Research Article

Generalized Bell Numbers and Peirce Matrix via Pascal Matrix

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With the Stirling matrix S and the Pascal matrix T , we show that $T^k S$ ($k \geq 0$) satisfies a type of generalized Stirling recurrence. Then, by expressing the sum of components of each row of $T^k S$ as k -Bell number, we investigate properties of k -Bell numbers as well as k -Peirce matrix.

1. Introduction

The Stirling number $s_{m,n}$ of the second kind is the number of ways to partition an m elements set into n nonempty subsets for any $m, n \geq 0$. $s_{m,n}$ can be expressed by $n!s_{m,n} = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^m$, and the Stirling matrix $S = [s_{m,n}]$ ($m, n \geq 0$) satisfies the Stirling recurrence $s_{m,n} = s_{m-1,n-1} + ns_{m-1,n}$ [1]. Some researches including [2–4] were devoted to investigating the Stirling matrix S with the Pascal matrix T and binomial expressions. On the other hand, the sum $B_m = \sum_{k=0}^m s_{m,k}$ of numbers of the m th row of S is called a Bell number, so $\{B_m \mid m \geq 0\} = \{1, 1, 2, 5, 15, \dots\}$ counts the number of partitions of an m elements set. One effective way for generating Bell numbers is to use the Peirce matrix (often called Bell table or Aitken's array), on which Bell numbers appear along both borders [5–7].

A main purpose of work is to study $T^k S$ for $k \geq 1$. We show that $T^k S$ satisfies a kind of generalized Stirling recurrence, and then by expressing the sum of components of each row of $T^k S$ as a k -Bell number we investigate k -Bell numbers as well as k -Peirce matrix. We discuss recurrence rules of a k -Peirce matrix and then interrelationships between each k -Peirce matrices.

2. Generalized k -Stirling Matrix and k -Bell Numbers

Throughout the work, we write T for the Pascal matrix and r_m for the m th row of T . And M^{tr} means

a transpose matrix of M . Let the Stirling matrix be

$$S = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \\ \vdots & & & & \end{bmatrix}$$

$$\text{and } S^* = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 7 & 6 & 1 & \\ 1 & 15 & 25 & 10 & 1 \\ \vdots & & & & \end{bmatrix}. \quad (1)$$

$$\text{Clearly } S \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \end{bmatrix}.$$

Theorem 1. We have the following:

$$(1) S \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix} = S^* \begin{bmatrix} 1 \\ 1 \\ 1 \\ \dots \end{bmatrix}, S^* \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix} = \begin{bmatrix} B_2-B_1 \\ B_3-B_2 \\ B_4-B_3 \\ \dots \end{bmatrix} \text{ and } S^* \begin{bmatrix} 2 \\ 3 \\ 4 \\ \dots \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix}.$$

$$(2) T \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \dots \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix} \text{ and } T \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix} = S^* \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix}.$$

Proof. The first two identities in (1) are easy to observe. And

$$S^* \begin{bmatrix} 2 \\ 3 \\ 4 \\ \dots \end{bmatrix} = S^* \begin{bmatrix} 1 \\ 1 \\ 1 \\ \dots \end{bmatrix} + S^* \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix}$$

$$= \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix} + \begin{bmatrix} B_2-B_1 \\ B_3-B_2 \\ B_4-B_3 \\ \dots \end{bmatrix} = \begin{bmatrix} B_2 \\ B_3 \\ B_4 \\ \dots \end{bmatrix}. \quad (2)$$

Clearly $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. And $B_m = \sum_{i=0}^{m-1} \binom{m-1}{i} B_i$ in [8] proves $r_m(B_0, \dots, B_{m-1})^{tr} = B_m$ for $m \geq 1$, so we have $T \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \dots \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix}$. Moreover since

$$r_{m+1} = \left(\binom{m-1}{0}, \dots, \binom{m-1}{m-1}, 0 \right) + \left(0, \binom{m-1}{0}, \dots, \binom{m-1}{m-1} \right) \quad (3)$$

$$= (r_m; 0) + (0; r_m),$$

it follows that

$$B_{m+1} = r_{m+1}(B_0, B_1, \dots, B_m)^{tr}$$

$$= (r_m; 0)(B_0, B_1, \dots, B_m)^{tr} + (0; r_m)(B_0, B_1, \dots, B_m)^{tr} \quad (4)$$

$$= r_m(B_0, \dots, B_{m-1})^{tr} + r_m(B_1, \dots, B_m)^{tr}$$

$$= B_m + r_m(B_1, \dots, B_m)^{tr}.$$

So $r_m(B_1, \dots, B_m)^{tr} = B_{m+1} - B_m$ and $T \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \dots \end{bmatrix} = \begin{bmatrix} B_2-B_1 \\ B_3-B_2 \\ B_4-B_3 \\ \dots \end{bmatrix} = S^* \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix}$. $\square$

Hence the next corollary follows immediately.

Corollary 2 ($TS = S^*$). As a generalization of $S^* = TS$, we shall consider $T^k S$ for $k > 0$. Write $S^{(k)} = T^k S = [s_{m,n}^{(k)}]$ for $m, n \geq 0$, and let $S^{(0)} = S$. Clearly $S^{(1)} = S^*$.

