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Research Article

Structure of *n*-Lie Algebras with Involutive Derivations

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We study the structure of n-Lie algebras with involutive derivations for $n \geq 2$. We obtain that a 3-Lie algebra A is a two-dimensional extension of Lie algebras if and only if there is an involutive derivation D on $A = A_1 + A_{-1}$ such that $\dim A_1 = 2$ or $\dim A_{-1} = 2$, where A_1 and A_{-1} are subspaces of A with eigenvalues 1 and -1, respectively. We show that there does not exist involutive derivations on nonabelian n-Lie algebras with $n = 2s$ for $s \geq 1$. We also prove that if A is a $(2s+2)$-dimensional $(2s+1)$-Lie algebra with $\dim A^1 = r$, then there are involutive derivations on A if and only if r is even, or r satisfies $1 \leq r \leq s+2$. We discuss also the existence of involutive derivations on $(2s+3)$-dimensional $(2s+1)$-Lie algebras.

1. Introduction

Derivation is an important tool in studying the structure of n-Lie algebras [1]. The derivation algebra $Der(A)$ of an n-Lie algebra A over the field of real numbers is the Lie algebra of the automorphism group $Aut(A)$, which is a Lie group if $\dim A < \infty$ [2]. Any n-Lie algebra-module (V, ρ) is a module of the inner derivation algebra $ad(A)$, which is a linear Lie algebra [3]. Also, derivations have close relationship with extensions of n-Lie algebras.

The concept of 3-Lie classical Yang-Baxter equations is introduced in [4]. It is known that if there is an involutive derivation D on A, then $(A, \{,,\}_D)$ is a 3-pre-Lie algebra, where $\{x, y, z\}_D = D(ad(x, y)D(z))$, $\forall x, y, z \in A$, and the 3-Lie algebra A is a subadjacent 3-Lie algebra of $(A, \{,,\}_D)$, and $r = \sum_i e_i^* \otimes D(e_i) - D(e_i) \otimes e_i^*$ is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in the 3-Lie algebra $A \ltimes_{ad^*} A^*$, where $\{e_1, \cdots, e_m\}$ is a basis of A and $\{e_1^*, \cdots, e_m^*\}$ is the dual basis of A^*.

Due to this importance of involutive derivations on 3-Lie algebras, we investigate in this paper the existence of involutive derivations on finite dimensional n-Lie algebras. More specifically, in Section 2, we discuss the properties of involutive derivations on n-Lie algebras. In Section 3, we study the existence of involutive derivations on $(2s+2)$-dimensional $(2s+1)$-Lie algebras. In Section 4, we consider the existence of involutive derivations on $(2s+3)$-dimensional $(2s+1)$-Lie algebras. In Section 5, we investigate a class of 3-Lie algebras with involutive derivations which are two-dimensional extension of Lie algebras.

In the following, we assume that all algebras are over an algebraically closed field $\mathbb{F}$ with characteristic zero, Id is the identity mapping, and $\mathbb{Z}$ is the set of integers. For $\lambda \in \mathbb{F}$ and an $\mathbb{F}$-linear mapping D on a vector space A, A_λ denotes the subspace $\{x \in A \mid D(x) = \lambda x\}$.

2. *n*-Lie Algebras with Involutive Derivations

An n-Lie algebra [1] is a vector space A over a field $\mathbb{F}$ equipped with a linear multiplication $[,\cdots,] : \wedge^n A \longrightarrow A$ satisfying, for all $x_1, \cdots, x_n, y_2, \cdots, y_n \in A$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \tag{1}$$

Equation (1) is usually called *the generalized Jacobi identity*, or *Filippov identity*.

The derived algebra of an n-Lie algebra A is a subalgebra of A generated by $[x_1, \cdots, x_n]$ for all $x_1, \cdots, x_n \in A$, and is denoted by A^1. We use $Z(A)$ to denote the center of A; that is, $Z(A) = \{x \mid x \in A, [x, A, \cdots, A] = 0\}$.

A derivation of A is an endomorphism of A satisfying

$$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n], \qquad \forall x_1, \cdots, x_n \in A. \tag{2}$$

If a derivation D satisfies that $D^2 = Id$, then D is called an *involutive derivation* on A. Der(A) denotes the derivation algebra of A.

For $x_1, \cdots, x_{n-1} \in A$, map $\operatorname{ad}(x_1, \cdots, x_{n-1})\colon A \longrightarrow A$,

$$\operatorname{ad}(x_1, \cdots, x_{n-1})(x) = [x_1, \cdots, x_{n-1}, x], \quad \forall x \in A \tag{3}$$

is called *a left multiplication* defined by elements $x_1, \cdots, x_{n-1}$. From (1), left multiplications are derivations.

The following lemma can be easily verified.

Lemma 1. *Let V be a finite dimensional vector space over $\mathbb{F}$ and D be an endomorphism of V with $D^2 = Id$. Then V can be decomposed into the direct sum of subspaces $V = V_1 \dotplus V_{-1}$, where $V_1 = \{v \in V \mid Dv = v\}$ and $V_{-1} = \{v \in V \mid Dv = -v\}$.*

If A is a finite dimensional n-Lie algebra with an involutive derivation D, then we have

$$A = A_1 \dotplus A_{-1}. \tag{4}$$

Lemma 2. *Let A be an n-Lie algebra over $\mathbb{F}$. If $D \in Der(A)$ is an involutive derivation, then, for all $x_1, \cdots, x_n \in A$,*

$$[x_1, \cdots, x_n] = \frac{-2}{n-1} \sum_{i<j} [x_1, \cdots, x_{i-1}, D(x_i), x_{i+1}, \cdots, x_{j-1}, D(x_j), x_{j+1}, \cdots, x_n], \tag{5}$$

$$[D(x_1), \cdots, D(x_n)] = \frac{-2}{n-1} \sum_{i<j} [Dx_1, \cdots, D(x_{i-1}), x_i, D(x_{i+1}), \cdots, D(x_{j-1}), x_j, D(x_{j+1}), \cdots, D(x_n)]. \tag{6}$$

Proof. If D is an involutive derivation on A, then, for all $x_1, \cdots, x_n \in A$,

$$\begin{aligned}[x_1, \cdots, x_n] &= D^2([x_1, \cdots, x_n]) \\ &= D\left(\sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n]\right) \\ &= n[x_1, \cdots, x_n] \\ &\quad + 2 \sum_{1 \le i < j \le n} [x_1, \cdots, D(x_i), \cdots, D(x_j), \cdots, x_n].\end{aligned} \tag{7}$$

Equation (5) follows. Equation (6) follows from (4) and $D^2 = Id$. □

Theorem 3. *Let A be a finite dimensional n-Lie algebra with $n = 2s$, $s \ge 1$. Then there is an involutive derivation D on A if and only if A is abelian.*

Proof. If A is abelian, then the result is trivial.

