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Research Article

On Edge Magic Total Labeling of (7, 3)-Cycle Books

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A graph G is called (a, b)-*cycle books* $B[(C_a, m), (C_b, n), P_t]$ if G consists of m cycles C_a and n cycles C_b with a common path P_t. In this article we show that the (7, 3)-cycle books $B[(C_7, 1), (C_3, n), P_2]$ admits edge-magic total labeling. In addition we prove that $(a, 3)$-cycle books $B[(C_a, 1), (C_3, n), P_2]$ admits super edge-magic total labeling for $a = 7$ and extends the values of a to $4x - 1$ for any positive integer x. Moreover we prove that the (7, 3)-cycle books $B[(C_7, 2), (C_3, n), P_2]$ admits super edge-magic total labeling.

1. Introduction

Let G be a graph such that |V (G)| = p and |E(G)| = q. An *edge-magic total labeling* of G is a bijective function f: V (G)∪E(G) ⟶ {1, 2,..., p+q} such that f (w) + f (wz) + f (z) = k for any edge wz ∈ E(G). Moreover G is said to be an *edge-magic total*. If f (V (G)) = {1, 2,..., p}, then f is *the super edge-magic total labeling* of G and G is said to be a *super edge-magic total*.

In this paper we generalize the definition of cycle books [1] and we investigate their edge-magic total labeling. Let m, n, a, b, and t be any positive integers and let C_a and C_b be the cycles of order a and b, respectively. A graph G is called (a, b)-*cycle books* $B[(C_a, m), (C_b, n), P_t]$ if G consists of m cycles C_a and n cycles C_b with a common path P_t.

From now on the graphs (a, b)-*cycle books* $B[(C_a, m), (C_b, n), P_t]$ is denoted by B(a, m, b, n, t). If a = b = 4, m + n = r, and t = 2, then the graphs B(4, m, 4, n, t) are the graphs cycle books B_r [2]. If a = b, m + n = r, and t = 2, then the graphs B(a, m, b, n, t) are the graphs cycle books $B_{a,r}$ [1] and it is the graphs cycle books $\Theta(C_a)^r$ [3]. If m = n = 1 and t = 2, then the graphs B(a,m,b,n,t) are a cycle C^x_{a+b-2} with a chord $x = \min(a, b) - 1$ [4]. If a = b = 2r + 1, m + n ≥ 2, and t = 2r, then the graphs B(a, m, b, n, t) are called a graph with many odd cycles $P_{2r}(+)N_{m+n}$ [5].

It is an open problem to determine the edge-magic total labeling of B(a, m, b, n, t) (see Research Problem 2.7, p.39 [1] for the case B(a, m, a, n, 2)). However some authors provided a partial solution to this problem. Figueroa-Centeno, Ichishima, and Muntaner-Batle [2] proved that B(4, m, 4, n, 2) is an edge-magic total. Moreover they proved that B(4, m, 4, n, 2) is not super edge-magic total for m + n ≡ 1, 3, 7 mod (8) and m + n = 4, B(4, m, 4, n, 2) is super edge-magic total for m + n = 2, 5, 6, 8, 10, 11 and conjecture that B(4, m, 4, n, 2) is super edge-magic total if and only if m + n is even or m + n ≡ 5(mod 8). Gallian [6] pointed out that Yuansheng et al. [7] proved this conjecture when m + n is even. Later on Yegnanarayanan and Vaidhyanathan [8] proved that B(4, m, 4, n, 2) is an edge-magic total.

Note that $C_{(2x+1)(2y+1)}$ [+] N_n ([9], p.7) is isomorphic to B((2x + 1)(2y + 1), 1, 4, n, 3) and $P_{2x}(+)N_n$ [5] is isomorphic to B(2x+1, μ, 2x+1, ν, 2x), $\mu+\nu$ = n. Singgih provided a new method to construct super edge-magic total labeling of graph $C_{(2x+1)(2y+1)}$ [+] N_n ([9], Theorem 4.13.1,p.93) from the super edge-magic total labeling of $P_{2x}(+)N_n$. Hence super edge-magic total labeling of B((2x+1)(2y+1), 1, 4, n, 3) is deduced from the super edge-magic total labeling of B(2x + 1, μ, 2x + 1, ν, 2x), $\mu + \nu$ = n.

This paper is organized as follows. In Section 3.1 we prove that the graphs B(7, m, 3, n, 2) are an edge-magic total for any integer n ≥ 1 and m = 1. In Section 3.2 we prove that $B[(C_a, 1), (C_3, n), P_2]$ admits super edge-magic total for a = 7 and extends the values of a to 4x − 1 for any positive integers x. Moreover we prove that the graphs $B[(C_7, 2), (C_3, n), P_2]$ admit super edge- magic total. The results of this paper are developed from [10, 11].

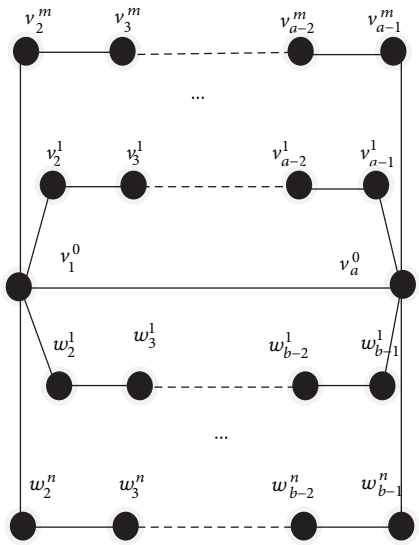

Figure 1: Graphs B(a,m,b,n,2).

2. Preliminary Notes

In this section we provide some previous results on super edge-magic total labeling of a graph. Figueroa-Centeno, Ichishima, and Muntaner-Batle [2] proved some necessary conditions for super edge-magic total labeling of a graph. We need them to prove the main results of this paper.