Theorem 3. $S^{(k)}$ satisfies the recurrence $s_{m+1,n+1}^{(k)} = s_{m,n}^{(k)} + (n+k)s_{m,n+1}^{(k)}$.

Proof. $S^{(1)} = [s_{m,n}^{(1)}] = S^*$ in Corollary 2 shows $s_{m+1,n+1}^{(1)} = s_{m,n}^{(1)} + (n+1)s_{m,n+1}^{(1)}$. And $S^{(2)} = \begin{bmatrix} 1 & 1 \\ 2 & 5 & 1 \\ 8 & 19 & 9 & 1 \\ 16 & 65 & 55 & 14 & 1 \end{bmatrix}$, $S^{(3)} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 7 & 1 \\ 27 & 37 & 12 & 1 \\ 81 & 175 & 97 & 18 & 1 \end{bmatrix}$ give identities $55 = 19 + (4)9$ and $97 = 37 + (5)12$, and it is easy to observe

$$s_{m+1,n+1}^{(k)} = s_{m,n}^{(k)} + (n+k)s_{m,n+1}^{(k)} \quad (5)$$

for $1 \leq n, m \leq 5$; $k = 2, 3$. Now we assume $s_{m+1,n+1}^{(k)} = s_{m,n}^{(k)} + (n+k)s_{m,n+1}^{(k)}$ for some k, m . Since

$$[s_{m,n}^{(k+1)}] = S^{(k+1)} = T^{k+1}S = TT^kS = TS^{(k)}, \quad (6)$$

any (m, n) th component $s_{m,n}^{(k+1)}$ comes from the m th row r_m of T and the n th column of $S^{(k)}$. So

$$s_{m,n}^{(k+1)} = \left(\underbrace{\binom{m-1}{0}, \dots, \binom{m-1}{n-2}, \binom{m-1}{n-1}}_{n-1}, \dots \right) \cdot \left(\underbrace{0, \dots, 0}_{n-1}, s_{n,n}^{(k)} \right) \quad (7)$$

$$= \binom{m-1}{n} s_{n,n}^{(k)} + \dots + \binom{m-1}{m-1} s_{m-1,n}^{(k)}$$

$$s_{n+1,n}^{(k)}, \dots, s_{m-1,n}^{(k)}, s_{m,n}^{(k)} \Big)^{tr} = \binom{m-1}{n-1} s_{n,n}^{(k)}$$

$$+ \binom{m-1}{n} s_{n+1,n}^{(k)} + \dots + \binom{m-1}{m-2} s_{m-1,n}^{(k)}$$

$$+ \binom{m-1}{m-1} s_{m,n}^{(k)}$$

and

$$s_{m,n+1}^{(k+1)} = \left(\underbrace{\binom{m-1}{0}, \dots, \binom{m-1}{n-1}, \binom{m-1}{n}}_n, \dots \right) \cdot \left(\underbrace{0, \dots, 0}_n, s_{n,n+1}^{(k)} \right) \quad (8)$$

$$= \binom{m-1}{n+1} s_{n,n+1}^{(k)} + \dots + \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} + s_{m,n+1}^{(k)}$$

$$s_{n+1,n+1}^{(k)}, s_{n+2,n+1}^{(k)}, \dots, s_{m-1,n+1}^{(k)}, s_{m,n+1}^{(k)} \Big)^{tr}$$

$$= \binom{m-1}{n} s_{n+1,n+1}^{(k)} + \binom{m-1}{n+1} s_{n+2,n+1}^{(k)} + \dots$$

$$+ \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} + s_{m,n+1}^{(k)}.$$

Therefore we have

$$\begin{aligned}
 s_{m,n}^{(k+1)} + (n+k+1)s_{m,n+1}^{(k+1)} &= \left[\binom{m-1}{n-1} s_{n,n}^{(k)} \right. \\
 &+ \binom{m-1}{n} s_{n+1,n}^{(k)} + \cdots + \binom{m-1}{m-2} s_{m-1,n}^{(k)} + s_{m,n}^{(k)} \Big] \\
 &+ (n+k+1) \left[\binom{m-1}{n} s_{n+1,n+1}^{(k)} + \cdots \right. \\
 &+ \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} + s_{m,n+1}^{(k)} \Big] = \left[\binom{m-1}{n-1} s_{n,n}^{(k)} \right. \\
 &+ \binom{m-1}{n} s_{n+1,n}^{(k)} + \cdots + \binom{m-1}{m-2} s_{m-1,n}^{(k)} + s_{m,n}^{(k)} \Big] \\
 &+ \left[\binom{m-1}{n} (n+k) s_{n+1,n+1}^{(k)} + \cdots \right. \\
 &+ \binom{m-1}{m-2} (n+k) s_{m-1,n+1}^{(k)} + (n+k) s_{m,n+1}^{(k)} \Big] \\
 &+ \left[\binom{m-1}{n} s_{n+1,n+1}^{(k)} + \cdots + \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} \right. \\
 &+ s_{m,n+1}^{(k)} \Big] = \binom{m-1}{n-1} s_{n,n}^{(k)} + \binom{m-1}{n} \left[s_{n+1,n}^{(k)} \right. \\
 &+ (n+k) s_{n+1,n+1}^{(k)} + \cdots + \binom{m-1}{m-2} \left[s_{m-1,n}^{(k)} \right. \\
 &+ (n+k) s_{m-1,n+1}^{(k)} + \left[s_{m,n}^{(k)} + (n+k) s_{m,n+1}^{(k)} \right] \\
 &+ \left[\binom{m-1}{n} s_{n+1,n+1}^{(k)} + \cdots + \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} \right. \\
 &+ s_{m,n+1}^{(k)} \Big] = \binom{m-1}{n-1} s_{n,n}^{(k)} + \binom{m-1}{n} s_{n+2,n+1}^{(k)} + \cdots \\
 &+ \binom{m-1}{m-2} s_{m,n+1}^{(k)} + s_{m+1,n+1}^{(k)} + \left[\binom{m-1}{n} s_{m,n+1}^{(k)} + s_{m+1,n+1}^{(k)} \right. \\
 &+ \cdots + \binom{m-1}{m-2} s_{m-1,n+1}^{(k)} + s_{m,n+1}^{(k)} \Big]
 \end{aligned} \tag{9}$$