Conversely, let D be an involutive derivation on A. By Lemma 1, $A = A_1 \dotplus A_{-1}$. Then, for any $i \in \mathbb{Z}$, $1 \le i \le n$, $x_1, \cdots, x_n \in A_1$, and $y_1, \cdots, , y_n \in A_{-1}$,

$$\begin{aligned} D([x_1, \cdots, x_i, y_1, \cdots, y_{n-i}]) &= i[x_1, \cdots, x_i, y_1, \cdots, y_{n-i}] \\ &\quad - (n-i)[x_1, \cdots, x_i, y_1, \cdots, y_{n-i}] \\ &= (2i - 2s)[x_1, \cdots, x_i, y_1, \cdots, y_{n-i}] \in A_{2i-2s}. \\ D([x_1, \cdots, x_n]) &= 2s[x_1, \cdots, x_n], \\ D([y_1, \cdots, y_n]) &= -2s[y_1, \cdots, y_n]. \end{aligned} \tag{8}$$

Thanks to $\pm 2s \ne \pm 1$ and $2i - 2s \ne \pm 1$, $A_{2i-n} = A_{\pm 2s} = 0$. Therefore, A is abelian. □

Theorem 4. *Let A be a finite dimensional n-Lie algebra with $n = 2s+1$, $s \ge 1$, and D be an involutive derivation on A. Then A_1 and A_{-1} are abelian subalgebras, and*

$$[\underbrace{A_1, \cdots, A_1}_{i}, \underbrace{A_{-1}, \cdots, A_{-1}}_{2s+1-i}] = 0, \qquad \forall 1 \le i \le 2s,\ i \ne s,\ s+1, \tag{9}$$

$$[\underbrace{A_1, \cdots, A_1}_{s}, \underbrace{A_{-1}, \cdots, A_{-1}}_{s+1}] \subseteq A_{-1}, \tag{10}$$

$$[\underbrace{A_1, \cdots, A_1}_{s+1}, \underbrace{A_{-1}, \cdots, A_{-1}}_{s}] \subseteq A_1. \tag{11}$$

Proof. Since $D \in DerA$, $[\underbrace{A_1, \cdots, A_1}_{i}, \underbrace{A_{-1}, \cdots, A_{-1}}_{2s+1-i}] \subseteq A_{2i-2s-1}, 0 \le i \le 2s+1$. If $[\underbrace{A_1, \cdots, A_1}_{i}, \underbrace{A_{-1}, \cdots, A_{-1}}_{2s+1-i}] \ne 0$, then $2i - 2s - 1 = \pm 1$, that is, $i = s+1$, or $i = s$. Therefore, $[A_1, \cdots, A_1] = [A_{-1}, \cdots, A_{-1}] = 0$. The result follows. □

Theorem 5. *Let A be an m-dimensional n-Lie algebra with $n = 2s+1$, $s \ge 1$. Then there is an involutive derivation on A if and only if A has the decomposition $A = B \dotplus C$ (as direct sum of subspaces), and*

$$[\underbrace{B, \cdots, B}_{i}, \underbrace{C, \cdots, C}_{2s+1-i}] = 0, \qquad 0 \le i \le 2s+1,\ i \ne s, s+1, \tag{12}$$

$$\begin{aligned}[\underbrace{B, \cdots, B}_{s}, \underbrace{C, \cdots, C}_{s+1}] &\subseteq C, \\ [\underbrace{B, \cdots, B}_{s+1}, \underbrace{C, \cdots, C}_{s}] &\subseteq B.\end{aligned} \tag{13}$$

Proof. If there is an involutive derivation D on A, then, by Theorem 4, $B = A_1$ and $C = A_{-1}$ satisfy (12) and (13).

Conversely, define an endomorphism D of A by $D(x) = x, D(y) = -y, \forall x \in B, y \in C$. Then $D^2 = Id$, $B = A_1$ and $C = A_{-1}$. By (12) and (13), D is a derivation. □

Corollary 6. *Let A be a $(2s+1)$-dimensional $(2s+1)$-Lie algebra with the multiplication $[e_1, \cdots, e_{2s+1}] = e_1$, where $\{e_1, \cdots, e_{2s+1}\}$ is a basis of A. Then the linear mapping $D : A \longrightarrow A$ defined by $D(e_i) = e_i$, $1 \le i \le s+1$, $D(e_j) = -e_j$, $s+2 \le j \le 2s+1$ is an involutive derivation on A.*

Proof. The result follows from a direct computation. □

3. Involutive Derivations on $(n+1)$-Dimensional n-Lie Algebras with $n = 2s+1$

In this section, we study involutive derivations on $(n+1)$-dimensional n-Lie algebras over $\mathbb{F}$. From Theorem 3, we only need to discuss the case of $n = 2s+1$, $s \ge 1$.

Lemma 7 (see [5]). *Let A be an $(n+1)$-dimensional non-abelian n-Lie algebra over $\mathbb{F}$, $n \ge 3$. Then up to isomorphisms A is one and only one of the following possibilities:*

$$(b_1)\ \ [e_2, e_3, \cdots, e_{n+1}] = e_1.$$

$$(b_2)\ \ [e_1, e_2, \cdots, e_n] = e_1.$$

$$(c_1)\ \ \begin{cases} [e_2, \cdots, e_{n+1}] = e_1, \\ [e_1, e_3, \cdots, e_{n+1}] = e_2. \end{cases}$$

$$(c_2)\ \ \begin{cases} [e_2, \cdots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \cdots, e_{n+1}] = e_2, \end{cases} \quad \alpha \in \mathbb{F},\ \alpha \neq 0. \tag{14}$$

$$(c_3)\ \ \begin{cases} [e_1, e_3, \cdots, e_{n+1}] = e_1, \\ [e_2, \cdots, e_{n+1}] = e_2. \end{cases}$$

$$(d_r)\ \ [e_1, \cdots, \widehat{e}_i, \cdots, e_{n+1}] = e_i, \quad 1 \le i \le r,$$

where $\{e_1, \cdots, e_{n+1}\}$ is a basis of A, $3 \le r \le n+1$, and $\widehat{e}_i$ means that e_i is omitted.