Theorem 1 (see [2]). *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function*

$$f : V(G) \longrightarrow \{1, 2, \ldots, p\} \tag{1}$$

such that the set $S = \{f(w) + f(z)$: $wz \in E(G)\}$ exists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $k = p + q + s$, where $s = \min(S)$ and $S = \{k - (p+1), k - (p + 2), \ldots, k - (p + q)\}$.

Theorem 2 (see [2]). *Let G be super edge-magic labeling (p, q)-graph and f be a super edge-magic labeling of G. Then*

$$\sum_{v \in V(G)} f(v) \deg(v) = qs + \frac{1}{2}(q-1)q. \tag{2}$$

In particular $2\sum_{v \in V(G)} f(v)\deg(v) = 0 \pmod q$.

3. Main Results

We sometimes interchange the role of vw and (v, w) as an element of E(G). From now on we assume that B(a, m, b, n, 2) is defined as follows. Let G be the graphs B(a, m, b, n, 2). Let A = $\{v_1^0, v_a^1\}$, B = $\{v_2^1, v_3^1, \ldots, v_{a-1}^1, v_2^2, v_3^2, \ldots, v_{a-1}^2, \ldots, v_2^m, v_3^m, \ldots, v_{a-1}^m\}$, and D = $\{w_2^1, w_3^1, \ldots, w_{b-1}^1, w_2^2, w_3^2, \ldots, w_{b-1}^2, \ldots, w_2^n, w_3^n, \ldots, w_{b-1}^n\}$. We define the vertex set V (G) = A ∪ B ∪ D and the edge set E(G) = $\{(v_1^0, v_a^1)\} \cup \{(v_{i-1}^j, v_i^j), 3 \le i \le a-1, 1 \le j \le m\} \cup \{(w_{i-1}^j, w_i^j), 3 \le i \le b-1, 1 \le j \le n\} \cup \{(v_1^0, v_2^i), (v_1^0, w_2^j), 1 \le i \le m, 1 \le i \le n\} \cup \{(v_a^0, v_{a-1}^i), (v_a^0, w_{b-1}^j), 1 \le i \le m, 1 \le j \le n\}$. A graph B(a, m, b, n, 2) is shown in Figure 1.

3.1. Edge Magic Total Labeling of (7,3)-Cycle Books. In this section we provide an edge-magic total labeling of (7, 3)-cycle books B(7, 1, 3, n, 2).

Theorem 3. *Let G be a (7, 3)-cycle books B(7, 1, 3, n, 2). Then G is an edge-magic with magic constant $k = 4n + 20$.*

Proof. Let G be a (7,3)-cycle books B(7, 1, 3, n, 2) in Figure 1 with |V(G)| = p and |E(G)| = q. We first notice that p = n + 7 and q = 2n + 7. We define the bijective function f: V (G) ∪ E(G) ⟶ {1, 2,..., p + q} or f: V (G) ∪ E(G) ⟶ {1, 2,..., 3n + 14} as in (3):

$$f(x) = \begin{cases} 1, & if\ x = v_1^0, \\ 3n + 14, & if\ x = v_1^0 v_2^1, \\ n + 5, & if\ x = v_2^1, \\ 2n + 11, & if\ x = v_2^1 v_3^1, \\ n + 4, & if\ x = v_3^1, \\ 2n + 9, & if\ x = v_3^1 v_4^1, \\ n + 7, & if\ x = v_4^1, \\ n + 3, & if\ x = v_4^1 v_5^1, \\ 2n + 10, & if\ x = v_5^1, \\ 2n + 8, & if\ x = v_5^1 v_6^1, \\ 2, & if\ x = v_6^1, \\ 3n + 12, & if\ x = v_6^1 v_7^0, \\ n + 6, & if\ x = v_7^0, \\ 3n + 13, & if\ x = v_7^0 v_1^0, \\ n + i + 7, & if\ x = v_1^0 w_2^i,\ 1 \le i \le n, \\ 3n - i + 12, & if\ x = w_2^i,\ 1 \le i \le n, \\ i + 2, & if\ x = v_7^0 w_2^i,\ 1 \le i \le n. \end{cases} \tag{3}$$

It is easy to verify that $f(w) + f(wz) + f(z) = 4n + 20$ for any edge $wz \in E(G)$, hence the theorem. □

3.2. Super Edge Magic Total Labeling of (7,3)-Cycle Books. In this section we prove that the graphs B(a, 1, 3, n, 2) are super edge-magic total for a = 4x−1, x ≥ 2, x is integer, and the graphs B(7, 2, 3, n, 2) are super edge-magic total for any positive integer n. First we provide some lemmas.

Lemma 4. *Let G be the graphs B(a, m, b, n, 2) in Figure 1, $|V(G)| = p$, $|E(G)| = q$, and let f be a super edge-magic total labeling of G.*

Let $S = \{f(w) + f(z)$: f or any edge $wz \in E(G)\}$ be q consecutive integers and $s = \min(S)$. Then

$$f(v_1^0) + f(v_a^0) = \frac{qs + (1/2)q(q-1) - p(p-1)}{m + n - 1}. \tag{4}$$

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1. Let |V(G)| = p and |E(G)| = q. We first notice that $\deg(v_1^0)$ =

$\deg(v_a^0) = m + n + 1$, $\deg(v_2^1) = \deg(v_3^1) = \ldots = \deg(v_{a-1}^1) = 2, \ldots$, $\deg(v_2^m) = \deg(v_3^m) = \ldots = \deg(v_{a-1}^m) = 2$, $\deg(w_2^1) = \deg(w_3^1) = \ldots = \deg(w_{b-1}^1) = 2, \ldots$, $\deg(w_2^n) = \deg(w_3^n) = \ldots = \deg(w_{b-1}^n) = 2$.