by the induction hypothesis on $S^{(k)}$. But since $s_{i,i}^{(k)} = 1$ for all i , it follows immediately from Pascal's rule that

$$\begin{aligned}
 s_{m,n}^{(k+1)} + (n+k+1)s_{m,n+1}^{(k+1)} &= \left[\binom{m-1}{n-1} + \binom{m-1}{n} \right] s_{n,n}^{(k)} \\
 &+ \left[\binom{m-1}{n} + \binom{m-1}{n+1} \right] s_{n+2,n+1}^{(k)} + \cdots \\
 &+ \left[\binom{m-1}{m-2} + 1 \right] s_{m,n+1}^{(k)} + s_{m+1,n+1}^{(k)} \\
 &= \binom{m}{n} s_{n,n}^{(k)} + \binom{m}{n+1} s_{n+2,n+1}^{(k)} + \cdots \\
 &+ \binom{m}{m-1} s_{m,n+1}^{(k)} + s_{m+1,n+1}^{(k)} \\
 &= \binom{m}{n} s_{n+1,n+1}^{(k)} + \binom{m}{n+1} s_{n+2,n+1}^{(k)} + \cdots \\
 &+ \binom{m}{m-1} s_{m,n+1}^{(k)} + s_{m+1,n+1}^{(k)} \\
 &= \left(\underbrace{\binom{m}{0}, \dots, \binom{m}{n-1}}_n, \binom{m}{n}, \dots, \binom{m}{m-1}, 1 \right) \\
 &\cdot \left(\underbrace{0, \dots, 0}_n, s_{n+1,n+1}^{(k)}, \dots, s_{m+1,n+1}^{(k)} \right)^{tr} = s_{m+1,n+1}^{(k+1)}.
 \end{aligned} \tag{10}$$

□

Theorem 3 agrees with the Stirling rule if $k = 1$, so we may call $T^k S = S^{(k)}$ the **generalized k -Stirling matrix** for $k > 0$. Like the Bell numbers which are the sum of each row of S , we will take sum of each row of the k -Stirling matrix $S^{(k)}$ and call this the **k -Bell number** and denote it by $B^{(k)} = \{B_i^{(k)} \mid i \geq 0\}$. The first few terms of the k -Bell numbers are obtained by examining $S^{(k)}$.

	$B^{(k)} = \{B_i^{(k)}\}$								
i	0	1	2	3	4	5	6	7	8
$B^{(0)}$	1	1	2	5	15	52	203	877	4140 ...
$B^{(1)}$	1	2	5	15	52	203	877	4140	21147 ...
$B^{(2)}$	1	3	10	37	151	674	3263	17007	94828 ...
$B^{(3)}$	1	4	17	77	372	1915	10481	60814	372939 ...
$B^{(4)}$	1	5	26	141	799	4736	29371	190497	1291020 ...
$B^{(5)}$	1	6	37	235	1540	10427	73013	529032	3967195 ...

(11)

Obviously $B_i^{(0)} = B_i$ the (original) Bell numbers, and $B_i^{(1)} = B_{i+1}^{(0)}$, so $B_i^{(2)} = B_{i+1}^{(1)} - B_i^{(1)} = B_{i+2}^{(0)} - B_{i+1}^{(0)}$ ($i \geq 0$) by Theorem 1. Note that $B^{(2)}$ is listed in A005493-OEIS as the numbers of partitions with a distinguished block. And $B^{(k)}$ ($3 \leq k \leq 5$) are coefficients of $\exp(kx + \exp(x) - 1)$ (A005494, A045379, and A196834). On the other hand, the k -Bell numbers $B^{(k)}$ were studied in [9] by using certain binomial expressions $B_m = \sum_{i=0}^m \{m \atop i\}$ and $B_m^{(k)} = \sum_{i=0}^m \{m+k \atop i+k\}$. A key feature of our study is to have these k -Bell numbers from multiplications of T and S explicitly. Hence it enables us to find relations between k -Bell numbers $B^{(k)}$ and $B^{(k+1)}$, as follows.