Theorem 8. *Let A be a $(2s+2)$-dimensional $(2s+1)$-Lie algebra over $\mathbb{F}$ and $\dim A^1 = r$. Then there exists an involutive derivation D on A if and only if r is even, or $0 \le r \le s+2$.*

Proof. If $\dim A^1 = r \le s+2$, then, by Lemma 7, and a direct computation, the linear mapping $D : A \longrightarrow A$ defined by $D(e_i) = e_i, D(e_j) = -e_j, 1 \le i \le s+2, s+3 \le j \le 2s+2$, is an involutive derivation on A.

Now we discuss the case $\dim A^1 = r \ge s+3$. Let $\{e_1, \cdots, e_{2s+2}\}$ be a basis of A and the multiplication in the basis be as follows:

$$e^i = (-1)^{2s+2+i} [e_1, \cdots, \widehat{e}_i, \cdots, e_{2s+2}] = \sum_{l=1}^{2s+2} \beta_{il} e_l, \qquad \beta_{il} \in F,\ 1 \le i \le 2s+2, \tag{15}$$

where $\beta_{il} \in \mathbb{F}, 1 \le i, l \le 2s+2$. Thanks to Theorem 3 in [1], A is a 3-Lie algebra if and only if the $(2s+2)\times(2s+2)$-matrix $B = (\beta_{il})$ is symmetric.

If $r = 2t > 3$, $2 \le t \le s+1$, then define the multiplication on A by

$$[e_1, \cdots, \widehat{e}_i, \cdots, e_{2s+2}] = (-1)^i e_{2s+3-i}, \qquad 1 \le i \le t,\ 3-t < i-2s \le 2, \tag{16}$$
$$[e_1, \cdots, \widehat{e}_j, \cdots, e_{2s+2}] = 0, \quad t < j \le 2s+3-t,$$

that is, $\beta_{i,2s+3-i} = \beta_{2s+3-i,i} = 1$ for $1 \le i \le t$, or $2s+3-t < i \le 2s+2$, and others are zero. Then, $B = (\beta_{il})$ is symmetric. Therefore, A is a $(2s+1)$-Lie algebra with the multiplication (16).

Define an endomorphism D of A by $De_i = e_i$, $1 \le i \le s+1$, and $De_j = -e_j$ for $s+2 \le j \le 2s+2$. Then D is an involutive derivation on A.

For the case $\dim A^1 = r = 2t+1 \ge s+3$. Suppose $l = \dim A_1$, $l' = \dim A_{-1}$.

If there is an involutive derivation D on A, then, by Theorem 4, $l + l' = 2s+2$, $s \le l \le s+2$ and $s \le l' \le s+2$. Since $\dim A^1 = r = 2t+1 \ge s+3$, $A^1 \cap A_1 \neq 0$ and $A^1 \cap A_{-1} \neq 0$. Therefore, $\dim A_1 = \dim A_{-1} = s+1$. Without loss of generality, we can suppose $\{e_1, \cdots, e_{s+1}\} \subseteq A_1$, and $\{e_{s+2}, \cdots, x_{2s+2}\} \subseteq A_{-1}$. By (10) and (11), the $(2s+2)\times(2s+2)$-matrix $B = (\beta_{il})$ defined by (15) is nonsymmetric, which is a contradiction. Therefore, if $\dim A^1 = r = 2t+1 \ge s+3$, then there do not exist involutive derivations on A. □

By Theorem 8, if A is a 10-dimensional 9-Lie algebra with $\dim A^1 = 7$, or 9, then there does not exist involutive derivation on A. If $1 \le \dim A^1 = r \le 10$ and $r \neq 7, 9$, then there are involutive derivations on A.

4. Involutive Derivations on $(n+2)$-Dimensional n-Lie Algebras with $n = 2s+1$

By Theorem 3, we only need to discuss the case where n is odd. So we suppose that A is a $(2s+3)$-dimensional $(2s+1)$-Lie algebra over $\mathbb{F}$, $s \ge 1$, and that $E_t = \mathrm{Diag}(1, \cdots 1)$ is the $(t \times t)$-unit matrix.

Lemma 9 (see [6]). *Let A be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra over $\mathbb{F}$ with a basis $\{e_1, \cdots, e_{2s+3}\}$. Then A is isomorphic to one and only one of the following possibilities:*

(a) *A is an abelian.*

(b) $\dim A^1 = 1$:

(b^1) $[e_2, \cdots, e_{2s+2}] = e_1$;

(b^2) $[e_1, \cdots, e_{2s+1}] = e_1$.

(c) $\dim A^1 = 2$:

$$(c^1) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_3, \cdots, e_{2s+3}] = e_2; \end{cases}$$

$$(c^2) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_1, e_4, \cdots, e_{2s+3}] = e_1; \end{cases}$$

$$(c^3) \begin{cases} [e_2, \cdots, e_{2s+2}] = \alpha e_1 + e_2, \\ [e_1, e_3, \cdots, e_{2s+2}] = e_2, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_1, e_4 \cdots, e_{2s+3}] = e_1; \end{cases}$$

$$(c^4) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_2, e_3, \cdots, e_{2s+2}] = e_2, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_1, e_4, \cdots, e_{2s+3}] = e_1; \end{cases}$$

$$(c^5) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_1, e_3, \cdots, e_{2s+2}] = e_2; \end{cases}$$

$$(c^6) \begin{cases} [e_2, \cdots, e_{2s+2}] = \alpha e_1 + e_2, \\ [e_1, e_3, \cdots, e_{2s+2}] = e_2; \end{cases} \quad \alpha \in \mathbb{F}, \ \alpha \neq 0.$$

$$(c^7) \begin{cases} [e_1, e_3, \cdots, e_{2s+2}] = e_1, \\ [e_2, e_3, \cdots, e_{2s+2}] = e_2; \end{cases}$$