By Theorem 2 we have $(m+n+1)[f(v_1^0)+f(v_a^0)] + \sum_{v\in V(G)-\{v_1^0,v_a^0\}} f(v)2 = qs + q(q-1)/2$. Hence $(m+n-1)[f(v_1^0)+f(v_a^0)] + \sum_{v\in V(G)} f(v)2 = qs + q(q-1)/2$. Moreover $(m+n-1)[f(v_1^0)+f(v_a^0)] + 2(1+2+\ldots+p) = qs + q(q-1)/2$. From the last equation we conclude that

$$f\left(v_1^0\right) + f\left(v_a^0\right) = \frac{qs + (1/2)\,q\,(q-1) - p\,(p+1)}{m+n-1} \tag{5}$$

hence the lemma. □

Lemma 5. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, and $b = 3$ and let f be a super edge-magic total labeling of G. Let $S = \{f(w)+f(z)$: f or any edge $wz \in E(G)\}$ be q consecutive integers and s and $s = \min(S)$. Then*

$$\begin{aligned} \frac{3\,(m+n-1) + p\,(p+1)}{q} - \frac{1}{2}q\,(q-1) &\le s \\ \le \frac{(m+n-1)\,(2p-1) + p\,(p+1)}{q} - \frac{1}{2}q\,(q-1) \end{aligned} \tag{6}$$

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, and $b = 3$. Let $|V(G)| = p$ and $|E(G)| = q$. We first prove the upper bound of s. Notice that $f(v_1^0) + f(v_7^0) \le p + p - 1$. By Lemma 4 and the last inequality we have

$$s \le \frac{(m+n-1)\,(2p-1) + p\,(p+1)}{q} - \frac{1}{2}q\,(q-1) \tag{7}$$

hence the upper bound of s. Next we prove the lower bound of s. Notice that $f(v_1^0) + f(v_7^0) \ge 3$. By Lemma 4 and the last inequality we have $s \ge (3(m+n-1)+p(p+1))/q - (1/2)q(q-1)$, hence the lemma. □

Lemma 6. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $b = 3$, $m = 1$, and $n = 1$ and let f be an edge-magic total labeling of G. Let $S = \{f(w) + f(z)$: f or any edge $wz \in E(G)\}$ be q consecutive integers and $s = \min(S)$. Then G is a super edge-magic total if and only if*

(i) $f(v_1^0) + f(v_7^0) = 9s - 36$,

(ii) $3 \le s \le 5$,

(iii) $k = p + q + s$.

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $b = 3$, $m = 1$, and $n = 1$ and let G be a super edge-magic total.

Note that $p = (a-2)m+2+(b-2)n$ and $q = (a-3)+2m+1+(b-3)n+2n$. Let f be an edge-magic total labeling of G. If f is a super edge-magic total labeling of G, the conditions (i) and (ii) follow from (4) and (6) of Lemmas 4 and 5, respectively, and the condition (iii) follows from Theorem 1.

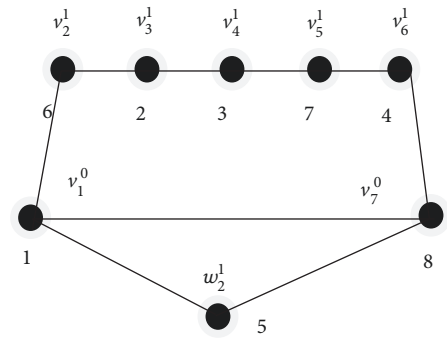

Figure 2: Bijective function $f_1(x)$.

Let f satisfy the conditions (i), (ii), and (iii). We will prove that G is super edge-magic total for a = 7, b = 3, m = 1, and n = 1. We first notice that p = 8 and q = 9.

Claim 1 (s = 5). If s = 4, then $f(v_1^0) + f(v_7^0) = 0$, a contradiction to $f(v_1^0) + f(v_7^0) \ge 3$, hence s = 5 and the claim.

By (iii) and Claim 1, we have k = 22 and by (i) we have $f(v_1^0) + f(v_7^0) = 9$. Hence either $f(v_1^0) = 1$ and $f(v_7^0) = 8$, $f(v_1^0) = 2$ and $f(v_7^0) = 7$, $f(v_1^0) = 3$ and $f(v_7^0) = 6$, or $f(v_1^0) = 4$ and $f(v_7^0) = 5$.

Case 1 ($f(v_1^0) = 1$ and $f(v_7^0) = 8$). Using k = 22, we define the bijection $f_1(x)$ in (8) such that $f_1(v_1^0) = f(v_1^0) = 1$ and $f_1(v_7^0) = f(v_7^0) = 8$.

$$f_1(x) = \begin{cases} 1, & if\ x = v_1^0, \\ 6, & if\ x = v_2^1, \\ 2, & if\ x = v_3^1, \\ 3, & if\ x = v_4^1, \\ 7, & if\ x = v_5^1, \\ 4, & if\ x = v_6^1, \\ 8, & if\ x = v_7^0, \\ 5, & if\ x = w_2^1, \end{cases} \tag{8}$$

The function $f_1(x)$ is depicted in Figure 2.

Using the definition of $f_1(x)$ we get the following: $f_1(v_3^1) + f(v_4^1) = 5$, $f_1(v_1^0) + f_1(w_2^1) = 6$, $f_1(v_1^0) + f_1(v_2^1) = 7$, $f_1(v_2^1) + f_1(v_3^1) = 8$, $f_1(v_1^0) + f_1(v_7^0) = 9$, $f_1(v_4^1) + f_1(v_5^1) = 10$, $f_1(v_5^1) + f_1(v_6^1) = 11$, $f_1(v_6^1) + f_1(v_7^0) = 12$, and $f_1(v_7^0) + f_1(w_2^1) = 13$. Hence $S = \{f_1(u) + f_1(w)$: $uw \in E(G\} = \{5, 6, \ldots, 13\}$ consists of q = 9 consecutive integers. Thus by Theorem 1 and Claim 1, we conclude that G is super edge-magic total. Moreover we conclude $f_1(uv) = k - (f_1(u) + f_1(v))$ for all $uv \in E(G)$.