Theorem 4. $S^{(k)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = T \begin{bmatrix} B_0^{(k-1)} \\ B_1^{(k-1)} \\ B_2^{(k-1)} \\ \vdots \end{bmatrix}$ and $S^{(k)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(k+1)} \\ B_1^{(k+1)} \\ B_2^{(k+1)} \\ \vdots \end{bmatrix}$.

Proof. Since $S^{(k)} = T^k S = TS^{(k-1)}$, it is clear that

$$\begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ \vdots \end{bmatrix} = S^{(k)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = TS^{(k-1)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = T \begin{bmatrix} B_0^{(k-1)} \\ B_1^{(k-1)} \\ B_2^{(k-1)} \\ \vdots \end{bmatrix}. \quad (12)$$

And, due to Theorem 1, we have

$$S^{(0)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_1^{(0)} \\ B_2^{(0)} \\ B_3^{(0)} \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(1)} \\ B_1^{(1)} \\ B_2^{(1)} \\ \vdots \end{bmatrix} \quad (13)$$

$$\text{and } S^{(1)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_2^{(0)} - B_1^{(0)} \\ B_3^{(0)} - B_2^{(0)} \\ B_4^{(0)} - B_3^{(0)} \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(2)} \\ B_1^{(2)} \\ B_2^{(2)} \\ \vdots \end{bmatrix}.$$

So if we assume $S^{(k-1)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ \vdots \end{bmatrix}$ for some k then we have

$$S^{(k)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = TS^{(k-1)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \end{bmatrix} = T \begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(k+1)} \\ B_1^{(k+1)} \\ B_2^{(k+1)} \\ \vdots \end{bmatrix}. \quad (14)$$

□

Theorem 4 says $T \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix}$, so $T \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix} = T^2 \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \end{bmatrix}$. Thus the expression $r_m(B_1, \dots, B_m)^{tr}$ in the proof of Theorem 1, which is the multiplication of the m th row of T

with $(B_1, \dots, B_m)^{tr}$, is equal to multiplication of the m th row of T^2 with $(B_0, \dots, B_{m-1})^{tr}$. Note that T^2 equals the arithmetic table $T^{(2x+1)}$ of the polynomial $(2x+1)^n$, while Pascal's table T is the arithmetic table of $(x+1)^n$. Hence if we let $r_m^{(2x+1)}$ be the m th row of $T^{(2x+1)} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix}$ of $(2x+1)^n$ then

$$\begin{aligned} B_{m+1} - B_m &= \sum_{i=0}^m \binom{m-1}{i} B_{i+1} \\ &= \left(\binom{m-1}{0}, \binom{m-1}{1}, \dots, \binom{m-1}{m-1} \right) \\ &\quad \cdot (B_1, \dots, B_m)^{tr} = r_m^{(2x+1)} (B_0, \dots, B_{m-1})^{tr}, \end{aligned} \quad (15)$$

which implies $T^2 \begin{bmatrix} B_0 \\ B_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_2 - B_1 \\ B_3 - B_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} B_0^{(2)} \\ B_1^{(2)} \\ \vdots \end{bmatrix} = S^{(2)} \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$. Thus, owing to Theorem 4, it is clear to have generalizations to k -Bell numbers ($k \geq 0$) that

$$\begin{aligned} T^u \begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ \vdots \end{bmatrix} &= T^u S^{(k)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = S^{(u+k)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} B_0^{(u+k)} \\ B_1^{(u+k)} \\ B_2^{(u+k)} \\ \vdots \end{bmatrix}. \end{aligned} \quad (16)$$

3. Matrix of General Bell Numbers

With all the j -Bell numbers $B^{(j)}$ ($j \geq 1$), we make a table $\widehat{GB}$ in which each j th column is composed of j -Bell numbers. We call it a **matrix of general Bell numbers** and denote it by $\widehat{GB} = [b_{i,j}]$ ($i, j \geq 1$).

$$\begin{array}{c} \widehat{GB} = \left[\begin{array}{c|c|c|c|c|c|c} B^{(1)} & \dots & B^{(7)} & \dots & \end{array} \right] \\ \hline \begin{array}{cccccccc} 1 & & & & & & & \\ 2 & 1 & & & & & & \\ 5 & 3 & 1 & & & & & \\ 15 & 10 & 4 & 1 & & & & \\ 52 & 37 & 17 & 5 & 1 & & & \\ 203 & 151 & 77 & 26 & 6 & 1 & & \\ 877 & 674 & 372 & 141 & 37 & 7 & 1 & \end{array} \end{array} \quad (17)$$

Theorem 5. The entries of the $(j+1)$ th column in $\widehat{GB} = [b_{i,j}]$ satisfy $b_{i,j+1} = b_{i,j} - j b_{i-1,j}$. So $B_i^{(j+1)} = B_{i+1}^{(j)} - j B_i^{(j)}$.