(d) $\dim A^1 = 3$:

$$(d^1) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_3, \cdots, e_{2s+3}] = e_3; \end{cases}$$

$$(d^2) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_3, \cdots, e_{2s+3}] = e_3 + \alpha e_2, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_3, \\ [e_1, e_4, \cdots, e_{2s+3}] = e_1; \end{cases}$$

$$(d^3) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_3, e_4, \cdots, e_{2s+3}] = e_3, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_1, e_4, \cdots, e_{2s+3}] = 2e_1; \end{cases}$$

$$(d^4) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_1, e_3, \cdots, e_{2s+2}] = e_2, \\ [e_1, e_2, e_4, \cdots, e_{2s+2}] = e_3; \end{cases}$$

$$(d^5) \begin{cases} [e_1, e_4, \cdots, e_{2s+3}] = e_1, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_3, \\ [e_3, e_4, \cdots, e_{2s+3}] = \beta e_2 + (1+\beta) e_3,; \end{cases}$$

$$(d^6) \begin{cases} [e_1, e_4, \cdots, e_{2s+3}] = e_1, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_3, e_4, \cdots, e_{2s+3}] = e_3; \end{cases}$$

$$(d^7) \begin{cases} [e_1, e_4, \cdots, e_{2s+3}] = e_2, \\ [e_2, e_4, \cdots, e_{2s+3}] = e_3, \\ [e_3, e_4, \cdots, e_{2s+3}] = se_1 + te_2 + ue_3, \end{cases}$$

$$\beta, s, t, u \in \mathbb{F}, \ \beta \neq 0, 1, \ s \neq 0. \tag{17}$$

And n-Lie algebras corresponding to the case (d^7) with coefficients s, t, u and s', t', u' are isomorphic if and only if there exists a nonzero element $\lambda \in \mathbb{F}$ such that $s = \lambda^3 s', t = \lambda^2 t'$, $u = \lambda u'$, $s, s', t, t', u, u' \in \mathbb{F}$.

(r) $\dim A^1 = r$,

$$4 \leq r < 2s+3, \ \textit{for } 2 \leq j \leq r, \ 1 \leq i \leq r,$$

$$(r^1) \begin{cases} [e_2, \cdots, e_{2s+2}] = e_1, \\ [e_2, \cdots, \widehat{e}_j, \cdots, e_{2s+3}] = e_j; \end{cases} \tag{18}$$

$$(r^2) \ [e_1, \cdots, \widehat{e}_i, \cdots, e_{2s+2}] = e_i.$$

Theorem 10. *If A is a $(2s+3)$-dimensional $(2s+1)$-Lie algebra over $\mathbb{F}$ with $\dim A^1 = r < s+3$, then there are involutive derivations on A.*

Proof. Define linear mappings $D_j : A \longrightarrow A$, $1 \leq j \leq 6$ by

$$D_1(e_i) = \begin{cases} e_i, & 1 \leq i \leq s+2, \text{ or } i = 2s+3, \\ -e_i, & \text{otherwise}; \end{cases}$$

$$D_2(e_i) = \begin{cases} e_i, & 1 \leq i \leq s+2, \\ -e_i, & \text{otherwise}; \end{cases}$$

$$D_3(e_i)=\begin{cases}-e_i, & s+2\le i\le 2s+1,\\ e_i, & \text{otherwise;}\end{cases}$$

$$D_4(e_i)=\begin{cases}e_i, & 1\le i\le s+1,\ \text{or } i=2s+2,\\ -e_i, & \text{otherwise;}\end{cases}$$

$$D_5(e_i)=\begin{cases}e_i, & 1\le i\le s+1,\ \text{or } i=2s+3,\\ -e_i, & \text{otherwise;}\end{cases}$$

$$D_6(e_i)=\begin{cases}e_i, & 1\le i\le s+3,\\ -e_i, & \text{otherwise.}\end{cases} \tag{19}$$

Since $\dim A^1 = r \le s+2$, it is easy to verify that D_1 is an involutive derivation on the 3-Lie algebras of the cases of (b^1), (c^i), (d^j), and (r^k), where $1\le i\le 7$, $1\le j\le 4$ and $1\le k\le 2$. D_2 is an involutive derivation on the 3-Lie algebras of the cases of (b^1), (d^4), and (c^i), where $5\le i\le 7$. D_3, D_4, and D_5 are involutive derivations on the 3-Lie algebras of the case of (b^2). And D_6 is an involutive derivation on the 3-Lie algebras of the cases of $(d^5), (d^6)$, and (d^7). Also D_i are involutive derivations on abelian algebras for $1\le i\le 6$. □

Next, we discuss the case of $\dim A^1 = r \ge s+3$. Let D be an endomorphism of A,

$$De_i=\sum_{j=1}^{2s+3} b_{ij}e_j,\quad b_{ij}\in\mathbb{F},\ 1\le i\le 2s+3, \tag{20}$$

and $B=(b_{ij})$ be the $(2s+3)\times(2s+3)$-matrix. Then

$$D(e_1,\cdots,e_{2s+3})^T = B(e_1,\cdots,e_{2s+3})^T = \begin{pmatrix}B_1 & B_0\\ B_2 & B_3\end{pmatrix}(e_1,\cdots,e_{2s+3})^T, \tag{21}$$

where $\left(\begin{smallmatrix}B_1 & B_0\\ B_2 & B_3\end{smallmatrix}\right)$ is the block matrix of B. First we discuss $(2s+3)$-dimensional $(2s+1)$-Lie algebras of the case (r^1) in Lemma 9.