Case 2 ($f(v_1^0) = 2$ and $f(v_7^0) = 7$). Using k = 22, we define the bijection $f_2(x)$ in (9) such that $f_2(v_1^0) = f(v_1^0) = 2$ and $f_2(v_7^0) = f(v_7^0) = 7$.

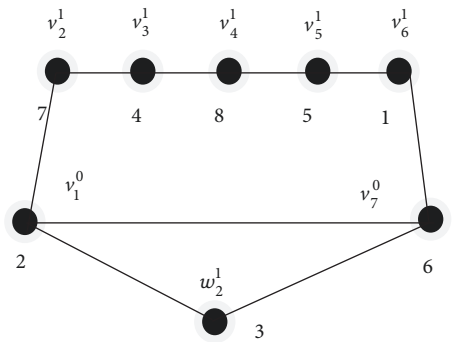

Figure 3: Bijective function $f_2(x)$.

$$f_2(x) = \begin{cases} 2, & if\ x = v_1^0, \\ 6, & if\ x = v_2^1, \\ 1, & if\ x = v_3^1, \\ 5, & if\ x = v_4^1, \\ 8, & if\ x = v_5^1, \\ 4, & if\ x = v_6^1, \\ 7, & if\ x = v_7^0, \\ 3, & if\ x = w_2^1. \end{cases} \tag{9}$$

Using the definition of $f_2(x)$ we get the following: $f_2(v_1^0) + f_2(w_2^1) = 5$, $f_2(v_3^1) + f_2(v_4^1) = 6$, $f_2(v_2^1) + f_2(v_3^1) = 7$, $f_2(v_1^0) + f_2(v_2^1) = 8$, $f_2(v_1^0) + f_2(v_7^0) = 9$, $f_2(v_7^0) + f_2(w_2^1) = 10$, $f_2(v_6^1) + f_2(v_7^0) = 11$, $f_2(v_5^1) + f_2(v_6^1) = 12$, and $f_2(v_4^1) + f_2(v_5^1) = 13$. Hence $S = \{f_2(u) + f_2(w): uw \in E(G)\} = \{5, 6, \ldots, 13\}$ consists of $q = 9$ consecutive integers. Thus by Theorem 1 and Claim 1, we conclude that G is super edge-magic total. Moreover we conclude $f_2(uv) = k - (f_2(u) + f_2(v))$ for all $uv \in E(G)$ and it is shown in Figure 3.

Case 3 ($f(v_1^0) = 3$ and $f(v_7^0) = 6$). Using $k = 22$, we define the bijection $f_3(x)$ in (10) such that $f_3(v_1^0) = f(v_1^0) = 3$ and $f_3(v_7^0) = f(v_7^0) = 6$.

$$f_3(x) = \begin{cases} 3, & if\ x = v_1^0, \\ 7, & if\ x = v_2^1, \\ 4, & if\ x = v_3^1, \\ 8, & if\ x = v_4^1, \\ 5, & if\ x = v_5^1, \\ 1, & if\ x = v_6^1, \\ 6, & if\ x = v_7^0, \\ 2, & if\ x = w_2^1, \end{cases} \tag{10}$$

The function $f_3(x)$ is depicted in Figure 4. Using the definition of $f_3(x)$ we get the following: $f_3(v_1^0) + f_3(w_2^1) = 5$, $f_3(v_5^1) + f_3(v_6^1) = 6$, $f_3(v_6^1) + f_3(v_7^0) = 7$, $f_3(v_7^0) + f_3(w_2^1) = 8$, $f_3(v_1^0) + f_3(v_7^0) = 9$, $f_3(v_1^0) + f_3(v_2^1) = 10$, $f_3(v_2^1) + f_3(v_3^1) = 11$, $f_3(v_3^1) + f_3(v_4^1) = 12$, and $f_3(v_4^1) + f_3(v_5^1) = 13$. Hence $S = \{f_3(u) + f_3(w): uw \in E(G)\} = \{5, 6, \ldots, 13\}$ consists of $q = 9$ consecutive integers. Thus by Theorem 1 and Claim 1, we conclude that G is super edge-magic total. Moreover we conclude $f_3(uv) = k - (f_3(u) + f_3(v))$ for all $uv \in E(G)$.

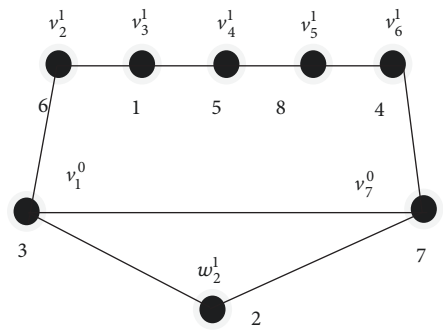

Figure 4: Bijective function $f_3(x)$.

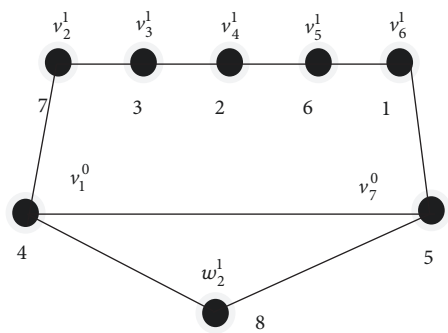

Figure 5: Bijective function $f_4(x)$.

