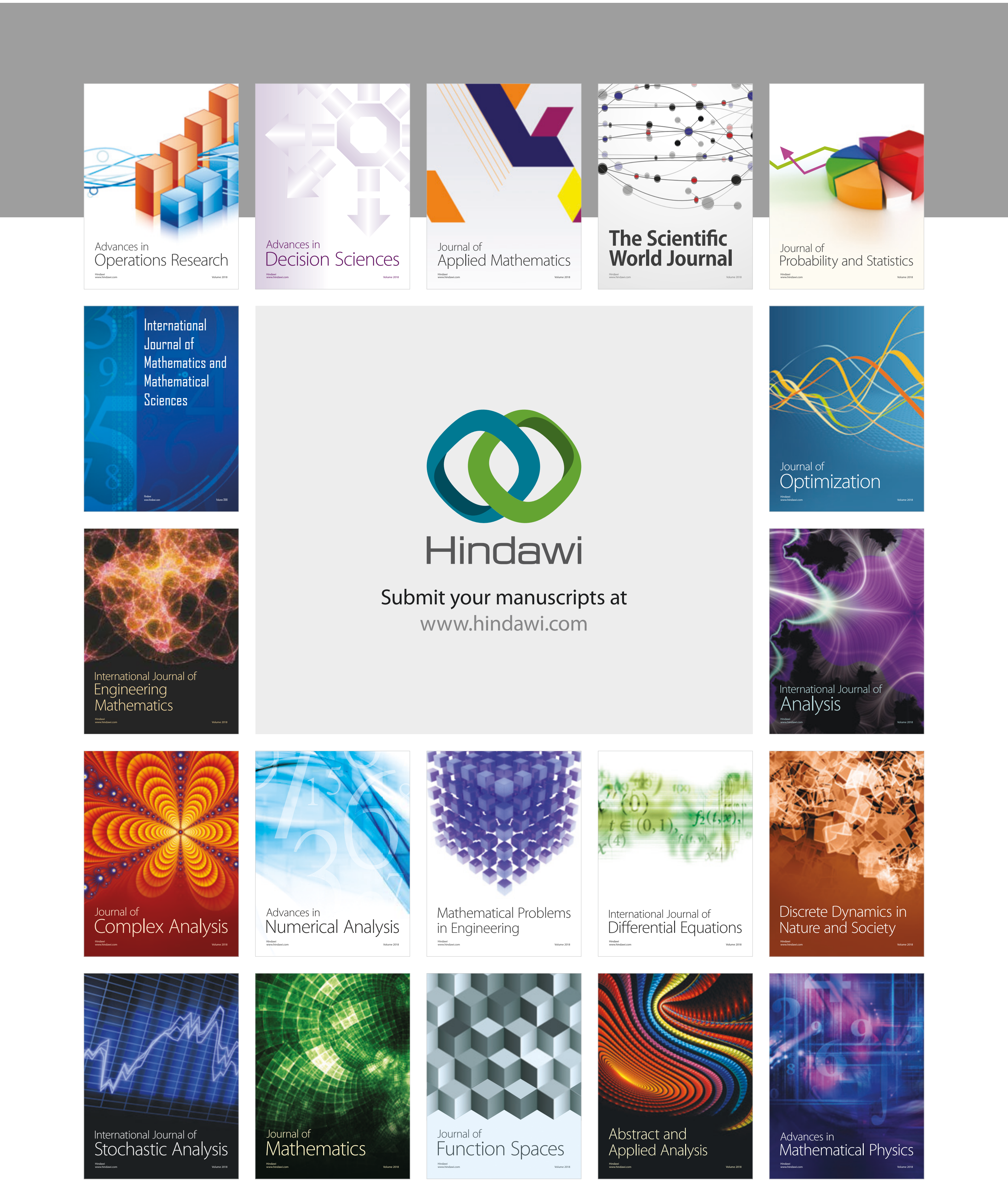
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