Lemma 11. *If A is a $(2s+3)$-dimensional $(2s+1)$-Lie algebra of the case (r^1) with $\dim A^1 = r\ge s+3, s\ge 1$. Then the linear mapping D is an involutive derivation on A if and only if the block matrix $B=\left(\begin{smallmatrix}B_1 & B_0\\ B_2 & B_3\end{smallmatrix}\right)$ satisfies that $B_0=O_{r\times(2s+3-r)}$ (which is the zero $(r\times(2s+3-r))$-matrix), and*

$$B_1^2=E_r,\quad B_3^2=E_{2s+3-r},\quad B_2B_1+B_3B_2=0, \tag{22}$$

$$\begin{gathered}\sum_{j=2}^{2s+2} b_{jj}=b_{11},\\ \sum_{j=2, j\ne i}^{2s+3} b_{jj}=b_{ii},\quad 2\le i\le r,\\ b_{2s+3,i}=(-1)^{i+1}b_{i,1},\quad 2\le i\le r,\\ b_{j,i}=(-1)^{j-i-1}b_{ij},\quad 2\le i,j\le r,\ i\ne j.\end{gathered} \tag{23}$$

Proof. By (2), and a direct computation, D is a derivation of A if and only if matrix B has the property:

$$\begin{gathered}\sum_{j=2}^{2s+2} b_{jj}=b_{11},\\ \sum_{j=2, j\ne i}^{2s+3} b_{jj}=b_{ii},\quad 2\le i\le r,\\ b_{1l}=0,\quad 2\le l\le 2s+3,\\ b_{2s+3,i}=(-1)^{i+1}b_{i,1},\quad b_{il}=0,\ 2\le i\le r,\ l\ge r+1,\\ b_{j,i}=(-1)^{j-i-1}b_{ij},\quad 2\le i,j\le r,\ i\ne j.\end{gathered} \tag{24}$$

Therefore, matrix B satisfies (23) and $B_0=O_{r\times(2s+3-r)}$. And $D^2=Id$ if and only if

$$B^2=\begin{pmatrix}B_1^2 & B_1B_0+B_0B_3\\ B_2B_1+B_3B_2 & B_2B_0+B_3^2\end{pmatrix}=\begin{pmatrix}E_r & O\\ O & E_{2s+3-r}\end{pmatrix}. \tag{25}$$

Thanks to $B_0=O_{r\times(2s+3-r)}$, (22) holds. □

Theorem 12. *Let A be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra of the case (r^1) with $\dim A^1 = r\ge s+3$, $s\ge 1$. If r is odd, then there are involutive derivations on A.*

Proof. Let $r=2t+1\ge s+3$. Then $t\ge 2$ and $r\ge 5$. Suppose D is an endomorphism of A and the matrix of D with respect to the basis $\{e_1,\cdots e_{2s+3}\}$ is $B=(b_{ij})=\left(\begin{smallmatrix}B_1 & B_0\\ B_2 & B_3\end{smallmatrix}\right)$ which satisfies (22) and (23), and $B_0=O_{r\times(2s+3-r)}$. Then

$$B_1 = \begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0\\ b_{21} & b_{22} & b_{23} & \cdots & b_{2,r-1} & b_{2,r}\\ b_{31} & b_{23} & b_{33} & \cdots & b_{3,r-1} & b_{3,r}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ b_{r-1,1} & (-1)^r b_{2,r-1} & (-1)^{r-1}b_{3,r-1} & \cdots & b_{r-1,r-1} & b_{r-1,r}\\ b_{r,1} & (-1)^{r+1}b_{2,r} & (-1)^r b_{3,r} & \cdots & b_{r-1,r} & b_{r,r}\end{pmatrix},$$

$$B_2 = \begin{pmatrix} b_{r+1,1} & b_{r+1,2} & b_{r+1,3} & \cdots & b_{r+1,r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2s+2,1} & b_{2s+2,2} & b_{2s+2,3} & \cdots & b_{2s+2,r} \\ b_{2s+3,1} & -b_{2,1} & b_{3,1} & \cdots & (-1)^{r+1} b_{r,1} \end{pmatrix}. \tag{26}$$

Since $\sum_{j=2}^{2s+2} b_{jj} = b_{11}, \sum_{j=2, j\neq i}^{2s+3} b_{jj} = b_{ii}, 2 \leq i \leq r$, we have $-b_{11} + 2b_{22} - b_{2s+3,2s+3} = 0$, $(r-3)b_{22} + \sum_{2s+3}^{r+1} b_{ii} = 0$, $b_{22} = b_{ii}, 3 \leq i \leq r$. Therefore,

$$\begin{aligned} b_{11} &= \frac{-1}{r-3}\left((r-1)k_1 + 2\sum_{j=2}^{2s+3-r} k_j\right), \\ b_{ii} &= \frac{-1}{r-3}\sum_{j=1}^{2s+3-r} k_j, \quad 2 \leq i \leq r, \\ b_{jj} &= k_{2s+3-j+1}, \quad r+1 \leq j \leq 2s+3, \ k_{2s+3-j+1} \in \mathbb{F}. \end{aligned} \tag{27}$$

Suppose

$$B_1^2 = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1i} & \cdots & c_{1j} & \cdots & c_{1r} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ii} & \cdots & c_{ij} & \cdots & c_{ir} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{j1} & c_{j2} & \cdots & c_{ji} & \cdots & c_{jj} & \cdots & c_{jr} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{ri} & \cdots & c_{rj} & \cdots & c_{rr} \end{pmatrix}. \tag{28}$$

By (23),

$$\begin{aligned} c_{11} &= b_{11}^2, \\ c_{1l} &= 0, \quad 2 \leq l \leq r, \\ c_{ii} &= \sum_{l=1}^{i} (-1)^{i+l-1} b_{li}^2 + \sum_{l=i+1}^{r} (-1)^{i+l-1} b_{il}^2, \quad 2 \leq i \leq r, \\ c_{ij} &= \sum_{l=1}^{i} (-1)^{l+i+1} b_{li} b_{lj} + \sum_{l=i+1}^{j} b_{il} b_{lj} \\ &\quad + \sum_{l=j+1}^{r} (-1)^{l+j+1} b_{il} b_{jl}, \quad 2 \leq i < j \leq r. \\ c_{ij} &= \sum_{l=1}^{j} (-1)^{l+i+1} b_{lj} b_{li} + \sum_{l=j+1}^{i} (-1)^{i+j} b_{jl} b_{li} \\ &\quad + \sum_{l=i+1}^{r} (-1)^{l+j+1} b_{jl} b_{il}, \quad 1 \leq j < i \leq r. \end{aligned} \tag{29}$$