Case 4 ($f(v_1^0) = 4$ and $f(v_7^0) = 5$). Using $k = 22$, we define the bijection $f_4(x)$ in (11) such that $f_4(v_1^0) = f(v_1^0) = 4$ and $f_4(v_7^0) = f(v_7^0) = 5$.

$$f_4(x) = \begin{cases} 4, & if\ x = v_1^0, \\ 7, & if\ x = v_2^1, \\ 3, & if\ x = v_3^1, \\ 2, & if\ x = v_4^1, \\ 6, & if\ x = v_5^1, \\ 1, & if\ x = v_6^1, \\ 5, & if\ x = v_7^0, \\ 8, & if\ x = w_2^1, \end{cases} \tag{11}$$

The function $f_4(x)$ is depicted in Figure 5. Using the definition of $f_4(x)$ we get the following: $f_4(v_3^1) + f_4(v_4^1) = 5$, $f_4(v_6^1) + f_4(v_7^0) = 6$, $f_4(v_5^1) + f_4(v_6^1) = 7$, $f_4(v_4^1) + f_4(v_5^1) = 8$, $f_4(v_1^0) + f_4(v_7^0) = 9$, $f_4(v_2^1) + f_4(v_3^1) = 10$, $f_4(v_1^0) + f_4(v_2^1) = 11$, $f_4(v_1^0) + f_4$

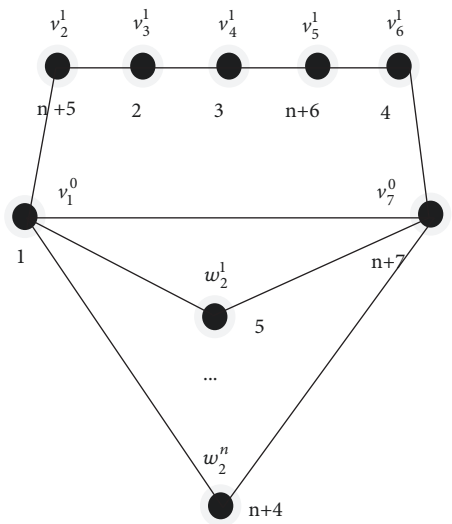

Figure 6: Bijective function $g_1(x)$.

$(w_2^1) = 12$, and $f_4(v_7^0) + f_4(w_2^1) = 13$. Hence $S = \{f_4(u) + f_4(w): uw \in E(G\} = \{5, 6,..., 13\}$ consists of $q = 9$ consecutive integers. Thus by Theorem 1 and Claim 1, we conclude that G is super edge-magic total. Moreover we conclude $f_4(uv) = k - (f_4(u) + f_4(v))$ for all $uv \in E(G)$, hence the lemma. □

Now we are ready to state our main result.

Theorem 7. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $m = 1$, and $b = 3$. Then G is a super edge-magic total. Moreover there are two bijective functions $g_1(V(G))$ and $g_2(V(G))$ with the magic constant $k = 3n + 19$.*

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $m = 1$, and $b = 3$. We first notice that $p = (a - 2)m + (b - 2)n + 2$ and $q = (a - 3)m + 2m + 1 + (b - 3)n + 2n$. For $a = 7$, $m = 1$, and $b = 3$ and $n = 1$ we have $p = 8$ and $q = 9$. Using the definition of the bijective function $f_1(x)$ in Case 1 of Lemma 6 together with $s = 5$ in Claim 1 of Lemma 6, we define the following bijective function $g_1(x)$ in (12):

$$g_1(x) = \begin{cases} 1, & if\ x = v_1^0, \\ n+5, & if\ x = v_2^1, \\ 2, & if\ x = v_3^1, \\ 3, & if\ x = v_4^1, \\ n+6, & if\ x = v_5^1, \\ 4, & if\ x = v_6^1, \\ n+7, & if\ x = v_7^0, \\ i+4, & if\ x = w_2^i,\ 1 \le i \le n \end{cases} \tag{12}$$

The function $g_1(x)$ is depicted in Figure 6. Using the definition of $g_1(x)$ we get the following: $g_1(v_3^1) + g_1(v_4^1) = 5$, $g_1(v_1^0) + g_1(w_2^1) = 6$, $g_1(v_1^0) + g_1(w_2^i) = i + 5$, $i \in \{2, 3,..., n\}$, $g_1(v_1^0) + g_1(v_2^1) = n + 6$, $g_1(v_2^1) + g(v_3^1) = n+7$, $g_1(v_1^0) + g_1(v_7^0) = n+8$, $g_1(v_4^1) + g(v_5^1) = n+9$, $g_1(v_5^1) + g_1(v_6^1) = n + 10$, $g_1(v_6^1) + g_1(v_7^0) = n + 11$, $g_1(v_7^0) + g_1(w_2^i) = n + 11 + i$, $i \in \{1, 2, 3,..., n\}$. Hence $S = \{g_1(u) + g_1(w): uw \in E(G\} = \{5, 6,..., i + 5, n + 7, n + 8, n+9, n+10, n+11, n+i+11, i \in \{1,..., n\}\}$ consists of $q = 2n+7$ consecutive integers. Thus by Theorem 1 and Claim 1 of Lemma 6, we conclude that G is super edge-magic total with magic constant $k = p+q+s = n+7+2n+7+5 = 3n + 19$. Moreover, by Theorem 1, we conclude $g_1(uv) = k - (g_1(u) + g_1(v))$ for all $uv \in E(G)$.

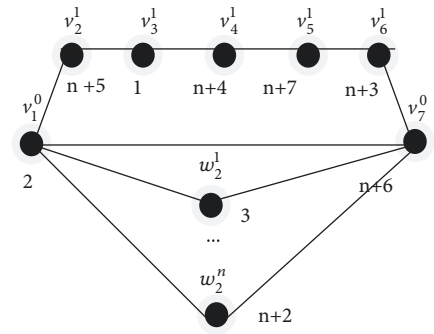

Figure 7: Bijective function $g_2(x)$.