Proof. Observe that each j th column ($2 \leq j \leq 5$) satisfies the following:

2nd col	3rd col	4th col	5th col
$2 - 1 = 1$	$3 - (2) 1 = 1$	$4 - (3) 1 = 1$	$5 - (4) 1 = 1$
$5 - 2 = 3$	$10 - (2) 3 = 4$	$17 - (3) 4 = 5$	$26 - (4) 5 = 6$
$15 - 5 = 10$	$37 - (2) 10 = 17$	$77 - (3) 17 = 26$	$141 - (4) 26 = 37$
$52 - 15 = 37$	$151 - (2) 37 = 77$	$372 - (3) 77 = 141$	$799 - (4) 141 = 235$

(18)

Suppose $b_{i,j+1} = b_{i,j} - j b_{i-1,j}$ for some $i, j > 1$. By looking at

$$\begin{aligned}
 [b_{i,j}] &= \begin{bmatrix} b_{1,1} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} & b_{3,3} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ \dots \end{bmatrix} \\
 &= \begin{bmatrix} B_0^{(1)} \\ B_1^{(1)} & B_0^{(2)} \\ B_2^{(1)} & B_1^{(2)} & B_0^{(3)} \\ B_3^{(1)} & B_2^{(2)} & B_1^{(3)} & B_0^{(4)} \\ \dots \end{bmatrix}
 \end{aligned} \tag{19}$$

it is enough to prove the following two cases.

(i) For any $i \geq 0$, assume $B_i^{(j)} = B_{i+1}^{(j-1)} - (j-1)B_i^{(j-1)}$ in some j th column and show $B_i^{(j+1)} = B_{i+1}^{(j)} - jB_i^{(j)}$ is true at $(j+1)$ th column.

(ii) For any $j \geq 1$, assume $B_{t-1}^{(j)} = B_t^{(j-1)} - (j-1)B_{t-1}^{(j-1)}$ at every t th row ($1 \leq t \leq i$) and show $B_i^{(j)} = B_{i+1}^{(j-1)} - (j-1)B_i^{(j-1)}$ is true at $(i+1)$ th row.

Using the i th row r_i of T , the induction hypothesis in (i) with (16) implies

$$\begin{aligned}
 B_i^{(j+1)} &= r_{i+1} (B_0^{(j)}, B_1^{(j)}, \dots, B_i^{(j)})^{tr} = r_{i+1} (B_1^{(j-1)} \\
 &\quad - (j-1)B_0^{(j-1)}, \dots, B_{i+1}^{(j-1)} - (j-1)B_i^{(j-1)})^{tr} \\
 &= r_{i+1} (B_1^{(j-1)}, \dots, B_{i+1}^{(j-1)})^{tr} - (j-1) \\
 &\quad \cdot r_{i+1} (B_0^{(j-1)}, \dots, B_i^{(j-1)})^{tr} = \left[\binom{i}{0} B_1^{(j-1)} \right. \\
 &\quad \left. + \binom{i}{1} B_2^{(j-1)} + \dots + \binom{i}{i} B_{i+1}^{(j-1)} \right] - (j-1) \\
 &\quad \cdot \left[\binom{i}{0} B_0^{(j-1)} + \binom{i}{1} B_1^{(j-1)} + \dots + \binom{i}{i} B_i^{(j-1)} \right] \\
 &= \left(-(j-1) \binom{i}{0}, \binom{i}{0} \right)
 \end{aligned}$$

$$\begin{aligned}
 &-(j-1) \binom{i}{1}, \dots, \binom{i}{i-1} - (j-1) \binom{i}{i}, \binom{i}{i} \\
 &\cdot (B_0^{(j-1)}, B_1^{(j-1)}, \dots, B_{i+1}^{(j-1)})^{tr}.
 \end{aligned} \tag{20}$$

On the other hand, we also have

$$\begin{aligned}
 B_{i+1}^{(j)} - jB_i^{(j)} &= r_{i+2} (B_0^{(j-1)}, B_1^{(j-1)}, \dots, B_{i+1}^{(j-1)})^{tr} \\
 &\quad - j r_{i+1} (B_0^{(j-1)}, B_1^{(j-1)}, \dots, B_i^{(j-1)})^{tr} \\
 &= \left[\binom{i+1}{0} B_0^{(j-1)} + \binom{i+1}{1} B_1^{(j-1)} \right. \\
 &\quad \left. + \binom{i+1}{2} B_2^{(j-1)} + \dots + \binom{i+1}{i+1} B_{i+1}^{(j-1)} \right] \\
 &\quad - j \left[\binom{i}{0} B_0^{(j-1)} + \binom{i}{1} B_1^{(j-1)} + \dots + \binom{i}{i} B_i^{(j-1)} \right] \\
 &= \left(\binom{i+1}{0} - j \binom{i}{0}, \binom{i+1}{1} - j \binom{i}{1}, \binom{i+1}{2} \right. \\
 &\quad \left. - j \binom{i}{2}, \dots, \binom{i+1}{i} - j \binom{i}{i}, \binom{i+1}{i+1} \right) \\
 &\quad \cdot (B_0^{(j-1)}, \dots, B_i^{(j-1)}, B_{i+1}^{(j-1)})^{tr} \\
 &= \left(-(j-1) \binom{i}{0}, \binom{i}{0} \right) \\
 &\quad - (j-1) \binom{i}{1}, \dots, \binom{i}{i-1} - (j-1) \binom{i}{i}, \binom{i}{i} \\
 &\quad \cdot (B_0^{(j-1)}, B_1^{(j-1)}, \dots, B_{i+1}^{(j-1)})^{tr},
 \end{aligned} \tag{21}$$

for $\binom{i+1}{u} - j \binom{i}{u} = \binom{i}{u-1} + \binom{i}{u} - j \binom{i}{u} = \binom{i}{u-1} - (j-1) \binom{i}{u}$ for all $0 \leq u \leq i$. Hence identities (20) and (21) prove (i).