Therefore, the endomorphisms D of A, which are defined by

$$\begin{aligned} &De_1 = e_1 \\ \text{or } &De_1 = -e_1, \\ &De_2 = \sum_{k=3}^{r} e_k, \\ &De_i = (-1)^{i-1} e_2 + \sum_{k=3}^{i-1} (-1)^i e_k + (-1)^{i-1} \sum_{k=i+1}^{r} e_k, \\ &\qquad 3 \leq i \leq r, \\ &De_j = (-1)^j e_j, \quad r+1 \leq j \leq 2s+3, \\ &De_1 = e_1 \\ \text{or } &De_1 = -e_1, \\ &De_2 = \sum_{k=3}^{r} e_k, \\ &De_i = (-1)^{i-1} e_2 + \sum_{k=3}^{i-1} (-1)^i e_k + (-1)^{i-1} \sum_{k=i+1}^{r} e_k, \\ &\qquad 3 \leq i \leq r, \\ &De_j = (-1)^{j-1} e_j, \quad r+1 \leq j \leq 2s+3, \end{aligned} \tag{30}$$

are involutive derivations on A. □

Theorem 13. *Let A be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra of the case (r^1) with $\dim A^1 = r = 2s+2$ $(s \geq 1)$, then there does not exist an involutive derivation on A.*

Proof. If D is an involutive derivation on A, then, by Lemma 11 and (23),

$$\begin{aligned} b_{11} &= \frac{-(2s+1)k_1}{2s-1}, \\ b_{2s+3,2s+3} &= k_1, \\ b_{ii} &= \frac{-k_1}{2s-1}, \\ &\qquad 2 \leq i \leq r, \ k_1 \in \mathbb{F}. \end{aligned} \tag{31}$$

Thanks to (22), $b_{2s+3,2s+3}^2 = b_{11}^2 = k_1^2 = 1$. Therefore, $(-(2s+1)k_1/(2s-1))^2 = ((2s+1)/(2s-1))^2 = 1$, which is a contradiction. □

Now we discuss case (r^2).

Theorem 14. *Let A be a $(2s+3)$-dimensional $(2s+1)$-Lie algebra of the case (r^2) with $\dim A^1 = r \geq s+3$. Then there exist involutive derivations on A if and only if r is even.*

Proof. By Lemma 7, $A = A_1 \dotplus \mathbb{F}e_{2s+3}$, where $e_{2s+3} \in Z(A)$, and A_1 is a $(2s+2)$-dimensional $(2s+1)$-Lie subalgebra of A with $\dim A^1 = \dim A_1^1 = r$. Then there exist involutive derivations on A if and only if there exist involutive derivations on A_1.

By Theorem 3 in [1], there is a basis $\{e_1, \cdots, e_{2s+2}\}$ of A_1 such that

$$\begin{aligned} &[e_1, \cdots, \widehat{e}_j, \cdots, e_{2s+2}] = 0, \quad t < j \le 2s+3-t, \\ &[e_1, \cdots, \widehat{e}_i, \cdots, e_{2s+2}] = (-1)^i e_{2s+3-i}, \\ &\qquad 1 \le i \le t, \ \text{ or } 2s+3-t < i \le 2s+2. \end{aligned} \tag{32}$$

If r is even, then $r = 2t \ge 4$. By Theorem 8 and (32), endomorphism D_1 of A_1 defined by

$$D_1(e_i) = \begin{cases} e_i, & i = 1, \cdots, s+1, \\ -e_i, & i = s+2, \cdots, 2s+2 \end{cases} \tag{33}$$

is an involutive derivation on A_1. Therefore, the endomorphism D of A defined by

$$\begin{aligned} &D(e_i) = e_i, \quad 1 \le i \le s+1, \\ &D(e_j) = -e_j, \quad s+2 \le j \le 2s+2, \\ &D(e_{2s+3}) = \pm e_{2s+3} \end{aligned} \tag{34}$$

is involutive derivation on A.

If $\dim A^1 = r$ is odd and endomorphism D of A is an involutive derivation on A, then $r = 2t+1 > 4$. Suppose $D(e_i) = \sum_{j=1}^{2s+3} a_{ij} e_j$, $1 \le i \le 2s+3$. Then

$$\begin{aligned} &[D(e_{2s+3}), e_{i_1}, \cdots, e_{i_{2s}}] \\ &\quad = D[e_{2s+3}, e_{i_1}, \cdots, e_{i_{2s}}] \\ &\quad - \sum_{j=1}^{2s} [e_{2s+3}, \cdots, De_{i_j}, \cdots, e_{i_{2s}}] = 0. \end{aligned} \tag{35}$$

We get $D(e_{2s+3}) \in Z(A) = \mathbb{F}e_{2s+3}$. Since $D^2 = Id$, $D(e_{2s+3}) = \pm e_{2s+3}$. By (32), $A^1 = \mathbb{F}e_1 + \cdots + \mathbb{F}e_t + \mathbb{F}e_{2s+1} + \cdots + \mathbb{F}e_{2s+3-t}$, and

$$\begin{aligned} &(-1)^i D(e_{2s+3-i}) \\ &\quad = \sum_{k=1, k\ne i}^{2s+2} [e_1, \cdots, \widehat{e}_i, \cdots, D(e_k), \cdots, e_{2s+2}], \end{aligned} \tag{36}$$

where $1 \le i \le t$, and $2s+3-t < i \le 2s+2$. Then $a_{i,2s+3} = 0$, for $1 \le i \le t$, or $2s+3-t < i \le 2s+2$, and $DA^1 \subseteq A_1$. Then the endomorphism D_2 of A_1 defined by

$$\begin{aligned} &D_2(e_i) = D(e_i), \\ &\qquad 1 \le i \le t, \ \text{ or } 2s+3-t < i \le 2s+2, \\ &D_2(e_j) = D(e_j) - a_{j,2s+3} e_{2s+3} = \sum_{j=1}^{2s+2} a_{ij} e_j, \\ &\qquad t < j \le 2s+3-t \end{aligned} \tag{37}$$

is an involutive derivation on the $(2s+2)$-dimensional $(2s+1)$-Lie algebra A_1, contradiction (Theorem 8). Therefore, there does not exist involutive derivation on A. □

5. Structure of 3-Lie Algebras with Involutive Derivations

Let $(L, [,])$ be a Lie algebra over $\mathbb{F}$, and p be an element which is not contained in L. Then $A = L \dotplus \mathbb{F}p$ is a 3-Lie algebra in the multiplication

$$\begin{aligned} &[x, y, z] = 0, \\ &[p, x, y] = [x, y], \\ &\qquad \text{for all } x, y, z \in L. \end{aligned} \tag{38}$$

And the 3-Lie algebra $(A, [,,])$ is called *one-dimensional extension of L.*

Theorem 15. *Let A be a 3-Lie algebra, then A is one-dimensional extension of a Lie algebra if and only if there exists an involutive derivation D on A such that* $\dim A_1 = 1$*, or* $\dim A_{-1} = 1$.