Using the definition of the bijective function $f_2(x)$ in Case 2 of Lemma 6 together with $s = 5$ in Claim 1 of Lemma 6, we define the following bijective function $g_2(x)$ in (13):

$$g_2(x) = \begin{cases} 2, & if\ x = v_1^0, \\ n+5, & if\ x = v_2^1, \\ 1, & if\ x = v_3^1, \\ n+4, & if\ x = v_4^1, \\ n+7, & if\ x = v_5^1, \\ n+3, & if\ x = v_6^1, \\ n+6, & if\ x = v_7^0, \\ i+2, & if\ x = w_2^i,\ 1 \le i \le n \end{cases} \tag{13}$$

The function $g_2(x)$ is depicted in Figure 7. Using the definitions of $g_2(x)$ we get the following: $g_2(v_1^0) + g_2(w_2^i) = i + 4$, $i \in \{1, 2,..., n\}$, $g_2(v_3^1) + g_2(v_4^1) = n + 5$, $g_2(v_2^1) + g_2(v_3^1) = n + 6$, $g_2(v_1^0) + g_2(v_2^1) = n+7$, $g_2(v_1^0) + g_2(v_7^0) = n+8$, $g_2(v_7^0) + g_2(w_2^i) = n+8+i$, $i \in \{1, 2,..., n\}$, $g_2(v_7^0) + g_2(v_6^1) = 2n + 9$, $g_2(v_6^1) + g_2(v_5^1) = 2n + 10$, $g_2(v_5^1) + g_2(v_4^1) = 2n + 11$. Hence $S = \{g_2(u) + g_2(w): uw \in E(G\} = \{5, 6,..., i + 4, n + 5, n + 6, n + 7, n + 8, n + 8 + i, 2n + 9, 2n + 10, 2n + 11, i \in \{1, 2,..., n\}\}$ consists of $q = 2n + 7$ consecutive integers. Thus by Theorem 1 and Claim 1 of Lemma 6, we conclude that G is super edge-magic total with magic constant $k = p + q + s = n + 7 + 2n + 7 + 5 = 3n + 19$.

Moreover, by Theorem 1, we conclude $g_2(uv) = k - (g_2(u) + g_2(v))$ for all $uv \in E(G)$, hence the theorem. □

Theorem 8. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 such that $|V(G)| = p$, $|E(G)| = q$, $a = 4x - 1$, $m = 1$, and $b = 3$.*

Let α_2: $V(G) \longrightarrow \{1, 2, \ldots, p\}$ be a bijective function such that

$$\begin{aligned}
&\alpha_2\left(v_i^j\right) = n - 1 + 2x + \frac{1}{2}(i+2), \\
&\qquad \text{if } i \text{ is even}, \ 2 \le i \le 2x - 2, \ j = 1, \\
&\alpha_2\left(v_i^j\right) = \frac{1}{2}(i+2), \\
&\qquad \text{if } i \text{ is even}, \ 2x \le i \le 4x - 2, \ j = 1, \\
&\alpha_2\left(v_i^j\right) = \frac{1}{2}(i+2), \quad i = 1, \ j = 0, \\
&\alpha_2\left(v_i^j\right) = \frac{1}{2}(i+1), \\
&\qquad \text{if } i \text{ is odd}, \ 1 \le i \le 2x - 1, \ j = 1, \\
&\alpha_2\left(v_i^j\right) = n - 1 + 2x + \frac{1}{2}(i+1), \\
&\qquad \text{if } i \text{ is odd}, \ 2x + 1 \le i \le 4x - 3, \ j = 1, \\
&\alpha_2\left(v_i^j\right) = n - 1 + 2x + \frac{1}{2}(i+1), \\
&\qquad \text{if } i = 4x - 1, \ j = 0, \\
&\alpha_2\left(w_i^j\right) = 2x + j, \quad 2 \le i \le b - 2, \ 1 \le j \le n.
\end{aligned} \tag{14}$$

Then $\alpha_2(v)$ is a super edge-magic total labeling of G with the magic constant $k = 10x + 3n - 1$, $v \in V(G)$.

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$ and $|E(G)| = q$. Let $a = 4x - 1$, $m = 1$, and $b = 3$. Notice that $p = 4x - 1 + n$ and $q = 2n + 4x - 1$. Let $S = \{\alpha_2(u) + \alpha_2(w): uw \in E(G\}$. We show that S consists of q consecutive integers.

Using the definition of $\alpha_2(v)$ in (14), $v \in V(G)$, we get the following:

$$\begin{aligned}
&\alpha_2\left(v_{2x-1}^1\right) + \alpha_2\left(v_{2x}^1\right) = 2x + 1, \\
&\alpha_2\left(v_1^0\right) + \alpha_2\left(w_2^i\right) = 2x + 1 + i, \quad 1 \le i \le n, \\
&\alpha_2\left(v_1^0\right) + \alpha_2\left(v_2^1\right) = n + 2x + 2, \\
&\alpha_2\left(v_2^1\right) + \alpha_2\left(v_3^1\right) = n + 2x + 3, \\
&\cdots \\
&\alpha_2\left(v_{x-2}^1\right) + \alpha_2\left(v_{x-1}^1\right) = n + 3x - 1, \\
&\alpha_2\left(v_{x-1}^1\right) + \alpha_2\left(v_x^1\right) = n + 3x, \\
&\alpha_2\left(v_x^1\right) + \alpha_2\left(v_{x+1}^1\right) = n + 3x + 1, \\
&\cdots \\
&\alpha_2\left(v_{2x-2}^1\right) + \alpha_2\left(v_{2x-1}^1\right) = n + 4x - 1, \\
&\alpha_2\left(v_1^0\right) + \alpha_2\left(v_{4x-1}^0\right) = n + 4x, \\
&\alpha_2\left(v_{2x}^1\right) + \alpha_2\left(v_{2x+1}^1\right) = n + 4x + 1, \\
&\cdots \\
&\alpha_2\left(v_{3x-2}^1\right) + \alpha_2\left(v_{3x-1}^1\right) = n + 5x - 1, \\
&\alpha_2\left(v_{3x-1}^1\right) + \alpha_2\left(v_{3x}^1\right) = n + 5x, \\
&\alpha_2\left(v_{3x}^1\right) + \alpha_2\left(v_{3x+1}^1\right) = n + 5x + 1, \\
&\cdots \\
&\alpha_2\left(v_{4x-2}^1\right) + \alpha_2\left(v_{4x-1}^1\right) = n + 6x - 1, \\
&\alpha_2\left(v_{4x-1}^0\right) + \alpha_2\left(w_2^i\right) = n + 6x - 1 + i, \quad 1 \le i \le n.
\end{aligned} \tag{15}$$

Hence by 3.2 we have S = {2x + 1, 2x + 2, 2x + 3,..., 2x + n + 1, 2x + n + 2,..., 3x + n −1, 3x + n, 3x + n + 1,..., 4x + n − 1, 4x + n, 4x + n + 1,..., 5x + n − 1, 5x + n, 5x +n + 1,..., 6x + n − 1, 6x + n, 6x + n + 1,..., 6x + 2n − 2, 6x + 2n −1} consists of q consecutive integers. Thus by Theorem 1 we conclude that G is super edge-magic total. Notice that s = 2x + 1. Hence k = p + q + s = 10x + 3n − 1. Moreover, by Theorem 1, we conclude $\alpha_2(uv) = k - (\alpha_2(u) + \alpha_2(v))$ for all $uv \in E(G)$, hence the theorem. □

Lemma 9. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $m = 2$, $b = 3$, and $n = 1$ and let f be an edge-magic total labeling of G. Let $S = \{f(w) + f(z)$: f or any edge $wz \in E(G)\}$ be q consecutive integers and $s = \min(S)$. Then G is a super edge-magic total if and only if*

(i) $f(v_1^0) + f(v_7^0) = (15s - 77)/2$,

(ii) $6 \le s \le 7$,

(iii) $k = p + q + s$.

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with $|V(G)| = p$, $|E(G)| = q$, $a = 7$, $m = 2$, $b = 3$, and $n = 1$ and let G be a super edge-magic total. Let f be an edge-magic total labeling of G. If f is a super edge-magic total labeling of G, the conditions (i) and (ii) follow from Lemmas 4 and 5, respectively, and the condition (iii) follows from Theorem 1.

Let f satisfy the conditions (i), (ii), and (iii). We first notice that $p = (a - 2)m + (b - 2)n + 2$ and $q = (a - 3)m + 2m + 1 + (b - 3)n + 2n$. We will prove that G is super edge-magic total for $a = 7$, $m = 2$, $b = 3$, and $n = 1$. Notice that $p = 13$ and $q = 15$.

Claim 1 (s = 7). If s = 6, then $f(v_1^0) + f(v_7^0) = 6{,}5$ (by (i)); a contradiction to $f(v_1^0) + f(v_7^0)$ is an integer, hence s = 7 and the claim. By (iii) and Claim 1, we have k =35 and by (i) we have $f(v_1^0) + f(v_7^0) = 14$.

Next we will show that there is a bijection $h_1(x)$ such that $h_1(v_1^0) = f(v_1^0)$, $h_1(v_7^0) = f(v_7^0)$, $h_1(v_1^0) + h_1(v_7^0) = 14$, and $h_1(x)$ is a super edge-magic total labeling of B(7, 2, 3, 1). Let $h_1(v_1^0) =$

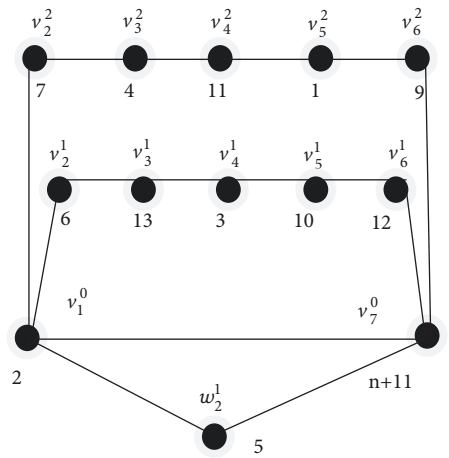

Figure 8: Bijective function $h_1(x)$.

$f(v_1^0) = 2$ and $h_1(v_7^0)=f(v_7^0)=12$. Using k = 35, we define the bijection $h_1(x)$ in (16):

$$h_1(x) = \begin{cases} 2, & if\ x = v_1^0, \\ 6, & if\ x = v_2^1, \\ 13, & if\ x = v_3^1, \\ 3, & if\ x = v_4^1, \\ 10, & if\ x = v_5^1, \\ 8, & if\ x = v_6^1, \\ 12, & if\ x = v_7^0, \\ 7, & if\ x = v_2^2, \\ 4, & if\ x = v_3^2, \\ 11, & if\ x = v_4^2, \\ 1, & if\ x = v_5^2, \\ 9, & if\ x = v_6^2, \\ 5, & if\ x = w_2^1. \end{cases} \tag{16}$$