Now, for (ii), the induction hypothesis $B_t^{(j-1)} + (j-2)B_t^{(j-2)} = B_{t+1}^{(j-1)}$ ($1 \leq t \leq i$) in (ii) with again (16) implies

$$\begin{aligned}
 B_i^{(j)} + (j-1)B_i^{(j-1)} &= B_i^{(j)} + (j-2)B_i^{(j-1)} + B_i^{(j-1)} \\
 &= r_{i+1} (B_0^{(j-1)}, \dots, B_i^{(j-1)})^{tr} + (j-2)
 \end{aligned}$$

$$\begin{aligned}
& \cdot r_{i+1} (B_0^{(j-2)}, \dots, B_i^{(j-2)})^{tr} \\
& + r_{i+1} (B_0^{(j-2)}, \dots, B_i^{(j-2)})^{tr} = r_{i-1} (B_0^{(j-1)}) \\
& + (j-2) B_0^{(j-2)}, \dots, B_i^{(j-1)} + (j-2) B_i^{(j-2)} \\
& + r_{i+1} (B_0^{(j-2)}, \dots, B_i^{(j-2)})^{tr} \\
& = r_{i+1} (B_1^{(j-2)}, \dots, B_{i+1}^{(j-2)})^{tr} \\
& + r_{i+1} (B_0^{(j-2)}, \dots, B_i^{(j-2)})^{tr}
\end{aligned} \tag{22}$$

Moreover

$$\begin{aligned}
B_{i+1}^{(j-1)} &= r_{i+2} (B_0^{(j-2)}, B_1^{(j-2)}, \dots, B_{i+1}^{(j-2)})^{tr} \\
&= [(0; r_{i+1}) + (r_{i+1}; 0)] (B_0^{(j-2)}, B_1^{(j-2)}, \dots, B_{i+1}^{(j-2)})^{tr} \\
&= r_{i+1} (B_1^{(j-2)}, \dots, B_{i+1}^{(j-2)})^{tr} \\
&+ r_{i+1} (B_0^{(j-2)}, \dots, B_i^{(j-2)})^{tr}
\end{aligned} \tag{23}$$

Then we complete the proof by comparing (22) and (23). $\square$

Therefore Theorem 5 with (16) shows

$$\begin{aligned}
T^u \begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ \dots \end{bmatrix} &= \begin{bmatrix} B_0^{(u+k)} \\ B_1^{(u+k)} \\ B_2^{(u+k)} \\ \dots \end{bmatrix} \\
&= \begin{bmatrix} B_1^{(u+k-1)} - (k+u-1) B_0^{(u+k-1)} \\ B_2^{(u+k-1)} - (k+u-1) B_1^{(u+k-1)} \\ B_3^{(u+k-1)} - (k+u-1) B_2^{(u+k-1)} \\ \dots \end{bmatrix}.
\end{aligned} \tag{24}$$

As an example, the 6-Bell numbers $B^{(6)} = \{B_i^{(6)}\} = \{1, 7, 50, 365, 2727, \dots\}$ and then 7-Bell numbers $B^{(7)} = \{B_i^{(7)}\} = \{1, 8, 65, 537, 4516, \dots\}$ are obtained sequentially from the 5-Bell numbers.

5th col.	6th col.	7th col.
6	$6 - (5)1 = 1$	$7 - (6)1 = 1$
37	$37 - (5)6 = 7$	$50 - (6)7 = 8$
235	$235 - (5)37 = 50$	$365 - (6)50 = 65$
1540	$1540 - (5)235 = 365$	$2727 - (6)365 = 537$

The next theorem gives another way to have k -Bell numbers $B^{(k)}$. Let $[0]_j$ denote the j -tuple $\{0, \dots, 0\}$.

Theorem 6. Let $X_j = ([0]_{j-1}; j, j+1, \dots, n)^{tr}$ and $\rho_n = (b_{n,1}, \dots, b_{n,n})$ be the n th row of $\widehat{GB}$. Then $\rho_n X_j + 1 = b_{n+1,j}$ for any $1 \leq j \leq n$.

Proof. Clearly $\widehat{GB}X_1 + 1 = \widehat{GB} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \dots \end{bmatrix} + 1 = \begin{bmatrix} 2 \\ 5 \\ 15 \\ \dots \end{bmatrix}$ and $\widehat{GB}X_2 + 1 = \widehat{GB} \begin{bmatrix} 0 \\ 2 \\ 3 \\ \dots \end{bmatrix} + 1 = \begin{bmatrix} 1 \\ 3 \\ 10 \\ \dots \end{bmatrix}$. Similarly $\widehat{GB}X_j + 1$ ($j = 3, 4$) equals $\begin{bmatrix} 1 \\ 4 \\ \dots \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ \dots \end{bmatrix}$. Thus, for instance, when $j = 4$, with n th row ρ_n of $\widehat{GB}$, we have $\rho_3 X_4 + 1 = d_{4,4} = 1$, $\rho_4 X_4 + 1 = d_{5,4} = 5$, $\rho_5 X_4 + 1 = d_{6,4} = 26$, $\rho_6 X_4 + 1 = d_{7,4} = 141$, and so on. Hence we can say that $\rho_n X_4 + 1 = b_{n+1,4}$ for $n \geq 3$.