Proof. If A is an one-dimensional extension of a Lie algebra L, then $A = L \dotplus \mathbb{F}p$. Define the endomorphism D of A by $D(p) = -p$ (or p), and $D(x) = x$ (or $-x$), $\forall x \in L$. Thanks to (38), $D^2 = Id$, and $D([x, y, z]) = 0 = [Dx, y, z] + [x, Dy, z] + [x, y, Dz]$, $D([p, x, y]) = [p, x, y] = [Dp, x, y] + [p, Dx, y] + [p, x, Dy]$, for all $x, y, z \in L$. Therefore, D is an involutive derivation on A, and $\dim A_{-1} = 1$ (or $\dim A_1 = 1$).

Conversely, let D be an involutive derivation on a 3-Lie algebra A, and $\dim A_{-1} = 1$ (or $\dim A_1 = 1$). Let $A_{-1} = \mathbb{F}p$, and $A_1 = L$ (or $A_{-1} = L$, $A_1 = \mathbb{F}p$), where $p \in A - L$. Thanks to Theorem 3, L is a Lie algebra with the multiplication $[x, y] = [p, x, y]$, for all $x, y \in L$, and A is one-dimensional extension of L. □

Let $(L, [,]_1)$ and $(L, [,]_2)$ be Lie algebras and $\{x_1, \cdots, x_m\}$ be a basis of L. For convenience, denote Lie algebras $(L, [,]_k)$ by L_k, $k = 1, 2$, respectively. Suppose p_1 and p_2 are two distinct elements which are not contained in L, and 3-Lie algebras $(B, [,,]_1)$ and $(C, [,,]_2)$ are one-dimensional extensions of Lie algebras L_1 and L_2, respectively, where $B = L \dotplus \mathbb{F}p_1$, $C = L \dotplus \mathbb{F}p_2$. Then $Der(L_1)$ and $Der(L_2)$ are subalgebras of $gl(L)$.

Definition 16. Let $L_1 = (L, [,]_1)$ and $L_2 = (L, [,]_2)$ be two Lie algebras, and p_1, p_2 be two distinct elements which are not contained in L, and $A = L \dotplus \mathbb{F}p_1 \dotplus \mathbb{F}p_2$. Then 3-algebra $(A, [,,])$ is called a two-dimensional extension of Lie algebras L_k, $k = 1, 2$, where $[,,] : A \wedge A \wedge A \longrightarrow A$ defined by

$$\begin{aligned} &[x, y, p_1] = [x, y]_1, \\ &[x, y, p_2] = [x, y]_2, \\ &[x, y, z] = 0, \\ &[p_1, p_2, x] = \lambda_x p_1 + \mu_x p_2, \\ &\qquad \forall x, y, z \in L, \ \lambda_x, \mu_x \in \mathbb{F}. \end{aligned} \tag{39}$$

If A is a 3-Lie algebra, then A is called a two-dimensional extension 3-Lie algebra of Lie algebras L_k, $k = 1, 2$.

Let $A = L \dotplus W$ be a two-dimensional extension of Lie algebras L_k, $k = 1, 2$, where $W = \mathbb{F}p_1 \dotplus \mathbb{F}p_2$. Define linear mappings $D_1, D_2 : L \longrightarrow End(L)$ and $D : L \longrightarrow W$ by

$$\begin{aligned} D_1(x) &= ad(p_1, x), \\ D_2(x) &= ad(p_2, x), \\ D(x) &= ad(p_1, p_2)(x), \\ &\qquad \forall x \in L, \end{aligned} \tag{40}$$

that is, for all $y \in L$, $D_1(x)(y) = [p_1, x, y] = [x, y]_1$, $D_2(x)(y) = [p_2, x, y] = [x, y]_2$, $D(x) = [p_1, p_2, x]$. We have the following result.

Theorem 17. *Let 3-algebra A be a two-dimensional extension of Lie algebras L_1 and L_2. Then A is a 3-Lie algebra if and only if linear mappings D_1, D_2, and D satisfy that $D_1 : L_1 \longrightarrow Der(L_1)$, $D_2 : L_2 \longrightarrow Der(L_2)$ are Lie homomorphisms, and*

$$\begin{aligned} D_1(x_3)([x_1, x_2]_2) &= [D_1(x_3)(x_1), x_2]_2 \\ &\quad + [x_1, D_1(x_3)(x_2)]_2 \\ &\quad - \lambda_{x_3}[x_1, x_2]_1 - \mu_{x_3}[x_1, x_2]_2, \end{aligned} \tag{41}$$

$$\begin{aligned} D_2(x_3)([x_1, x_2]_1) &= [D_2(x_3)(x_1), x_2]_1 \\ &\quad + [x_1, D_2(x_3)(x_2)]_1 \\ &\quad + \lambda_{x_3}[x_1, x_2]_1 + \mu_{x_3}[x_1, x_2]_2, \end{aligned} \tag{42}$$

$$\begin{aligned} D([x_1, x_2]_1) &= (\mu_{x_1}\lambda_{x_2} - \lambda_{x_1}\mu_{x_2}) p_1, \\ D([x_1, x_2]_2) &= (\mu_{x_1}\lambda_{x_2} - \lambda_{x_1}\mu_{x_2}) p_2, \end{aligned} \tag{43}$$

$$\begin{aligned} \lambda_{[x_1,x_2]_1} &= \mu_{[x_1,x_2]_2} = \mu_{x_1}\lambda_{x_2} - \lambda_{x_1}\mu_{x_2}, \\ \mu_{[x_1,x_2]_1} &= \lambda_{[x_1,x_2]_2} = 0, \end{aligned} \tag{44}$$

$$\begin{aligned} D_k(x_1)(x_2) &= -D_k(x_2)(x_1), \\ &\qquad \textit{for all } x_1, x_2 \in L,\ k = 1, 2, \end{aligned} \tag{45}$$

where $x_1, x_2, x_3 \in L$, $D(x_i) = \lambda_{x_i} p_1 + \mu_{x_i} p_2$, $i = 1, 2, 3$.