Using the definition of $h_1(x)$ we get the following: $h_1(v_1^0) + h_1(w_2^1) = 7$, $h_1(v_1^0)+h_1(v_2^1) = 8$, $h_1(v_1^0)+h_1(v_2^2) = 9$, $h_1(v_5^2)+h_1(v_6^2) = 10$, $h_1(v_2^2)+h_1(v_3^2) = 11$, $h_1(v_4^2) + h_1(v_5^2) = 12$, $h_1(v_4^1) + h_1(v_5^1) = 13$, $h_1(v_1^0) + h_1(v_7^0) = 14$, $h_1(v_3^2) + h_1(v_4^2) = 15$, $h_1(v_3^1) + h_1(v_4^1) = 16$, $h_1(v_7^0) + h_1(w_2^1) = 17$, $h_1(v_5^1) + h_1(v_6^1) = 18$, $h_1(v_2^1) + h_1(v_3^1) = 19$, $h_1(v_6^1) + h_1(v_7^0) = 20$, $h_1(v_6^2) + h_1(v_7^0) = 21$. Hence $S = \{h_1(u) + h_1(w): uw \in E(G\} = \{7, 8,..., 21\}$ consists of q = 15 consecutive integers. Thus by Theorem 1 and Claim 1, we conclude that G is super edge-magic total. Moreover we conclude $h_1(uv) = k - (h_1(u) + h_1(v))$ for all $uv \in E(G)$ and it is shown in Figure 8, hence the theorem. □

Theorem 10. *Let G be the graphs B(a, m, b, n, 2) in Figure 1 with |V(G)| = p and |E(G)| = q, a = 7, m = 2, and b = 3. Then G is a super edge-magic total with the magic constant k = 3n + 32.*

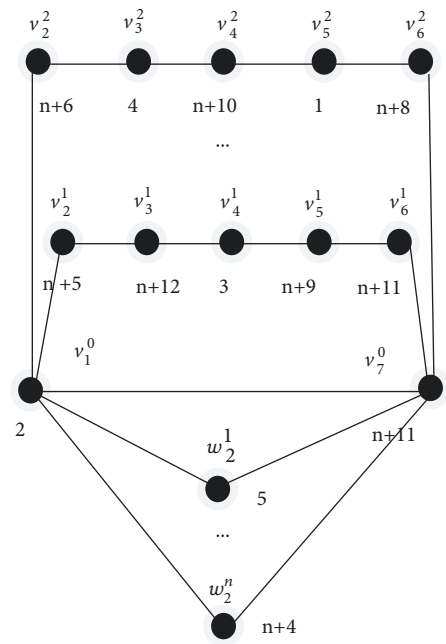

Figure 9: Bijective function $h_2(x)$.

Proof. Let G be the graphs B(a, m, b, n, 2) in Figure 1 with |V(G)| = p, |E(G)| = q, a = 7, m = 2, and b = 3. We first notice that p = (a − 2)m + (b − 2)n + 2 and q = (a − 3)m + 2m + 1 + (b − 3)n + 2n. Using the definition of the bijective function $h_1(x)$ and s = 7 in Lemma 9, we define the following bijective function $h_2(x)$ in (17):

$$h_2(x) = \begin{cases} 2, & if\ x = v_1^0, \\ n+5, & if\ x = v_2^1, \\ n+12, & if\ x = v_3^1, \\ 3, & if\ x = v_4^1 \\ n+9, & if\ x = v_5^1 \\ n+7, & if\ x = v_6^1, \\ n+11, & if\ x = v_7^0, \\ n+6, & if\ x = v_2^2, \\ 4, & if\ x = v_3^2, \\ n+10, & if\ x = v_4^2, \\ 1, & if\ x = v_5^2, \\ n+8, \cdots & if\ x = v_6^2 \\ i+4, & 1 \le i \le n. \end{cases} \tag{17}$$

The function $h_2(x)$ is depicted in Figure 9. Using the definition of $h_2(x)$ we get the following: $h_2(v_1^0)+h_2(w_2^i) = i + 6$,

$i \in \{1,\ldots, n\}$, $h_2(v_1^0) + h_2(v_2^1) = n + 7$, $h_2(v_1^0)+h_2(v_2^2) = n+8$, $h_2(v_5^2)+h_2(v_6^2) = n+9$, $h_2(v_2^2) + h_2(v_3^2) = n + 10$, $h_2(v_4^2) + h_2(v_5^2) = n + 11$, $h_2(v_4^1) + h_2(v_5^1) = n + 12$, $h_2(v_1^0) + h_2(v_7^0) = n + 13$, $h_2(v_3^2) + h_2(v_4^2) = n + 14$, $h_2(v_3^1) + h_2(v_4^1) = n + 15$, $h_2(v_7^0) + h_2(w_2^i) = n + 15 + i$, $i \in \{1,\ldots, n\}$, $h_2(v_5^1) + h_2(v_6^1) = 2n + 16$, $h_2(v_2^1) + h_2(v_3^1) = 2n + 17$, $h_2(v_7^0) + h_2(v_6^1) = 2n + 18$, $h_2(v_7^0) + h_2(v_6^2) = 2n + 19$. Hence $S = \{h_2(u) + h_2(w): uw \in E(G\} = \{7, 8,\ldots, n + 6, n + 7, n + 8, n + 9, n +10, n + 11, n + 12, n + 13, n + 14, n + 15, n + 16,\ldots, 2n + 15, 2n + 16, 2n + 17, 2n +18, 2n + 19\}$ consists of $q = 2n + 13$ consecutive integers. Thus by Theorem 1 and Claim 1 of Lemma 9, we conclude that G is super edge-magic total with magic constant $k = p + q + s = n + 12 + 2n + 13 + 7 = 3n + 32$. Moreover, by Theorem 1, we conclude $h_2(uv) = k - (h_2(u) + h_2(v))$ for all $uv \in E(G)$, hence the theorem. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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