Now assume $\rho_n X_j + 1 = b_{n+1,j} - 1$ for $n \geq j-1$. Then since $X_{j+1} = X_j - ([0]_{j-1}; j, 0, \dots, 0)^{tr}$, Theorem 5 implies

$$\begin{aligned}
\rho_n X_{j+1} &= \rho_n X_j - \rho_n ([0]_{j-1}; j, 0, \dots, 0)^{tr} \\
&= b_{n+1,j} - 1 - j b_{n,j} = b_{n+1,j+1} - 1.
\end{aligned} \tag{26}$$

$\square$

Theorem 6 can be restated in terms of k -Bell numbers.

Corollary 7. $(B_n^{(1)}, B_{n-1}^{(2)}, \dots, B_1^{(n)}) ([0]_{j-1}; j, j+1, \dots, n)^{tr} = B_{n-j+2}^{(j)} - 1$.

Hence $\widehat{GB}(1, 2, \dots, n)^{tr} = (B_{n+1}^{(1)}, \dots, B_{n+1}^{(n)})^{tr}$ and $\widehat{GB}([0]_{j-1}; j, \dots, n)^{tr} = ([0]_{j-1}; B_{n+1}^{(j)}, \dots, B_{n+1}^{(n)})^{tr}$. Precisely, with $\rho_6 = (203, 151, 77, 26, 6, 1)$, we have $\rho_6(0, 0, 0, 0, 0, 6)^{tr} + 1 = 7$, $\rho_6(0, 0, 0, 0, 5, 6)^{tr} + 1 = 37$, $\rho_6(0, 0, 0, 4, 5, 6)^{tr} + 1 = 141$, $\rho_6(0, 0, 3, 4, 5, 6)^{tr} + 1 = 372$, $\rho_6(0, 2, 3, 4, 5, 6)^{tr} + 1 = 674$, and $\rho_6(1, 2, 3, 4, 5, 6)^{tr} + 1 = 877$, where these yield the 7th row $\rho_7 = (877, 674, 372, 141, 37, 7, 1)$.

4. k -Peirce Matrix

The Peirce matrix $[p_{i,j}]$ ($i, j \geq 0$) was designed to generate Bell numbers.

$$P = [p_{i,j}] = \begin{bmatrix} 1 & & & & \\ 1 & 2 & & & \\ 2 & 3 & 5 & & \\ 5 & 7 & 10 & 15 & \\ 15 & 20 & 27 & 37 & 52 \end{bmatrix} \tag{27}$$

where it holds a recurrence $p_{i,j} = p_{i,j-1} + p_{i-1,j-1}$ with $p_{0,0} = B_0$, $p_{n,0} = p_{n-1,n-1} = B_n$. So the 6th row of P begins with 52 followed by 67, 87, 114, and 151 and then reach 203, the next Bell number of 52. Thus all Bell numbers are on both borders of P . The matrix is often called the Bell matrix or Aitken's array named after E. T. Bell and A. Aitken. Here we call it Peirce matrix after C. S. Peirce [10] to avoid confusion with the matrix of general Bell numbers in Section 3.

Now, for any $k \geq 1$, let $P^{(k)} = [p_{i,j}^{(k)}]$ ($i, j \geq 1$) be a matrix satisfying the following two rules.

(i) Each i th row begins with the Bell number $p_{i,1}^{(k)} = B_{i-1}^{(0)}$.

(ii)

$$p_{i,j}^{(k)} = p_{i,j-1}^{(k)} + k p_{i-1,j-1}^{(k)} \quad \text{for all } i > 1, \tag{28}$$

and $p_{i,j}^{(k)} = 0$ for $j > i$.

Obviously $P^{(1)}$ is the Peirce matrix, and $P^{(k)}$ ($k = 2, 3$) are equal to

$$P^{(2)} = \begin{bmatrix} 1 & & & \\ 1 & 3 & & \\ 2 & 4 & 10 & \\ 5 & 9 & 17 & 37 \end{bmatrix} \quad (29)$$

$$\text{and } P^{(3)} = \begin{bmatrix} 1 & & & & \\ 1 & 4 & & & \\ 2 & 5 & 17 & & \\ 5 & 11 & 26 & 77 & \end{bmatrix}$$

Notice that the left border is always comprised of Bell numbers, while the right diagonals (r.diag.) of $P^{(2)}$ and $P^{(3)}$ are of 2 and 3-Bell numbers, respectively. And the second right diagonal (2nd r.diag.) of $P^{(2)}$ equals the r.diag. of $P^{(3)}$. The next theorem shows that the r.diag. of $P^{(k)} = [p_{i,j}^{(k)}]$ ($k \geq 1$) is composed of k -Bell numbers $B_i^{(k)}$.

Theorem 8. (1) The entries of r.diag. of $P^{(k)}$ are k -Bell numbers; i.e., $p_{i,i}^{(k)} = B_{i-1}^{(k)}$, while the entries of left border are Bell numbers.