Proof. If A is a two-dimensional extension 3-Lie algebra, then, by Definition 16, linear mappings D_k satisfy that $D_k(L_k) \subseteq Der(L_k)$, and D_k are Lie homomorphisms, $k = 1, 2$. Thanks to (39),

$$\begin{aligned} D_1(x_3)([x_1, x_2]_2) &= [p_2, [p_1, x_3, x_1], x_2] \\ &\quad + [p_2, x_1, [p_1, x_3, x_2]] \\ &\quad + [[p_1, x_3, p_2], x_1, x_2] \\ &= [D_1(x_3)(x_1), x_2]_2 \\ &\quad + [x_1, D_1(x_3)(x_2)]_2 \\ &\quad - \lambda_{x_3}[x_1, x_2]_1 - \mu_{x_3}[x_1, x_2]_2, \end{aligned} \tag{46}$$

for all $x_1, x_2, x_3 \in L$, (41) holds. Similarly, we have (42).

Thanks to (39) and (40),

$$\begin{aligned} D([x_1, x_2]_1) &= ad(p_1, p_2)[x_1, x_2]_1 \\ &= \lambda_{[x_1,x_2]_1} p_1 + \mu_{[x_1,x_2]_1} p_2 \\ &= (\mu_{x_1}\lambda_{x_2} - \lambda_{x_1}\mu_{x_2}) p_1, \\ D([x_1, x_2]_2) &= ad(p_1, p_2)[x_1, x_2]_2 \\ &= \lambda_{[x_1,x_2]_2} p_1 + \mu_{[x_1,x_2]_2} p_2 \\ &= (\mu_{x_1}\lambda_{x_2} - \lambda_{x_1}\mu_{x_2}) p_2, \end{aligned} \tag{47}$$

Equations (43) and (44) hold. Equation (45) follows from (39) and (40), directly.

Conversely, by (39), $\forall x_1, x_2, x_3, x \in L$,

$$\begin{aligned} [x_1, x_2, x_3] &= 0, \\ [p_1, x_1, x_2] &= D_1(x_1)(x_2) = [x_1, x_2]_1, \\ [p_2, x_1, x_2] &= D_2(x_1)(x_2) = [x_1, x_2]_2, \\ [p_1, p_2, x] &= D(x) = \lambda(x) p_1 + \mu_x p_2. \end{aligned} \tag{48}$$

Since $D_k(L) \subseteq Der(L_k)$ and D_k are Lie homomorphisms, $B = L \dotplus \mathbb{F}p_1$ and $C = L \dotplus \mathbb{F}p_2$ are 3-Lie algebras, which are one-dimensional extension 3-Lie algebras of Lie algebras L_k, $k = 1, 2$, respectively.

Next we only need to prove that the multiplication on A defined by (39) satisfies (1). For all $x_i \in L$, $1 \le i \le 5$, that products $[[x_1, x_2, x_3], x_4, x_5]$, $[[p_j, x_2, x_3], x_4, x_5]$, $[[x_1, x_2, x_3], x_4, p_j]$ and $[[x_1, x_2, p_j], x_4, p_j]$ satisfy (1), $j = 1, 2$ follow from that B and C are one-dimensional extension 3-Lie algebras of L_k and (39), directly.

From (41) and (42), it follows that products $[[p_i, x_1, x_2], p_j, x_3]$, $1 \le i \ne j \le 2$, satisfy (1). It follows from (43)–(45) that products $[p_1, p_2, [p_i, x_1, x_2]]$, $[x_1, x_2, [p_i, p_2, x_3]]$, and $[p_i, x_1, [p_1, p_2, x_2]]$, $i = 1, 2$, satisfy (1). We omit the computation process. □

Theorem 18. *Let $(A, [, ,])$ be a 3-Lie algebra. Then A is a two-dimensional extension 3-Lie algebra of Lie algebras if and only if there is an involutive derivation T on A such that $\dim A_1 = 2$ or $\dim A_{-1} = 2$.*

Proof. If A is a two-dimensional extension 3-Lie algebra of Lie algebras. Then by Theorem 15, there are Lie algebras $L_1 = (L, [,]_1)$ and $L_2 = (L, [,]_2)$, such that $A = L \dotplus W$ and the multiplication of A is defined by (39), where $W = \mathbb{F}p_1 + \mathbb{F}p_2$.

Define the endomorphism T of A by $T(x) = x, T(p_1) = -p_1, T(p_2) = -p_2$, or $T(x) = -x, T(p_1) = p_1, T(p_2) = p_2, \forall x \in L$. Then $T^2 = Id$, and $A_1 = L$, $A_{-1} = W$, or $A_{-1} = L$, $A_1 = W$. Thanks to (38) and (41)-(45), T is a derivation of A.

Conversely, if there is an involutive derivation T on the 3-Lie algebra A such that $\dim A_{-1} = 2$ (or $\dim A_1 = 2$). By Theorem 4, $[A_1, A_1, A_1] = 0$, $[A_1, A_1, A_{-1}] \subseteq A_1$, $[A_1, A_{-1}, A_{-1}] \subseteq A_{-1}$. Let $L = A_1$ and $A_{-1} = \mathbb{F}p_1 \dotplus \mathbb{F}p_2$. Then $[L, L, p_1] \subseteq L, [L, L, p_2] \subseteq L$, and $(L, [,]_1)$ and

$(L, [,]_2)$ are Lie algebras, where $[x, y]_1 = [x, y, p_1]$, $[x, y]_2 = [x, y, p_2]$, $\forall x, y \in L$. Thanks to Theorem 17, the 3-Lie algebra A is a two-dimensional extension 3-Lie algebra of Lie algebras L_1 and L_2. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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