(2) The 2nd r.diag. is composed of the $(k+1)$ -Bell numbers; i.e., $p_{i,i-1}^{(k)} = B_i^{(k+1)}$.

Proof. Following the rules in (28), $P^{(k)} = [p_{i,j}^{(k)}]$ is equal to

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & k+1 & & & & & \\ 2 & k+2 & k^2+2k+2 & & & & \\ 5 & 2k+5 & k^2+4k+5 & k^3+3k^2+6k+5 & & & \\ 15 & 5k+15 & 2k^2+10k+15 & k^3+6k^2+15k+15 & k^4+4k^3+12k^2+20k+15 & & \\ 52 & 15k+52 & 5k^2+30k+52 & 2k^3+15k^2+45k+52 & k^4+8k^3+30k^2+60k+52 & \dots & \end{array} \quad (30)$$

When $k = 1, 2, 3$, the r.diag. of $P^{(k)}$ yields $\{1, 2, 5, 15, \dots\}$, $\{1, 3, 10, 37, \dots\}$, and $\{1, 4, 17, 77, \dots\}$, where these correspond to $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$.

Assume the r.diag

$$\{1, k+1, k^2+2k+2, k^3+3k^2+6k+5, k^4+4k^3+12k^2+20k+15, \dots\} \quad (31)$$

of $P^{(k)}$ equals the k -Bell numbers $B^{(k)}$. Then, due to (16), we have

$$\begin{bmatrix} B_0^{(k+1)} \\ B_1^{(k+1)} \\ B_2^{(k+1)} \\ B_3^{(k+1)} \\ B_4^{(k+1)} \end{bmatrix} = T \begin{bmatrix} B_0^{(k)} \\ B_1^{(k)} \\ B_2^{(k)} \\ B_3^{(k)} \\ B_4^{(k)} \end{bmatrix}$$

$$= T \begin{bmatrix} 1 \\ k+1 \\ k^2+2k+2 \\ k^3+3k^2+6k+5 \\ k^4+4k^3+12k^2+20k+15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 & 2 & 1 \\ 5 & 6 & 3 & 1 \\ 15 & 20 & 12 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 15 & 15 & 6 & 1 & \\ 52 & 60 & 30 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \end{bmatrix}$$

(32)

And it is not hard to see that it corresponds to the r.diag. of $P^{(k+1)}$

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & k+2 & & & & & \\ 2 & k+3 & k^2+4k+5 & & & & \\ 5 & 2k+7 & k^2+6k+10 & k^3+6k^2+15k+15 & & & \\ 15 & 5k+20 & 2k^2+14k+27 & k^3+9k^2+30k+37 & k^4+8k^3+30k^2+60k+52 & & \\ 52 & 15k+67 & 5k^2+40k+87 & 2k^3+21k^2+81k+114 & \dots & & \end{array} \quad (33)$$

Moreover, by comparing $P^{(k)}$ with $P^{(k+1)}$, we also notice that the r.diag. of $P^{(k+1)}$ equals the 2nd r.diag. of $P^{(k)}$. $\square$

Owing to Theorem 8, we may call $P^{(k)}$ the **Peirce k -matrix**. Write $\text{diag}[a, b, c, \dots]$ for the diagonal matrix having diagonal entries $a, b, c, \dots$. Every k -Bell number can be obtained by Bell numbers as follows.

Theorem 9. $B_n^{(k)} = r_n \text{diag}[B_0, B_1, \dots, B_{n-1}](k^{n-1}, \dots, k, 1)^{tr}$.

Proof. When $k = 1$, it is due to Theorem 1. And, by Theorem 8, we have

$$\begin{aligned} \begin{bmatrix} B_1^{(k)} \\ B_2^{(k)} \\ B_3^{(k)} \\ B_4^{(k)} \\ B_5^{(k)} \end{bmatrix} &= \begin{bmatrix} 1 \\ 1+k \\ 2+2k+k^2 \\ 5+6k+3k^2+k^3 \\ 15+20k+12k^2+4k^3+k^4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 5 & 6 & 3 & 1 & \\ 15 & 20 & 12 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \end{bmatrix}. \end{aligned} \quad (34)$$

But we notice

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 6 & 5 \\ 1 & 4 & 12 & 20 & 15 \end{bmatrix} &= \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 15 \end{bmatrix} \\ &= T \text{diag}[B_0, B_1, \dots, B_4]. \end{aligned} \quad (35)$$

Hence, with the n th row r_n of T , the n th k -Bell number $B_n^{(k)}$ is equal to $r_n \text{diag}[B_0, \dots, B_{n-1}](k^{n-1}, \dots, k, 1)^{tr}$. $\square$

For example, with $r_6 = (1, 5, 10, 10, 5, 1)$, the 6th k -Bell number $B_6^{(k)}$ equals

$$\begin{aligned} &r_6 \text{diag}[B_0, \dots, B_5](k^5, \dots, 1)^{tr} \\ &= r_6 \text{diag}[1, 1, 2, 5, 15, 52](k^5, \dots, 1)^{tr} \\ &= k^5 + 5k^4 + 20k^3 + 50k^2 + 75k + 52. \end{aligned} \quad (36)$$

Data Availability

All data used to support the findings of this study are either obtained via the cited sources or included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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